$$a_n y^{(n)} + \dots + a_0 y = 0 \qquad a_0, \dots, a_n \in \mathbb{R}$$

$$\Updownarrow$$

$$p(\lambda) = a_n \lambda^n + \dots + a_0 = 0 \qquad \overset{\text{kořen}}{\lambda} \to y = e^{\lambda x}$$

$\lambda \in \mathbb{R}$ kořen nás. k : $\{\underbrace{e^{\lambda x}, xe^{\lambda x}, \dots, x^{k-1} e^{\lambda x}}_{k \text{ funkcí}}\} \to FS$

$\alpha \pm \beta i$ kořeny nás. k: $\{\underbrace{e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta, \dots, x^{k-1} e^{\alpha x}\cos\beta x}_{k}\} \to FS$

$\{\underbrace{e^{\alpha x}\sin\beta x, \dots, x^{k-1} e^{\alpha x}\sin\beta x}_{k}\} \to FS$

(1) $y'' - 3y' + 2y = 0$

$\lambda^2 - 3\lambda + 2 = 0 \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \end{matrix} \Big\} \to FS = \{e^x, e^{2x}\}$

$y = Ae^x + Be^{2x}$

(2) $y'' + 6y' + 9y = 0$

$\lambda^2 + 6\lambda + 9 = 0 \to \lambda_{1,2} = -3 \qquad FS = \{e^{-3x}, xe^{-3x}\}$

(3) $y'' + 2y' = 0$

(3) $y'' + 3y = 0$

$\lambda^2 + 3 = 0 \rightarrow \lambda_{1,2} = \pm\sqrt{3}\,i \qquad FS = \{\cos\sqrt{3}x, \sin\sqrt{3}x\}$

$\alpha = 0,\ \beta = \sqrt{3}$ nás 1

(4) $y'''' + 8y'' + 16y = 0$

$\lambda^4 + 8\lambda^2 + 16 = 0$

$t = \lambda^2 \rightarrow t^2 + 8t + 16 = 0 \rightarrow \lambda_{1,2,3,4} \overset{\text{nás. } 2}{=} \pm 2i \qquad \alpha = 0 \quad \beta = 2$

$t = -4$ nás. 2 $\qquad \{\sin 2x, \cos 2x, x\sin 2x, x\cos 2x\} = FS$

(5) $y^{(5)} - 6y^{(4)} + 12y''' - 8y'' = 0$

$\lambda^5 - 6\lambda^4 + 12\lambda^3 - 8\lambda^2 = 0$ $\quad \left| \begin{array}{l} \lambda = 0 \text{ nás } 2 \\ \lambda = 2 \text{ nás } 3 \end{array} \right. \qquad FS = \left\{ \begin{array}{l} 1, x, \\ e^{2x}, xe^{2x}, x^2e^{2x} \end{array} \right\}$

$\parallel$

$\lambda^2(\lambda^3 - 6\lambda^2 + 12\lambda - 8) = \lambda^2(\lambda - 2)^3$

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)(\lambda^2 - 4\lambda + 4) = (\lambda - 2)^3$

$\lambda^3 - 2\lambda^2 + 2\lambda^2 - 6\lambda^2 + 12\lambda - 8$

$\parallel \qquad \underbrace{-4\lambda^2 + 8\lambda}_{-4\lambda(\lambda - 2)} \qquad \underbrace{-8\lambda + 12\lambda - 8}_{4(\lambda - 2)}$

$\lambda^2(\lambda - 2)$

(6) $y''' - y'' - 3y' + 27y = 0 \qquad FS = \{e^{3x}, e^{2x}\sin(\sqrt{5}x), e^{2x}\cos(\sqrt{5}x)\}$

$\lambda^3 - \lambda^2 - 3\lambda + 27 = 0 \qquad \lambda = 3$

$$\lambda^3 - \lambda^2 - 3\lambda + 27 = 0 \qquad \lambda_1 = -3$$

$$\lambda^3 + 3\lambda^2 - 4\lambda^2 - 3\lambda + 27$$

$$\lambda^2(\lambda+3) \quad \underbrace{-4\lambda^2 - 12\lambda}_{-4\lambda(\lambda+3)} \quad + \underbrace{9\lambda + 27}_{9(\lambda+3)}$$

$$\lambda^3 - \lambda^2 - 3\lambda + 27 = (\lambda+3)(\lambda^2 - 4\lambda + 9)$$

$$\lambda_{2,3} = \frac{4 \pm \sqrt{16-36}}{2} = \underset{\alpha}{2} \pm \underset{\beta}{\sqrt{5}}\, i$$

$2\sqrt{5}$

$$e^{\alpha x} \sin\beta x, \quad \cos\beta x$$

Speciální pravá strana

$$a_n y^{(n)} + \dots + a_0 y = f$$

$$f(x) = \boxed{P(x)}\, \underbrace{e^{\alpha x} \cos \beta x}_{=1} + \underbrace{Q(x)\, e^{\alpha x} \sin \beta x}_{=0}$$

$(\alpha = 0,\ \beta = 0)$

P, Q polynomy

$$f(x) = x^4 \qquad P(x) = x^4,\ Q(x) = 0,\ \alpha = \beta = 0$$

Řešení hledáme ve tvaru

$$y = x^k \left[R(x)\, e^{\alpha x} \cos \beta x + S(x)\, e^{\alpha x} \sin \beta x \right]$$

R, S polynomy $\quad \deg_R, \deg_S \le \max(\deg_P, \deg_Q)$

R, S polynomy $\deg_R, \deg_S \leq \max(\deg_P, \deg_Q)$

k násobnost kořene $\alpha + \beta i$ v char. polynomu levé strany

$-1+0$

(7) • $y'' - 3y' + 2y = \boxed{xe^{-x}} \rightarrow \alpha = -1, \beta = 0, Q(x) = 0, P(x) = x$

$FS = \{e^x, e^{2x}\}$

$\lambda_{1,2} = 1, 2$

$x e^{-1 \cdot x} \cos(0 \cdot x) + 0 \ldots$

$\underbrace{\cos(0 \cdot x)}_{1}$

$\deg_Q = 0, \deg_P = 1 \rightarrow R(x) = Ax + B, S(x) = Cx + D$

$y_p = (Ax+B)e^{-x} + (Cx+D)e^{-x} \cdot 0$

$= (Ax+B)e^{-x}$

$C \cdot xe^{-x}, De^{-x}$ $\quad C = 1$, $D = 0$

(8) $y''' + 4y'' + 4y' = (x+3)e^{-2x}$ $\quad \deg_Q = 0$ $\quad -2+0i$

$\lambda^3 + 4\lambda^2 + 4\lambda = \lambda(\lambda+2)^2$ $\quad P(x) = x+3, Q(x) = 0, \alpha = -2, \beta = 0$

$\lambda_{2,3} = -2$ (nás. 2) $\quad \deg_P = 1$

" " $\lambda_{2,3}=-2$ (nás. 2) " deg p = 1

$$y_p = x^2(Ax+B)e^{-2x}$$

(9) $y''+9y = \sin 3x\, e^{5x}$ $\quad P, Q, \alpha, \beta = ?$

$\underbrace{\alpha = 5, \beta = 3}_{5+3i}$, $\underline{P(x)=0, \ Q(x) = 1}$

$\lambda^2+9=0 \Rightarrow \lambda_{1,2} = \boxed{\pm 3i}$ $\quad$ násobnost 0

$$y_p = \underbrace{A e^{5x}\cos 3x + B e^{5x}\sin 3x}$$

(10) $y''+9y = \underset{\deg 2}{x^2} \cdot \sin 3x\, e^{0x} \rightarrow P, Q, \alpha, \beta, k = ?$

$\beta = 3, \alpha = 0, P(x) = 0, Q(x) = x^2$

násobn. $1 = k$

$$y_p = x\left((Ax^2+Bx+C)\cos 3x + (Dx^2+Ex+F)\sin 3x\right)$$

(11) $y''-2y'+2y = \cos x(1+e^x) = \cos x + \cos x\, e^x$ $\quad$ deg = 0

(11) $y''-2y'+2y = \cos x\,(1+e^x) = \cos x + \cos x\, e^x$ deg $=0$

$= \boxed{\cos x}$ $\alpha=0,\ \beta=1,\ P(x)=1,\ Q(x)=0$

$= \boxed{\cos x\, e^x}$ $\alpha=1,\ \beta=1,\ P(x)=1,\ Q(x)=0$

$1\pm i \to k=0$

$$\begin{cases} y_p = A\cos x + B\sin x \\ y_p' = -A\sin x + B\cos x \\ y_p'' = -A\cos x - B\sin x \end{cases}$$

$y''-2y'+2y$

$$\boxed{\cos x = -A\cos x - B\sin x + 2A\sin x - 2B\cos x + 2A\cos x + 2B\sin x}$$

$0\sin x$

$$1\cos x = \cos x(-A-2B+2A) + \sin x(-B+2A+2B)$$

$$\left.\begin{aligned} 1 &= A-2B \\ 0 &= 2A+B \end{aligned}\right\} \begin{aligned} &B=-2A \to B=-\tfrac{2}{5} \\ &1=A+4A=5A \to A=\tfrac{1}{5} \end{aligned}$$

$$\boxed{y_p = \frac{1}{5}\cos x - \frac{2}{5}\sin x}$$

$y_q = \dots$

$y_s = \boxed{y_p} + y_q$ $\cos x + e^x\cos x$

$y_q = \dots$ $\qquad$ $y_s = \boxed{y_p} + y_q$

Variace konstant

$$FS = \{y_1, y_2\} \qquad \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \boxed{C_1'(x)} \\ \boxed{C_2'(x)} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{f}{a_2} \end{pmatrix} \to y_p = C_1(x) y_1(x) + C_2(x) y_2(x)$$

(13) $1 \cdot y'' - 2y' + y = \frac{e^x}{x} =: f(x)$

$\lambda_{1,2} = 1$

$\{e^x, xe^x\}$

$$\begin{pmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{pmatrix} \begin{pmatrix} C'(x) \\ D'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^x}{x} \end{pmatrix}$$

$$\left| \begin{array}{l} \cancel{e^x} C'(x) + x\cancel{e^x} D'(x) = 0 \\ \cancel{e^x} C'(x) + (x+1)\cancel{e^x} D'(x) = \frac{\cancel{e^x}}{x} \end{array} \right.$$

$$\begin{cases} C'(x) + x D'(x) = 0 \\ C'(x) + (x+1) D'(x) = \frac{1}{x} \end{cases} \to C'(x) = -x D'(x) = -1$$

$$\underline{D'(x) = \frac{1}{x}} \rightarrow D(x) \stackrel{c}{=} \log|x|$$

$$C(x) \stackrel{c}{=} -x$$

$$\underline{y_p = -x \cdot e^x + \log|x| \cdot xe^x}$$

$$y = \underbrace{Ae^x + Bxe^x}_{y_h} \underbrace{-xe^x + \log|x|\, xe^x}_{y_p}$$

$$x \in (0,\infty), (-\infty,0).$$