NMSA405: exercise 10 – convergence theorems

Exercise 10.1: Give an example of a martingale which converges to the random variable $X_\infty \in L_1$ almost surely but not in L_1.

Exercise 10.2: Let (Y_n) be a sequence of independent random variables such that

$$\mathbb{P}(Y_n = 2^n - 1) = 2^{-n}, \quad \mathbb{P}(Y_n = -1) = 1 - 2^{-n}, \quad n \in \mathbb{N}.$$

Check that $X_n = \sum_{k=1}^n Y_k$ is a martingale. Show that $X_n \xrightarrow[n\to\infty]{\text{a.s.}} -\infty$ and hence the assumptions of the martingale convergence theorems cannot be fulfilled.

Exercise 10.3: (martingale proof of the Kolmogorov 0-1 law) Let $X = (X_1, X_2, \dots)$ be a sequence of independent random variables and $F = [X \in T]$ where $T \in \mathcal{T}$ is a terminal set. Show that

$$\forall\, n \in \mathbb{N} \qquad \mathbb{E}[\mathbf{1}_F \mid \mathcal{F}_n] = \mathbb{P}(F) \quad \text{a.s.} \qquad \text{and at the same time} \qquad \mathbb{E}[\mathbf{1}_F \mid \mathcal{F}_n] \xrightarrow[n\to\infty]{\text{a.s.}} \mathbf{1}_F.$$

From this conclude that $\mathbb{P}(F)$ is either 0 or 1.

NMSA405: exercise 11 – backwards martingale

Definition: Let $(\dots, X_{-2}, X_{-1})$ be a random sequence indexed by negative integers. Let $\dots \subseteq \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1}$ be a non-decreasing sequence of σ-algebras (filtration). Assume that $X_{-n} \in L_1$ for any $n \in \mathbb{N}$ and $\sigma(\dots, X_{-n-1}, X_{-n}) \subseteq \mathcal{F}_{-n}$. We say that the sequence (X_{-n}) is an $\mathcal{F}_{-n}$*-martingale* if

$$\mathbb{E}[X_{-n} \mid \mathcal{F}_{-(n+1)}] = X_{-(n+1)} \quad \text{a.s. for all } n \in \mathbb{N}.$$

If $\mathcal{F}_{-n} = \sigma(\dots, X_{-n-1}, X_{-n})$, then we speak about a *backwards martingale*. Analogously we define $\mathcal{F}_{-n}$*-submartingale* and $\mathcal{F}_{-n}$*-supermartingale*.

Theorem: (Doob's backwards submartingale convergence theorem) Let (X_{-n}) be an $\mathcal{F}_{-n}$-submartingale. Then there exists a random variable $X_{-\infty}$(with values in $\mathbb{R} \cup \{-\infty, \infty\}$) such that $X_{-n} \xrightarrow[n\to\infty]{\text{a.s.}} X_{-\infty}$. The limiting random variable $X_{-\infty}$ is integrable provided that $\sup_{n\in\mathbb{N}} \mathbb{E}X_{-n}^- < \infty$.

Exercise 11.1: Let Y be an integrable random variable and $(\mathcal{F}_{-n})$ a filtration. We define $X_{-n} = \mathbb{E}[Y \mid \mathcal{F}_{-n}]$ for $n \in \mathbb{N}$. Show that (X_{-n}) is a uniformly integrable $\mathcal{F}_{-n}$-martingale.

Exercise 11.2: Let (X_n) be an iid random sequence of integrable random variables. We define

$$Z_{-n} = \frac{1}{n}\sum_{k=1}^n X_k, \quad n \in \mathbb{N}.$$

Show that (Z_{-n}) is a backwards martingale.

Exercise 11.3: (martingale proof of the strong law of large numbers) Argue that the backwards martingale from the previous exercise has an integrable limit in the a.s. and L_1 sense. Show that this limit must be constant and equal to $\mathbb{E}X_1$ a.s.