Stochastic processes 2 – topics for oral examination

1. Definition of a stochastic process, Daniell-Kolmogorov theorem. Example of a consistent system of distributions, example of a non-consistent system of distributions. Theorem 3 with proof.

2. Strict and weak stationarity, their relationship (Theorem 2 with proof). Example of a process which is weakly stationary but not strictly stationary, example of a process which is strictly stationary but not weakly stationary.

3. Definition of the autocovariance function, its properties (Theorems 4, 6 and 8 with proofs, Theorems 5 and 7 without proofs).

4. Hilbert space, scalar product, its continuity (Theorem 10 with proof). Construction of the space $L_2(\Omega, \mathcal{A}, \mathbb{P})$, mean square convergence.

5. Mean square continuity, necessary and sufficient conditions (Theorems 13, 14 and 15 with proofs), examples.

6. Riemann integral of a stochastic process, construction, sufficient condition for existence (Theorem 18 with proof), relationship to the mean square continuity (remark with proof).

7. Spectral decomposition of the autocovariance function for a random sequence (Theorem 19, proof of the implication that $R(t)$ with the appropriate structure is positive semidefinite). Spectral density and its relationship to the spectral distribution function. Symmetry of the spectral density implies that the autocovariance function is real-valued (proof).

8. Inverse formula for spectral density (Theorem 22 with proof, Theorem 23 without proof). Spectral density of the white noise sequence.

9. Orthogonal increment process, orthogonal distribution function, Theorem 24 with proof. Orthogonal distribution function of the Wiener process.

10. Integral with respect to an orthogonal increment process, construction for simple functions, properties (Theorem 25 with proof).

11. Integral with respect to an orthogonal increment process, construction for measurable functions, properties (Theorem 26 with proof).

12. Properties of the sequence $X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}Z(\lambda), t \in \mathbb{Z}$ (Theorem 27 with proof). Spectral decomposition of a random sequence (Theorem 28 without proof).

13. MA(n) model, definition, properties (Theorem 31 with proof).

14. MA(∞) model, definition, properties (Theorem 34 with proof).

15. AR(p) model, definition, necessary and sufficient condition for causality (Theorem 35 with proof).

16. ARMA(p,q) model, definition, conditions for causality and invertibility (Theorems 36 and 37 without proofs). Prediction based on infinite history for ARMA(p,q) models, prediction error.

17. Linear filters, definition, properties (Theorem 38 without proof), comparison to MA(∞) models. Properties of non-causal AR(1) model.

18. Mean square ergodicity, definition, necessary and sufficient condition for random sequences (Theorem 39 with proof), limiting behaviour of the variance (Theorem 40 without proof). Example of an ergodic sequence, example of a non-ergodic sequence.

19. Projections in a Hilbert space, projection theorem (Theorem 51 without proof), properties of the projection mapping (Theorem 52 with proof). Prediction based on infinite history for AR(p) models, prediction error.

20. Prediction based on finite history, principles, derivation, prediction error, regularity of Γ_n (Theorem 53 without proof). Filtration of signal and noise, comparison with the problem of prediction.

21. Partial autocorrelation function, two definitions and their equivalence, computation (Theorem 57 with proof).

22. Estimation of the autocovariance and autocorrelation functions, estimators, properties (Theorem 58 without proof). Estimating the parameter of a causal AR(1) model using $\hat{r}_n(1)$.

23. Parameter estimation for AR(p) models, method of moments, derivation of the equations, properties of the estimators (Theorem 59 without proof).

24. Parameter estimation for AR(p) models, least squares method and asymptotic properties of the estimators, maximum likelihood method, conditional maximum likelihood method.

25. Parameter estimation for MA(q) models, method of moments, example for MA(1).

26. Parameter estimation for ARMA(p, q) models, method of moments.

27. Periodogram and its properties (including but not limited to Theorem 60 with proof).