

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in U(a, \delta) : f(x) \in U(f(a), \varepsilon)$$

↑
f je spojitá v a

✓ IR

$$\sqrt{x^2 + y^2} < \varepsilon$$

✓ a=b=0

$$U((a,b), \varepsilon) = \{(x,y) \in \mathbb{R}^2 : \sqrt{(a-x)^2 + (b-y)^2} < \varepsilon\}$$

$$(1) \quad f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Mimo (0,0):

- polynomy a elementární funkce jsou spojité
- součet, součin, podíl a složené spojité funkce jsou spojité (pokud jsou definovány)

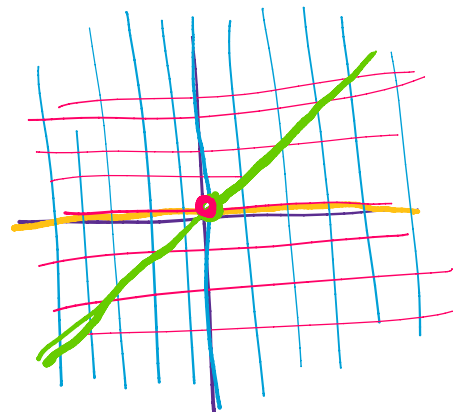
f je spojitá v bodech $\mathbb{R}^2 \setminus \{(0,0)\}$ (podíl polynomů)

v bode (0,0):

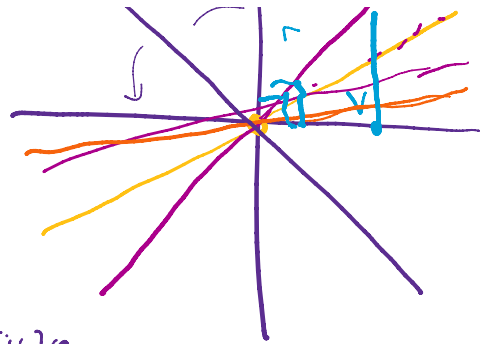
$$f(x,0) = \begin{cases} 0 & x \neq 0 \\ 0 & x = 0 \end{cases} = 0$$

$$f(0,y) = 0$$

$$f(x,x) = \begin{cases} \frac{x^2}{2x^2} = \frac{1}{2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$$f(x, kx) = \frac{x \cdot kx}{x^2 + k^2 x^2} = \frac{k}{1+k^2}$$



$$f(r \cos \varphi, r \sin \varphi) = \frac{r \cos \varphi \cdot r \sin \varphi}{r^2 \cos^2 \varphi + r^2 \sin^2 \varphi} = \frac{r^2 \frac{1}{2} \sin 2\varphi}{r^2} = \frac{1}{2} \sin 2\varphi$$

$$(2) \quad f(x, y) = \begin{cases} \frac{x y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Minim (0,0) : spojibá (podíl polynomů)

✓ bodě (0,0):

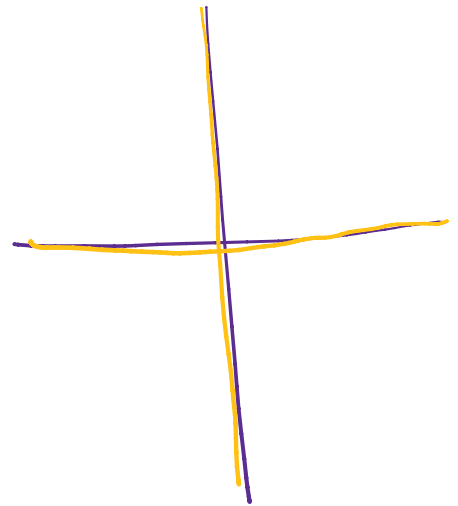
$$f(x, 0) = f(0, y) = 0$$

$$f(x, x) = \begin{cases} \frac{x^3}{2x^2} = \frac{x}{2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$x \neq 0 \quad f(x, kx) = \frac{x^3 k^2}{x^2 + k^2 x^2} = \frac{x k^2}{1+k^2}$$

$$f(x, x^2) = \dots$$

$$f(r \cos \varphi, r \sin \varphi) = \frac{r^3 \cos \varphi \sin^2 \varphi}{r^2 \cos^2 \varphi + r^2 \sin^2 \varphi} = r \cos \varphi \sin^2 \varphi$$

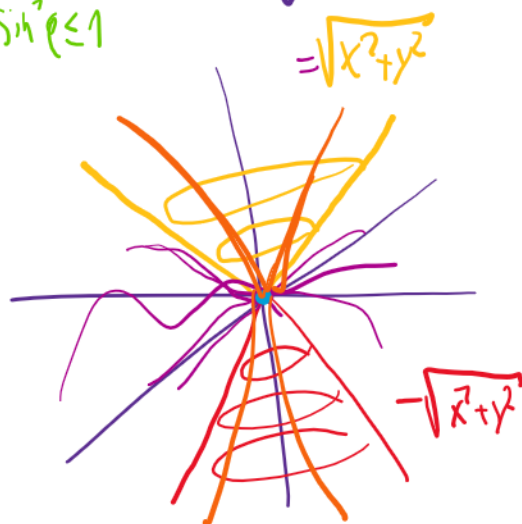


$$\underbrace{r \cos \varphi + r' \sin \varphi}_{r^2}$$

$$r = \sqrt{x^2 + y^2}$$

$$\left| \underbrace{f(r \cos \varphi, r \sin \varphi) - f(0,0)}_{\approx 0} \right| = \underbrace{r \cos \varphi \sin^2 \varphi}_{\substack{|\cos \varphi| \leq 1 \\ \sin^2 \varphi \leq 1}} \leq r = g(r)$$

$$\varepsilon = \delta$$



$$\downarrow$$

$$\left| f(r \cos \varphi, r \sin \varphi) - f(0,0) \right| \leq g(r), \quad \underbrace{g(r) \rightarrow 0}_{r \rightarrow 0+}$$

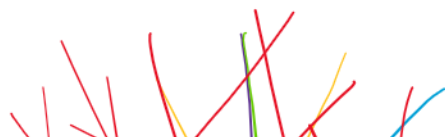
potom je f spojité v $(0,0)$

$$\lim_{x \rightarrow 0} f(x) = L$$

$$\left| f(x,y) - f(0,0) \right| \leq \frac{|x| y^2}{x^2 + y^2} \leq \sqrt{x^2 + y^2} < \delta = \varepsilon$$

$\delta = \varepsilon$

$$(3) \quad \lim_{x,y \rightarrow 0} \frac{x^2 y}{x^2 + y^2} = f(x,y)$$



$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^4+y^2}$$

$x, y \neq 0$

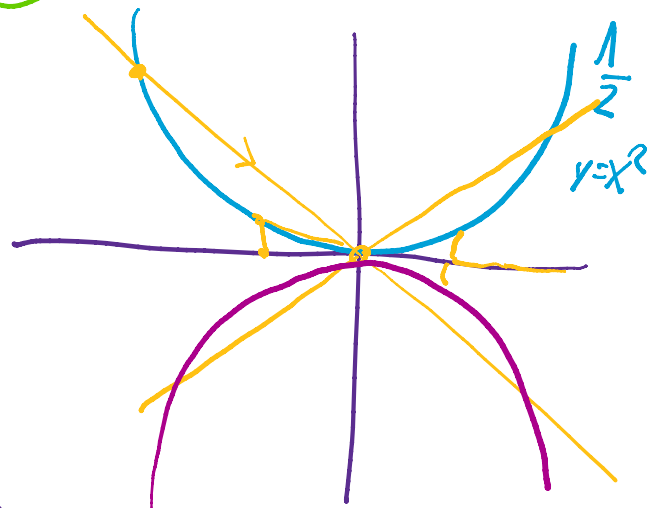
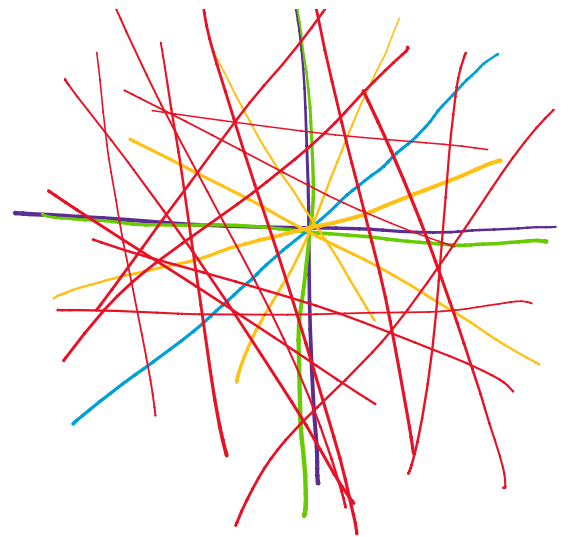
$$f(x,0) = f(0,y) = \underline{\underline{0}}$$

$$f(x,x) = \frac{x^3}{x^4+x^2} = \frac{x}{x^2+1} \rightarrow \underline{\underline{0}}$$

$$f(x,kx) = \frac{kx^3}{x^4+k^2x^2} = \frac{kx}{x^2+k^2} \rightarrow \underline{\underline{0}}$$

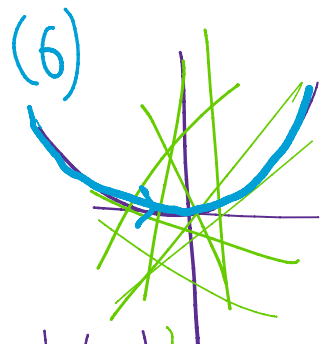
$$f(x,x^2) = \frac{x^3 \cdot x^2}{x^4+x^4} = \underline{\underline{\frac{1}{2}}}$$

$$f(x,-x^2) = \frac{x^3(-x^2)}{x^4+x^4} = \underline{\underline{-\frac{1}{2}}}$$



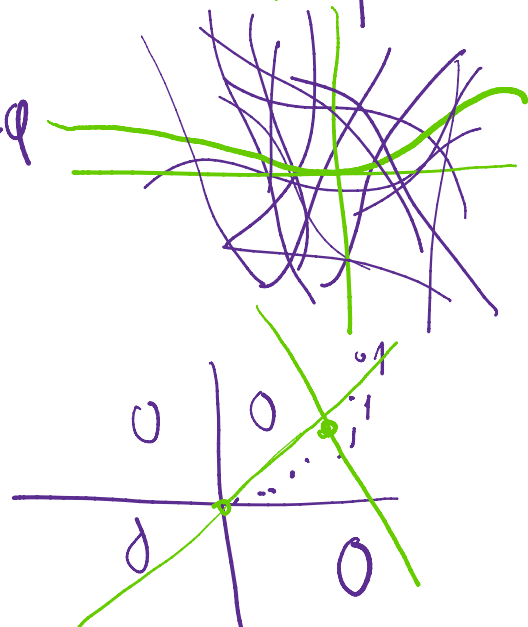
$$f(r \cos \theta, r \sin \theta) = \frac{r^3 \cos^3 \theta \sin \theta}{r^4 \cos^4 \theta + r^2 \sin^2 \theta}$$

$$= \frac{r \cos^3 \theta \sin \theta}{\underbrace{r^2 \cos^4 \theta + \sin^2 \theta}_{\leq g(r)}} \leq g(r)$$



$$\frac{xy}{\sqrt{x^2+y^2}} \rightarrow \frac{r^2 \cos \theta \sin \theta}{r} = r \cos \theta \sin \theta$$

$$\frac{xy}{\underbrace{1+y^2+x^2}_{\leq 1}} \leq g(r)$$



0 / 0

$$(5) \lim_{(x,y) \rightarrow (0,0)} \log \frac{x^2+y^2}{r^2} = \frac{\sqrt{x^2+y^2}}{r} \log(r^2) \cdot r$$

$$\lim_{r \rightarrow 0^+} \log(r^2) \cdot r = 2 \lim_{r \rightarrow 0^+} \frac{\log r}{\frac{1}{r}} = 2 \lim_{r \rightarrow 0^+} \frac{\frac{1}{r}}{-\frac{1}{r^2}} = 0$$

$$(7) \lim_{(x,y) \rightarrow (0,0)} (x^2+y^2)^{x^2y^2} = \lim_{(x,y) \rightarrow (0,0)} e^{x^2y^2 \log(x^2+y^2)}$$

$$(8) \frac{\sqrt{x^2+y^2+1} - 1}{x^2+y^2} \rightarrow a-b^2$$

$$(10) \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x,y) \right) = \underline{\underline{L}}$$

$\uparrow$
 $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = M$

$$f(x,y) = x^2 + y^2$$

$$\frac{\partial f}{\partial x} = 2x + 0 = 2x$$

$$\frac{\partial f}{\partial x} = 2x + 0 = 2x$$

$$S(x, y) = x^2 + C$$

$$(1) f(x, y, z) = xyz - x^3y + 3z^4$$

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\frac{\partial f}{\partial x} = yz - 3x^2y \quad \frac{\partial f}{\partial y} = xz - x^3 \quad \frac{\partial f}{\partial z} = xy + 12z^3$$

$$(2) S(x, y) = \sin(xy)$$

$$\frac{\partial f}{\partial x} = y \cos(xy) \quad \frac{\partial f}{\partial y} = x \cos(xy)$$

$$(3) f(x, y, z) = x^{(y^z)^{\alpha}}$$

$\bullet (x^a)^b = a x^{a-1}$
 $\frac{x^{(y^z)}}{x^{(y^z)}} (a^x)^b = \log a \cdot a^x$

$\nabla f = ?$

$$\frac{\partial f}{\partial x} = y^z x^{y^z-1} \quad \frac{\partial f}{\partial y} = \log x \cdot x^{(y^z)} \cdot z \cdot y^{z-1}$$

$$\frac{\partial f}{\partial z} = \log x \cdot x^{(y^z)} \cdot \log y \cdot y^z$$

$$(4) \cdot \underline{\underline{\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}}}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial x}$$

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) \neq \frac{\partial^2 f}{\partial x \partial y}(0, 0)$$

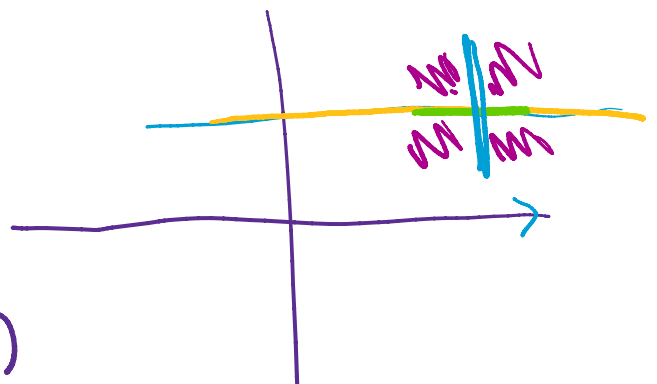
$\mathbb{C}^2 \rightarrow$ symplectic p.d. Züßchen $\rightarrow$ müssen rechnen

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \rightarrow \text{symmetrisch}$$

$$\frac{\partial f}{\partial x_i}(a) = \lim_{\delta \rightarrow 0} \frac{f(a_1, \dots, a_i + \delta, \dots, a_n) - f(a)}{\delta}$$

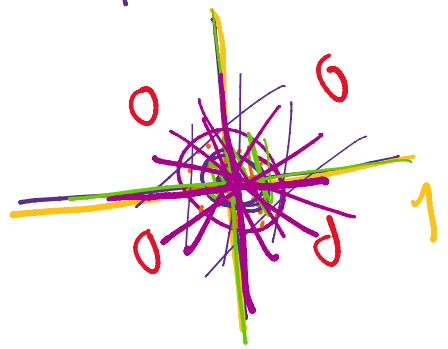
$$g(t) = f(\underline{a_1, \dots, a_{i-1}}, t, \underline{a_{i+1}, \dots, a_n})$$

$$g'(t) \rightarrow \frac{\partial f}{\partial x_i}(a) = g'(a_i)$$



$$g'(t) \rightarrow \frac{\partial}{\partial x_i}(a) = \underline{\underline{g'(a_i)}}$$

$$f(x,y) = \begin{cases} 1 & xy=0 \\ 0 & \text{jin } ax \end{cases}$$



$$D_v f(a) = \lim_{t \rightarrow 0} \frac{f(a+tv) - f(a)}{t}$$

$$v \in \mathbb{R}^d$$

$$v = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) = e_i \rightarrow \underline{\underline{\frac{\partial}{\partial x_i}}}$$

$$\sum \frac{1}{n2^n} \rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n} = f(x)$$

$$R=1$$

$$\underline{\underline{\frac{1}{2} \rightarrow x}}$$

$$\left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right)' = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

$$f(x) = \frac{1}{1-x} \rightarrow f(x) = -\log(1-x)$$

$$\sum \frac{1}{n2^n} = f\left(\frac{1}{2}\right) = -\log\left(1-\frac{1}{2}\right) = -\log\left(\frac{1}{2}\right)$$