NMTP438, topic 4: binomial, Poisson and Cox point process

1. Show that the mixed binomial point process with the Poisson distribution (with parameter λ) of the number of points N is a Poisson process with the intensity measure $\lambda\frac{\nu(\cdot)}{\nu(B)}$.

2. Let Φ be a Poisson point process with the intensity measure Λ and $B \in \mathcal{B}$ be a given Borel set. Show that $\Phi|_B$ is a Poisson point process and determine its intensity measure.

3. Consider two independent Poisson point processes Φ_1 and Φ_2 with the intensity measures Λ_1 and Λ_2. Show that $\Phi = \Phi_1 + \Phi_2$ is a Poisson process and determine its intensity measure.

4. Let Φ be a Poisson point process with the intensity measure Λ. Determine the covariance $\mathrm{cov}(\Phi(B_1), \Phi(B_2))$ for $B_1, B_2 \in \mathcal{B}$.

5. Let Φ be a binomial point process with n points in B and the measure ν. Determine the covariance $\mathrm{cov}(\Phi(B_1), \Phi(B_2))$ for $B_1, B_2 \in \mathcal{B}$.

6. Let Φ be a mixed Poisson point process with the driving measure $Y \cdot \Lambda$, where Y is a non-negative random variable and Λ is a locally finite diffuse measure. Determine the covariance $\mathrm{cov}(\Phi(B_1), \Phi(B_2))$ for $B_1, B_2 \in \mathcal{B}_0$ and show that it is non-negative.

7. Determine the second-order factorial moment measure of a binomial point process.

8. Determine the Laplace transform of a binomial point process.

9. Dispersion of a random variable $\Phi(B)$ is defined as

$$D(\Phi(B)) = \frac{\mathrm{var}\,\Phi(B)}{\mathbb{E}\,\Phi(B)}, \quad B \in \mathcal{B}_0.$$

Show that

a) for a Poisson process $D(\Phi(B)) = 1$,
b) a binomial process is underdispersed, i.e. $D(\Phi(B)) \leq 1$,
c) a Cox process is overdispersed, i.e. $D(\Phi(B)) \geq 1$.

10. Let Y be a random variable with a gamma distribution. Show that the corresponding mixed Poisson process Φ is a negative binomial process, i.e. that $\Phi(B)$ has a negative binomial distribution for every $B \in \mathcal{B}_0$.

NMTP438, topic 5: stationary point process

1. Show that a homogeneous Poisson point process is stationary and isotropic.

2. Based on the interpretation of the Palm distribution determine the Palm distribution and the reduced Palm distribution of a binomial point process.

3. Consider independent random variables U_1 a U_2 with uniform distribution on the interval $[0, a]$, $a > 0$, and the point process Φ in $\mathbb{R}^2$defined as

$$\Phi = \sum_{m,n \in \mathbb{Z}} \delta_{(U_1+ma, U_2+na)}.$$

Determine the Palm distribution and the reduced second-order moment measure of the process. Express its contact distribution function and the nearest-neighbour distribution function.

4. Show that for a homogeneous Poisson process with the intensity λ it holds that $\mathrm{PI} = \mathrm{CE} = 1$, $F(r) = G(r) = 1 - \mathrm{e}^{-\lambda \omega_d r^d}$ and $J(r) = 1$.

5. Determine the pair-correlation function of a binomial point process, provided it exists.

6. Let $Y = \{Y(x) : x \in \mathbb{R}^d\}$ be a weakly stationary Gaussian random field with the mean value μ and the autocovariance function $C(x, y) = \sigma^2 r(x - y)$, where σ^2 denotes the variance and r is the autocorrelation function of the random field Y. Consider the random measure

$$\Psi(B) = \int_B \mathrm{e}^{Y(x)}\, \mathrm{d}x, \quad B \in \mathcal{B}^d.$$

The Cox point process Φ with the driving measure Ψ is called a *log-Gaussian Cox process*. Show that the distribution of Φ is determined by its intensity and its pair-correlation function.

7. Determine the pair-correlation function of

a) the Thomas process,
b) the Matérn cluster process for $d = 2$.

8. For a point process with the hard-core distance $r > 0$ and the intensity λ we define the *coverage density* as $\tau = \lambda |b(o, r/2)|$. It is in fact the mean volume fraction of the union of balls with the centers in the points of the process and the radii $r/2$. Determine the maximum possible value of τ for the following models:

a) Matérn hard-core process type I,
b) Matérn hard-core process type II.