NMSA405: exercise 8 – optional sampling theorem

Exercise 8.1: Let (X_n) be a sequence of iid random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ and let $S_n = \sum_{k=1}^n 2^{k-1} X_k$, $n \in \mathbb{N}$. Consider the first hitting time T of the sequence (S_n) of the set $\{1\}$. Then for (S_n) and T the optional sampling theorem does not hold. Show that $\mathbb{E}S_1 \neq \mathbb{E}S_T$ and the condition $\lim_{n\to\infty} \int_{[T>n]} |S_n| \, \mathrm{d}\mathbb{P} = 0$ is not fulfilled.

Exercise 8.2: (remark to the Theorem 3.5) Let (X_n) be a $\mathcal{F}_n$-martingale and $T < \infty$ a.s. be a $\mathcal{F}_n$-stopping time. Show that the condition

$$\exists\, 0 < c < \infty : T > n \Longrightarrow |X_n| \leq c \quad \text{a.s.}$$

does not imply the condition

$$X_T \in L_1 \quad \text{and} \quad \int_{[T>n]} |X_n| \, \mathrm{d}\mathbb{P} \underset{n\to\infty}{\longrightarrow} 0$$

from the Theorem 3.3.
Hint: Consider the sequence $X_n = \sum_{k=1}^n 3^k Y_k$ where (Y_k) is a sequence of iid random variables with the uniform distribution on $\{-1, 0, 1\}$.

NMSA405: exercise 9 – random walks

Definition: Let (X_n) be an iid random sequence such that $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = -1) = 1 - p$ where $p \in [0, 1]$. We call the corresponding random walk (S_n) a *(simple) discrete random walk.* If $p = 1/2$ we get the symmetric simple random walk.

Exercise 9.1: Consider the stopping time $T^B = \min\{n \in \mathbb{N} : S_n \notin B\}$ defined as the first exit time of the discrete random walk S_n from the bounded set $B \in \mathcal{B}(\mathbb{R})$ and the stopping time $T_a = \min\{n \in \mathbb{N} : S_n = a\}$ defined as the first hitting time of the random walk S_n of the set $\{a\}$ for $a \in \mathbb{Z}$. Show that

1. $T^B < \infty$ a.s.,
2. $T_a < \infty$ a.s. if $p = 1/2$.

Exercise 9.2: Show that the discrete random walk fulfills

(i) $S_n \underset{n\to\infty}{\longrightarrow} \infty$ a.s. $\Longleftrightarrow p > 1/2$,

(ii) $S_n \underset{n\to\infty}{\longrightarrow} -\infty$ a.s. $\Longleftrightarrow p < 1/2$,

(iii) $\limsup_{n\to\infty} S_n = \infty$ a.s., $\liminf_{n\to\infty} S_n = -\infty$ a.s. $\Longleftrightarrow p = 1/2$.

Exercise 9.3: Consider a discrete symmetric random walk (S_n). For $a, b \in \mathbb{Z}$, $a < 0$, $b > 0$, we define $T_{a,b} = \min\{n \in \mathbb{N} : S_n \notin (a, b)\}$ as the first exit time of S_n from the interval (a, b). Show that in that case

$$\mathbb{P}(S_{T_{a,b}} = a) = \frac{b}{b-a} \quad \text{and} \quad \mathbb{E}T_{a,b} = -ab.$$

Corollary: (i) $\mathbb{E}T^B < \infty$ for any bounded set $B \in \mathcal{B}(\mathbb{R})$, (ii) $\mathbb{E}T_b = \infty$ for any $b \in \mathbb{Z}$, $b \neq 0$.

Exercise 9.4: Let (S_n) be a symmetric simple random walk and let $A < 0 < B$ be independent integrable random variables, independent of (S_n). Denote $T = \min\{n \in \mathbb{N} : S_n \notin (A, B)\}$. Show that in that case

$$\mathbb{P}(S_T = A) = \mathbb{E}\frac{B}{B-A} \quad \text{and} \quad \mathbb{E}T = -\mathbb{E}A \cdot \mathbb{E}B < \infty.$$

NMSA405: exercise 10 – convergence theorems

Exercise 10.1: Give an example of a martingale which converges to the random variable $X_\infty \in L_1$ almost surely but not in L_1.

Exercise 10.2: Let (Y_n) be a sequence of independent random variables such that

$$\mathbb{P}(Y_n = 2^n - 1) = 2^{-n}, \quad \mathbb{P}(Y_n = -1) = 1 - 2^{-n}, \quad n \in \mathbb{N}.$$

Check that $X_n = \sum_{k=1}^n Y_k$ is a martingale. Show that $X_n \xrightarrow[n\to\infty]{\text{a.s.}} -\infty$ and hence the assumptions of the martingale convergence theorems cannot be fulfilled.

Exercise 10.3: (martingale proof of the Kolmogorov 0-1 law) Let $X = (X_1, X_2, \dots)$ be a sequence of independent random variables and $F = [X \in T]$ where $T \in \mathcal{T}$ is a terminal set. Show that

$$\forall\, n \in \mathbb{N} \qquad \mathbb{E}[\mathbf{1}_F \mid \mathcal{F}_n] = \mathbb{P}(F) \quad \text{a.s.} \qquad \text{and at the same time} \qquad \mathbb{E}[\mathbf{1}_F \mid \mathcal{F}_n] \xrightarrow[n\to\infty]{\text{a.s.}} \mathbf{1}_F.$$

From this conclude that $\mathbb{P}(F)$ is either 0 or 1.

NMSA405: exercise 11 – backwards martingale

Definition: Let $(\dots, X_{-2}, X_{-1})$ be a random sequence indexed by negative integers. Let $\dots \subseteq \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1}$ be a non-decreasing sequence of σ-algebras (filtration). Assume that $X_{-n} \in L_1$ for any $n \in \mathbb{N}$ and $\sigma(\dots, X_{-n-1}, X_{-n}) \subseteq \mathcal{F}_{-n}$. We say that the sequence (X_{-n}) is an $\mathcal{F}_{-n}$-*martingale* if

$$\mathbb{E}[X_{-n} \mid \mathcal{F}_{-(n+1)}] = X_{-(n+1)} \quad \text{a.s. for all } n \in \mathbb{N}.$$

If $\mathcal{F}_{-n} = \sigma(\dots, X_{-n-1}, X_{-n})$, then we speak about a *backwards martingale*. Analogously we define $\mathcal{F}_{-n}$-*submartingale* and $\mathcal{F}_{-n}$-*supermartingale*.

Theorem: (Doob's backwards submartingale convergence theorem) Let (X_{-n}) be an $\mathcal{F}_{-n}$-submartingale. Then there exists a random variable $X_{-\infty}$(with values in $\mathbb{R} \cup \{-\infty, \infty\}$) such that $X_{-n} \xrightarrow[n\to\infty]{\text{a.s.}} X_{-\infty}$. The limiting random variable $X_{-\infty}$ is integrable provided that $\sup_{n\in\mathbb{N}} \mathbb{E}X_{-n}^- < \infty$.

Exercise 11.1: Let Y be an integrable random variable and $(\mathcal{F}_{-n})$ a filtration. We define $X_{-n} = \mathbb{E}[Y \mid \mathcal{F}_{-n}]$ for $n \in \mathbb{N}$. Show that (X_{-n}) is a uniformly integrable $\mathcal{F}_{-n}$-martingale.

Exercise 11.2: Let (X_n) be an iid random sequence of integrable random variables. We define

$$Z_{-n} = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad n \in \mathbb{N}.$$

Show that (Z_{-n}) is a backwards martingale.

Exercise 11.3: (martingale proof of the strong law of large numbers) Argue that the backwards martingale from the previous exercise has an integrable limit in the a.s. and L_1 sense. Show that this limit must be constant and equal to $\mathbb{E}X_1$ a.s.