Autocovariance function and stationarity

Definition 2.1: Let $\{X_t, t \in T\}$, where $T \subset \mathbb{R}$, be a stochastic process with finite second moments, i.e. $\mathbb{E}|X_t|^2 < \infty$ for all $t \in T$. (In general complex) function of two arguments defined on $T \times T$ by the formula

$$R(s,t) = \mathbb{E}(X_s - \mathbb{E}X_s)(\overline{X_t - \mathbb{E}X_t})$$

is called the *autocovariance function of the process* $\{X_t, t \in T\}$.

Definition 2.2: Let $\{X_t, t \in T\}$ be a stochastic process. We call the process

- *strictly stationary* if for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathbb{R}$, $t_1, \dots, t_n \in T$ and $h > 0$ such that $t_1+h, \dots, t_n + h \in T$ it holds that

$$\mathbb{P}(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n) = \mathbb{P}(X_{t_1+h} \leq x_1, \dots, X_{t_n+h} \leq x_n),$$

- *weakly stationary* if the process has finite second moments, a constant mean value $\mathbb{E}X_t = \mu$ and its autocovariance function $R(s,t)$ depends only on $t - s$,
- *covariance stationary* if the process has finite second moments and its autocovariance function $R(s,t)$ depends on $s - t$ only,
- *process of uncorrelated random variables* if the process has finite second moments and for its autocovariance function it holds that $R(s,t) = 0$ for all $s \neq t$,
- *centered* if $\mathbb{E}X_t = 0$ for all $t \in T$,
- *Gaussian* if for all $n \in \mathbb{N}$ and $t_1, \dots, t_n \in T$ the vector $(X_{t_1}, \dots, X_{t_n})^{\mathrm{T}}$ has n-dimensional normal distribution,
- *process with independent increments* if for all $t_1, \dots, t_n \in T$ fulfilling $t_1 < \dots < t_n$ the random variables $X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent,
- *process with stationary increments* if for all $s, t \in T$ fulfilling $s < t$ the distribution of increments $X_t - X_s$ depends on $t - s$ only.

Theorem 2.1: The following implications hold:

a) strictly stationary with finite second moments $\Rightarrow$ weakly stationary,
b) weakly stationary and Gaussian $\Rightarrow$ strictly stationary,
c) weakly stationary $\Rightarrow$ covariance stationary,
d) process of uncorrelated random variables $\Rightarrow$ covariance stationary,
e) centered process of uncorrelated random variables $\Rightarrow$ weakly stationary.

Theorem 2.2: The autocovariance function has the following properties:

- it is non-negative on the diagonal: $R(t,t) \geq 0$,
- it is Hermitian: $R(s,t) = \overline{R(t,s)}$,
- it fulfills the Cauchy-Schwarz inequality: $|R(s,t)| \leq \sqrt{R(s,s)}\sqrt{R(t,t)}$,
- it is positive semidefinite: for all $n \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{C}$ and $t_1, \dots, t_n \in T$ it holds that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} R(t_j, t_k) \geq 0.$$

Remark: The non-negative values on the diagonal and the Hermitian property follow from the positive semidefiniteness.

Theorem 2.3: For each positive semidefinite function R on $T \times T$ there is a stochastic process $\{X_t, t \in T\}$ with finite second moments such that R is its autocovariance function.

Corollary 2.4: Any complex valued function R on $T \times T$ is positive semidefinite if and only if it is an autocovariance function of some stochastic process.

Exercise 2.1: Let $X_t = a + bt + Y_t$, $t \in \mathbb{Z}$, where $a, b \in \mathbb{R}$, $b \neq 0$ and $\{Y_t, t \in \mathbb{Z}\}$ be a sequence of independent identically distributed random variables with zero mean and finite positive variance σ^2.

a) Determine the autocovariance function of the sequence $\{X_t, t \in \mathbb{Z}\}$ and discuss its stationarity.
b) For $q \in \mathbb{N}$ we define random variables V_t by the formula

$$V_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t+j}, \quad t \in \mathbb{Z}.$$

Determine the autocovariance function of the sequence $\{V_t, t \in \mathbb{Z}\}$ and discuss its stationarity.

Exercise 2.2: Let X be a random variable with a uniform distribution on the interval $(0, \pi)$. Consider the sequence of random variables $\{Y_t, t \in \mathbb{N}\}$ where $Y_t = \cos(tX)$. Discuss the properties of such a random sequence.

Exercise 2.3: Consider the stochastic process $X_t = \cos(t + B)$, $t \in \mathbb{R}$, where B is a random variable with a uniform distribution on the interval $(0, 2\pi)$. Check whether the process is weakly stationary.

Exercise 2.4: Let X be a random variable such that $\mathbb{E}X = 0$ and $\operatorname{var} X = \sigma^2 < \infty$. We define $X_t = (-1)^t X$, $t \in \mathbb{N}$. Discuss the properties of the sequence $\{X_t, t \in \mathbb{N}\}$.

Exercise 2.6: Let $\{N_t, t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$ and let A be a real-valued random variable with zero mean and unit variance, independent of the process $\{N_t, t \geq 0\}$. We define $X_t = A(-1)^{N_t}, t \geq 0$. Determine the autocovariance function of $\{X_t, t \geq 0\}$.

Exercise 2.8: Let $\{W_t, t \geq 0\}$ be a Wiener process. We define the so-called *Ornstein-Uhlenbeck process* $\{U_t, t \geq 0\}$ by the formula $U_t = \mathrm{e}^{-\alpha t/2} W_{\exp\{\alpha t\}}, t \geq 0$, where $\alpha > 0$ is a parameter. Decide whether $\{U_t, t \geq 0\}$ is weakly (strictly) stationary and determine its autocovariance function.

Exercise 2.11: Let $\{X_t, t \in \mathbb{Z}\}$ be a sequence of independent identically distributed random variables. Prove that the process is strictly stationary. Is it also weakly stationary?

Exercise 2.12: Let $\{X_t, t \in \mathbb{Z}\}$ be a sequence of uncorrelated random variables with zero mean and finite positive variance (so-called *white noise*). Prove that it is weakly stationary. Is it also strictly stationary?

Exercise 2.13: Let $X_0 = 0$, $X_t = Y_1 + \cdots + Y_t$ for $t = 1, 2, \ldots$, where $Y_1, Y_2, \ldots$ are independent identically distributed random variables with zero mean and finite positive variance. Show that $\{X_t, t \in \mathbb{N}_0\}$ is a Markov chain. Determine its autocovariance function. What can we say about the properties of such a random sequence?

Exercise 2.14: Let $\{X_t, t \in T\}$ a $\{Y_t, t \in T\}$ be uncorrelated weakly stationary processes, i.e. for all $s, t \in T$ the random variables X_s and Y_t are uncorrelated. Show that in such a case also the process $\{Z_t, t \in T\}$ with $Z_t = X_t + Y_t$ is weakly stationary.

Exercise 2.18: Determine the autocovariance function of the Wiener process $\{W_t, t \geq 0\}$. For $0 \leq t_1 < t_2 < \cdots < t_n$ determine the variance matrix of the random vector $(W_{t_1}, \ldots, W_{t_n})^{\mathrm{T}}$.