NMSA405: exercise 5 – symmetric simple random walk

**Definition 2.6:** Let $X_1, X_2, \ldots$ be an iid random sequence such that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. We call the corresponding random walk $(S_n)$ the symmetric simple random walk.

**Exercise 5.1:** (Proposition 2.9) (Reflection principle) Let $(S_n)$ be a symmetric simple random walk. Consider the stopping time $T$, the first hitting time of the set $\{a\}$ by the random walk for a given $a \in \mathbb{N}$. Denote

$$S_T^r = 2S_{\min\{k,T\}} - S_k, \quad k \in \mathbb{N}.$$ \hspace{1cm} (Proposition 2.11) (Martingale differences of an $L_2$-martingale are orthogonal in $L_2$)

Then

$$(S_I^r, S_{I+1}^r, \ldots) \overset{d}{=} (S_1, S_2, \ldots).$$

**Exercise 5.2:** (Proposition 2.10) (Maxima of the symmetric simple random walk) For a symmetric simple random walk $(S_n)$ denote $M_n = \max_{k=1,\ldots,n} S_k$, $n \in \mathbb{N}$. Consider the stopping time $T$, the first hitting time of the set $\{a\}$ by the random walk for a given $a \in \mathbb{N}$. Then

$$\mathbb{P}(T \leq n) = \mathbb{P}(M_n \geq a) = 2\mathbb{P}(S_n \geq a) - \mathbb{P}(S_n = a) \quad \text{and} \quad \lim_{n \to \infty} \mathbb{P}(M_n \geq a) = 1.$$

NMSA405: exercise 6 – martingales

**Exercise 6.1:** (Proposition 2.18) Let $(X_n)$ be a sequence of independent integrable random variables. Denote $S_n = X_1 + \ldots + X_n$ for $n \in \mathbb{N}$.

- c) If $\mathbb{E}X_n = 1$ for all $n \in \mathbb{N}$ then $Z_n = \prod_{j=1}^n X_j$ is a martingale.
- d) If $\mathbb{P}(X_n = -1) = q$ and $\mathbb{P}(X_n = 1) = p$ where $p \in (0,1)$ and $p + q = 1$ then $Y_n = (q/p)^{S_n}$ is a martingale.

**Exercise 6.2:** Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda|_{[0, 1]})$, a finite measure $\mu \ll \lambda$ on $([0, 1], \mathcal{B}([0, 1]))$ and an increasing sequence of sets $\{0 = t_0^0 < t_1^0 < \ldots < t_k^0 = 1\}$ such that

$$\max_{k \in \{0,1,\ldots,k_n-1\}} |t_{k+1}^n - t_k^n| \to 0.$$ \hspace{1cm} (Proposition 2.21) (Martingale differences of an $L_2$-martingale are orthogonal in $L_2$)

Denote $B_k^n = [t_k^n, t_{k+1}^n)$ and

$$D_n(x) = \frac{\mu(B_k^n)}{\lambda(B_k^n)}, \quad x \in B_k^n.$$ \hspace{1cm} (Definition 2.6) (Martingale differences of an $L_2$-martingale are orthogonal in $L_2$)

Show that $(D_n)$ is an $(\mathcal{F}_n)$-martingale where $\mathcal{F}_n = \sigma(B_1^n, \ldots, B_k^n)$. What is the a.s. limit of $D_n$ for $n \to \infty$?

**Exercise 6.3:** (Proposition 2.21) (Martingale differences of an $L_2$-martingale are orthogonal in $L_2$) Let $(M_n)$ be an $L_2$-martingale. Denote $D_1 = M_1$ and $D_{n+1} = M_{n+1} - M_n$ for $n \in \mathbb{N}$. Then $\mathbb{E}D_nD_m = 0$ for $m \neq n$, and hence $\mathbb{E}M_n^2 = \sum_{j=1}^n \mathbb{E}D_j^2$ and $\text{var} M_n = \sum_{j=1}^n \text{var} D_j$.

NMSA405: exercise 7 – martingales and Doob decomposition

**Exercise 7.1:** Let $Y$ be an integrable random variable and let $(\mathcal{F}_n)$ be a filtration. Consider the sequence $X_n = \mathbb{E}[Y \mid \mathcal{F}_n]$, $n \in \mathbb{N}$, and show that $(X_n)$ is a $\mathcal{F}_n$-martingale.

**Exercise 7.2:** (Pólya urn model) Consider an urn which at time $n = 0$ contains $b$ black and $w$ white balls, $b, w \in \mathbb{N}$. At each time $n \in \mathbb{N}$ we draw a ball from the urn at random, write down its color and put it back together with $\Delta \in \mathbb{N}$ new balls of the same color. Denote $X_n$ the relative frequency of the white balls in the urn at time $n$ (i.e. the ratio of the number of white balls to the number of all balls in the urn at the given time). Show that $(X_n)$ is a martingale. Consider also the case with $\Delta = 0$ or $\Delta = -1$. 

Exercise 7.3: A deck of cards contains $a$ black and $b$ red cards. The deck has been shuffled randomly and we start drawing the cards from the top one after another. Denote $X_n$ the relative number of black cards after drawing $n$ cards where $n \in \{0, \ldots, a + b - 1\}$. Let $X_n = X_{a+b-1}$ for $n \geq a+b$. Show that $(X_n)$ is a martingale.

Exercise 7.4: Let $(X_n)$ be a sequence of random variables such that the probability density function $f_n : \mathbb{R}^n \to (0, \infty)$ of the random vector $(X_1, \ldots, X_n)$ is positive on $\mathbb{R}^n$. Suppose we are given a consistent system of probability density functions $(g_n)$, i.e. $g_n : \mathbb{R}^n \to [0, \infty)$ fulfills $\int_{\mathbb{R}^n} g_n(x) \, dx = 1$ and $\int g_{n+1}(x, y) \, dy = g_n(x)$ for almost all $x \in \mathbb{R}^n$. We define the likelihood ratio

$$S_n = \frac{g_n(X_1, \ldots, X_n)}{f_n(X_1, \ldots, X_n)}, \quad n \in \mathbb{N}.$$ 

Show that $(S_n)$ is a martingale.

Exercise 7.5: Let $(F_n)$ be a filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(Q_n)$ a consistent system of $F_n$-probability measures, i.e. $Q_{n+1} \mid F_n = Q_n$ for $n \in \mathbb{N}$, such that $Q_n \ll \mathbb{P} \mid F_n$. We define $X_n = \frac{dQ_n}{d\mathbb{P}}$. Show that $(X_n)$ is a $F_n$-martingale.

Exercise 7.6: Let $X_n : (\Omega, \mathcal{F}) \to (S_n, \mathcal{S}_n)$, $n \in \mathbb{N}$, be a sequence of random variables. Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ and $(\nu_n)$ a consistent system of probability distributions such that $\nu_n \ll P_{X_1, \ldots, X_n} := \mu_n$. Similarly as above show that the likelihood ratio $T_n = \frac{d\nu_n}{d\mu_n}(X_1, \ldots, X_n)$ between $H_1 : (X_1, \ldots, X_n)^T \sim \nu_n$ and $H_0 : (X_1, \ldots, X_n)^T \sim \mu_n$ is a $\sigma(X_1, \ldots, X_n)$-martingale under the null hypothesis $H_0$.

Exercise 7.7: Let $(X_n)$ be an iid random sequence. Let $\alpha \in \mathbb{R}$ be such that $\beta = \ln\mathbb{E}e^{\alpha X_1} \in \mathbb{R}$. We define $Z_n = \exp\{\alpha S_n - \beta n\}$ where $S_n = X_1 + \ldots + X_n$. Show that $(Z_n)$ is a martingale.

Exercise 7.8: Let $(X_n)$ be a sequence of independent integrable random variables with zero mean. We define $M_n = \sum_{k=1}^n \prod_{i=1}^k X_i$ for $n \in \mathbb{N}$. Show that $(M_n)$ is a martingale.

Exercise 7.9: Let $(X_n)$ be an iid random sequence with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = \sigma^2 \in (0, \infty)$ and $\mathbb{E}\exp\{X_1\} = \gamma < \infty$. Consider the corresponding random walk $(S_n)$. Show that the following sequences are submartingales and determine their compensators:

- a) $S_n^2$,
- b) $V_n = X_1^2 + \ldots + X_n^2$,
- c) $\exp\{S_n\}$.

Exercise 7.10: Let $(X_n)$ be a $F_n$-martingale such that $X_n \in L_2$. Show that

$$I_n = \sum_{k=1}^n \text{var}(X_k \mid F_{k-1})$$

is the compensator of the sequence $Z_n = X_n^2$ where $F_0 = \{\emptyset, \Omega\}$.