(1) $\sum_{n=1}^{\infty} \frac{(z-3)^n}{n 5^n}$ $a_n = \frac{1}{n 5^n}$ $z_0 = 3$

$w = z-3$

$\sum_{n=1}^{\infty} \frac{w^n}{n 5^n}$ $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\frac{1}{n 5^n}}{\frac{1}{(n+1) 5^{n+1}}} = \lim_{n\to\infty} 5 \frac{n+1}{n} = 5$

řada konverguje absolutně pro $|w| < 5$

diverguje pro $|w| > 5$

$|w| = 5$? $w = 5(\cos\varphi + i \sin\varphi)$

$w^n = 5^n (\cos n\varphi + i \sin n\varphi)$

$$\sum_{n=1}^{\infty} \frac{w^n}{n 5^n} = \sum_{n=1}^{\infty} \left(\frac{\cos n\varphi}{n} + i \frac{\sin n\varphi}{n} \right) = \sum_{n=1}^{\infty} \frac{\cos n\varphi}{n} + i \sum_{n=1}^{\infty} \frac{\sin n\varphi}{n}$$

$\sum \cos n\varphi$ má OČS pro $\varphi \neq 2k\pi$

$\sum_{n=1}^{\infty} \frac{\cos n\varphi}{n}$ konverguje podle Dirichletova kritéria pro $\varphi \neq 2k\pi$, $k \in \mathbb{Z}$

$\frac{1}{n} \searrow 0$ $\varphi = 2k\pi \Rightarrow \cos n\varphi = 1 \Rightarrow \sum \frac{1}{n} = \infty$

$\sum \sin n\varphi$ má OČS $\varphi \in \mathbb{R}$

$\sum_{n=1}^{\infty} \frac{\sin n\varphi}{n}$ konverguje pro $\varphi \in \mathbb{R}$

$\frac{1}{n} \searrow 0$

$$\sum \frac{w^n}{n5^n} \quad \text{konverguje pro} \quad w = 5(\cos\varphi + i\sin\varphi) \quad \varphi \in \mathbb{R} \setminus \{2k\pi, k \in \mathbb{Z}\}$$

$$z = w + 3$$

(2) $\sum_{n=1}^{\infty} a^{(n^2)} z^n \quad a > 0$

$$R = \begin{cases} \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{a^{n^2}}{a^{n^2+2n+1}} = \lim_{n\to\infty} \frac{1}{a^{2n+1}} \begin{cases} \underline{\infty} & a<1 \\ 1 ? & a=1 \\ \underline{0} & a>1 \end{cases} \\ \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}} \end{cases}$$

$a=1 \quad \sum_{n=1}^{\infty} \underbrace{z^n}_{\alpha_n} \quad \underline{|z|=1}$ řady diverguje

$|\alpha_n| = 1 \not\to 0 \qquad \alpha_n \to 0 \Leftarrow\!\Rightarrow |\alpha_n| \to 0$

(4) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{(n^2)} (z-1)^n \Rightarrow w = z-1 \quad \sum_{n=1} \left|\left(1+\frac{1}{n}\right)^{(n^2)} w^n\right|$

$R = \frac{1}{e} \qquad \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \left(1+\frac{1}{n}\right)^n \to e$

$\uparrow \underline{e^{n\log(1+\frac{1}{n})}}$

$|w| = \frac{1}{e}$

$\sum^{\infty} \left(\ldots\right)^{(n^2)} \left(\frac{1}{\ldots}\right) \qquad \lim \alpha_n = \frac{1}{\ldots} \neq 0$

$$\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^{(n^2)} \frac{1}{e^n}}_{\alpha_n} \qquad \lim_{n\to\infty} \alpha_n = \frac{1}{\sqrt{e}} \neq 0$$

$$\alpha_n = e^{n^2 \log\left(1+\frac{1}{n}\right) - n} \quad \to -\frac{1}{2}$$

$$\lim_{n\to\infty}\left(n^2 \log\left(1+\frac{1}{n}\right) - n\right) = \lim_{x\to\infty} x^2 \log\left(1+\frac{1}{x}\right) - x$$

$$= \lim_{x\to 0+} \frac{1}{x^2}\log(1+x) - \frac{1}{x}$$

$$\log(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$= \lim_{x\to 0+} \frac{\log(1+x) - x}{x^2}$$

$$= \lim_{x\to 0} \frac{x - \frac{x^2}{2} - x + o(x^2)}{x^2} = -\frac{1}{2}$$

Řada diverguje všude na kruhu konvergence.

$$\frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}$$

(5)

$$\sum_{n=0}^{\infty} n z^n = z \sum_{n=1}^{\infty} 1 \cdot n z^{n-1} \qquad z^{2n} = (z^2)^n$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\left(\sum^{\infty} a_n z^n\right)' = \sum^{\infty} a_n n \cdot z^{n-1} \qquad R = 1 \qquad z \in \mathbb{D}$$

$$\left(\sum_{n=0} a_n z^n\right)' = \sum_{n=1} a_n\, n \cdot z^{n-1} \qquad R=1 \qquad z\in\mathbb{R}$$

$$\sum_{n=0}^{\infty} n z^n = z\left(\sum_{n=0}^{\infty} z^n\right)' = z\left(\frac{1}{1-z}\right)' = z\,\frac{(-1)\cdot(-1)}{(1-z)^2}$$

$$= \frac{z}{(1-z)^2} \qquad z\in(-1,1)$$

• (6) $\underline{R=1}$

$$\left(\overbrace{\sum_{n=1}^{\infty} \frac{x^n}{n}}^{f(x)}\right)' = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{1-x} \to f(x) \overset{c}{=} \int \frac{1}{1-x}\,dx = -\log(1-x)$$

$$f(0)=0 \qquad -\log(1-0)=0$$

$$f(x) = -\log(1-x), \quad x\in(-1,1).$$

(7)

$$\sum_{n=2}^{\infty} n(n-1) z^{n-2} = \sum_{n=2}^{\infty} n^2 z^n - \sum_{n=2}^{\infty} n z^n$$

$$\downarrow$$

$$z\sum \underbrace{n}_{a_n}\, n\, z^{n-1}$$

a_n

(8) $\sum_{n=1}^{\infty} \frac{1}{n2^n} \to f\left(\frac{1}{2}\right)$, $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

$\frac{1}{2} \in (-1,1)$

$= -\log\left(1-\frac{1}{2}\right) = -\log\left(\frac{1}{2}\right)$

$g(x) = \sum_{n=1}^{\infty} \frac{1}{n2^n} x^n \leftarrow g(1) = ?$

$y = \frac{x}{2}$

$\frac{1}{n}\left(\frac{x}{2}\right)^n$

$\sum_{n=1}^{\infty} \frac{n^2}{n!} x^{n-1}$

(9) $\sum_{n=0}^{\infty} \frac{n^2}{n!} \to \sum_{n=0}^{\infty} \frac{n^2}{n!} x^n = f(x)$

$f(1)$

$(x^n)'$

$f(x) = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^n = x \sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^{n-1}$

$= x\left(\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}\right)' = x\left(x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}\right)'$

$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = e^x$

$= x(xe^x)' = x(e^x + xe^x)$ $\quad x \in \mathbb{R}$

$\sum^{\infty} \frac{n^2}{n!} = 1 \cdot (e^1 + 1e^1) = 2e$

$$\sum_{n=0} \frac{n^2}{n!} = 1\cdot(e^1+1e^1) = \underline{\underline{2e}}$$

(10)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \to \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{2n-1} = S(x) \qquad S(0)=0$$

$$x^{2(n-1)} = (x^2)^{n-1} \qquad R=1$$

$$\left(\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{2n-1}\right)' = \sum_{n=1}^{\infty} (-1)^n x^{2n-2} = \sum_{n=1}^{\infty} (-1)^n (x^2)^{n-1}$$

$$= -\sum_{n=1}^{\infty} (-x^2)^{n-1} = -\sum_{n=0}^{\infty} (-x^2)^n$$

$$= -\frac{1}{1-(-x^2)} = -\frac{1}{1+x^2}$$

$$S'(x) = -\frac{1}{1+x^2} \to S(x) \overset{c}{=} -\operatorname{arctg} x$$

$$-\operatorname{arctg} 0 = 0$$

$$S(x) = -\operatorname{arctg} x$$

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ konv $\quad \frac{1}{2n-1} \searrow 0$

T. l. (podle Abelovy věty)

...

Tedy (podle Abelovy věty)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \lim_{x\to 1-} f(x) = -\operatorname{arctg} 1 = \underline{\underline{-\frac{\pi}{4}}}$$

(11) $$\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}}_{\text{AK}} x^{n+1} = f(x) \qquad R = ?$$

$$\frac{1}{n^2}$$

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n = \sum_{n=1}^{\infty} \frac{(\overset{z}{-x})^n}{n} = -\log(1-z) = -\log(1+x)$$

$$f(x) \overset{c}{=} -\int 1\cdot\log(1+x)\,dx = \left| \begin{matrix} u = \log(1+x) & u' = \frac{1}{1+x} \\ v' = 1 & v = 1+x \end{matrix} \right|$$

$f(0) = 0$

$$= -\log(1+x)\cdot(1+x) + \int 1 \overset{c}{=} -\log(1+x)\cdot(1+x) + x$$

$\uparrow 0$ pro $x=0$

$$f(x) = -\log(1+x)\cdot(1+x) + x$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = \lim_{x\to 1-} f(x) = \underline{\underline{-\log 4 + 1}}$$

$\sum_{n=1} n(n+1)$ $\quad x \to 1-$