NMSA409, topic 9: Prediction

Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence. $H_{-\infty}^n = \mathcal{H}\{\dots, X_{n-1}, X_n\}$ denotes the Hilbert space generated by the random variables $\{X_t, t \leq n\}$, i.e. by the history of the process $\{X_t, t \in \mathbb{Z}\}$ up to time n.

Prediction $\widehat{X}_{n+h}(n)$ of X_{n+h} (where $h \in \mathbb{N}$) based on the infinite history $X_n, X_{n-1}, \dots$ is the orthogonal projection of the random variable X_{n+h} into the space $H_{-\infty}^n$. We denote $\widehat{X}_{n+h}(n) = P_{H_{-\infty}^n}(X_{n+h})$.

Concerning the prediction based on the finite history we denote $H_1^n = \mathcal{H}\{X_1, \dots, X_n\}$ the Hilbert space generated by the random variables $X_1, \dots, X_n$.
The best linear prediction of X_{n+h} (for $h \in \mathbb{N}$) is the orthogonal projection into the space H_1^n, i.e. $\widehat{X}_{n+h}(n) = \sum_{j=1}^n c_j X_j \in H_1^n$ such that $X_{n+h} - \widehat{X}_{n+h}(n) \perp H_1^n$.

Prediction error (*residual variance*) is defined as $\mathbb{E}|X_{n+h} - \widehat{X}_{n+h}(n)|^2$.

Exercise 9.1: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t - \frac{1}{2}X_{t-1} + \frac{1}{16}X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise $WN(0, \sigma^2)$. Suppose we know the history of the process up to the time $t = 100$. Compute the predictions $\widehat{X_{101}}(100)$, $\widehat{X_{102}}(100)$ and $\widehat{X_{103}}(100)$ and the respective prediction errors for $\widehat{X_{101}}(100)$ and $\widehat{X_{102}}(100)$.

Exercise 9.2: Let $\{X_t, t \in \mathbb{Z}\}$ be a random sequence given by the equation

$$X_t = Y_t - 0.5Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is WN$(0, \sigma^2)$. Determine $\widehat{X}_4, \widehat{X}_5$ based on the observations X_1, X_2, X_3 and compute the prediction error.

Exercise 9.3: Consider a stationary AR(1) process $\{X_t, t \in \mathbb{Z}\}$ defined by the equation

$$X_t + \frac{1}{3}X_{t-1} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. Predict the values of X_{k+1} for $k \in \mathbb{N}$ if you have observed the values $X_0 = X_1 = 1$. Compute the prediction error.

Exercise 9.4: Consider a stationary AR(2) process $\{X_t, t \in \mathbb{Z}\}$ defined by the equation

$$X_t + \frac{1}{3}X_{t-1} + \frac{1}{3}X_{t-2} = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. Assume that you have observed the values of the process
a) $X_0 = X_1 = 1$,
b) $X_0 = 1$.
Predict the values of X_{k+1} for $k \in \mathbb{N}$ and compute the prediction error.

Exercise 9.5: Consider a stationary ARMA(1,1) process $\{X_t, t \in \mathbb{Z}\}$ given by the equation

$$X_t + \frac{1}{3}X_{t-1} = Y_t - Y_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a white noise. Predict the values of X_{k+1} for $k \in \mathbb{N}$ if you have observed the values $X_0 = -1$, $X_1 = 2$.

Exercise 9 .6: We know the values $X_1 = 5.9$, $X_2 = 4.9$, $X_3 = 2.2$, $X_4 = 2.0$, $X_5 = 4.9$ of the process

$$(X_t - 4) - 0.8(X_{t-1} - 4) = Y_t, \quad t \in \mathbb{Z},$$

where $\{Y_t, t \in \mathbb{Z}\}$ is a centered white noise with the variance $\sigma^2 = 0.7$. Find the prediction of X_6 and X_7. Compute the respective prediction errors.