Theory and Numerics for Problems of Fluid Dynamics

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PREFACE

This text should serve as a source for the course "Theory and Numerics for Problems of Fluid Dynamics", delivered at RWTH Aachen in April/May 2006. The main purpose of this course is to give a survey on the theory of incompressible Navier-Stokes equations. We also discuss the finite element method for the numerical solution of viscous incompressible flow. Moreover, we are concerned with some results in the theoretical and numerical analysis of compressible flow. More details can be found in the books

R. Temam: *Navier-Stokes Equations. Theory and Numerical Analysis.* North-Holland, Amsterdam, 1977.

V. Girault, P.-A. Raviart: *Finite Element Mathods for Navier-Stokes Equations. Theory and Algorithms.* Springer, Berlin, 1986.

M. Feistauer: *Mathematical Methods in Fluid Dynamics.* Longman Scientific & Technical, Harlow, 1993.

M. Feistauer, J. Felcman, I. Straškraba: *Mathematical and Computational Methods for Compressible Flow*, Clarendon Press, Oxford, 2004, ISBN 0 19 850588 4.

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CONTENTS

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BASIC EQUATIONS

Problems of fluid dynamics play an important role in many areas of science and technology. Let us mention, e.g.

airplane industry,
mechanical engineering,
turbomachninery,
ship industry,
civil engineering,
chemical engineering,
food industry,
environmental protection,
meteorology,
oceanology,
medicine.

The image of the flow can be obtained with the aid of

a) experiments (e.g. in wind tunnels), which are expensive and lengthy, sometimes impossible,

b) mathematical models and their realization with the use of numerical methods on modern computers. The numerical simulation of flow problems constitutes the Computational Fluid Dynamics (CFD). Its goal is to obtain results comparable with measurements in wind tunnels and to replace expensive and lengthy experiments.

1.1 Governing equations

Let $(0,T) \subset I\!R$ be a time interval, during which we follow the fluid motion, and let $\Omega \subset I\!R^N$, $N = 1, 2, 3$, denote the domain occupied by the fluid. (For simplicity we assume that it is independent of time t.)

1.1.1 *Description of the flow*

There are two possibilities for describing the fluid motion: Lagrangian and Eulerian.

We shall use here the *Eulerian description* based on the determination of the *velocity* $\boldsymbol{v}(x,t) = (v_1(x,t), \dots, v_N(x,t))$ of the fluid particle passing through the point x at time t.

We shall use the following basic notation: ρ – fluid density, p – pressure, θ – absolute temperature. These quantities are called state variables.

In what follows, we shall introduce the mathematical formulation of fundamental physical laws: the law of conservation of mass, the law of conservation of momentum and the law of conservation of energy, called in brief *conservation laws.*

1.1.2 *The continuity equation*

$$\frac{\partial \rho}{\partial t}(x,t) + \operatorname{div}(\rho(x,t)\boldsymbol{v}(x,t)) = 0, \qquad t \in (0,T),\ x \in \Omega, \tag{1.1.1}$$

is the differential form of the law of conservation of mass.

1.1.3 *The equations of motion*

Basic dynamical equations equivalent with the *law of conservation of momentum* have the form

$$\frac{\partial}{\partial t}(\rho v_i) + \operatorname{div}(\rho v_i \boldsymbol{v}) = \rho f_i + \sum_{j=1}^{N} \frac{\partial \tau_{ji}}{\partial x_j}, \qquad i = 1, \ldots, N. \tag{1.1.2}$$

This can be written as

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho \boldsymbol{f} + \operatorname{div}\mathcal{T}. \tag{1.1.3}$$

Here $\boldsymbol{v}\otimes\boldsymbol{v}$ is the tensor with components $v_i v_j, i,j = 1,\ldots,N$, τ_{ij} are components of the stress tensor $\mathcal{T}$ and f_i are components of the density of the outer volume force $\boldsymbol{f}$.

1.1.4 *The law of conservation of the moment of momentum*

Theorem 1.1 *The law of conservation of the moment of momentum is valid if and only if the stress tensor $\mathcal{T}$ is symmetric.*

1.1.5 *The Navier–Stokes equations*

The relations between the stress tensor and other quantities describing fluid flow, particularly the velocity and its derivatives, represent the so-called *rheological equations* of the fluid. The simplest rheological equation

$$\mathcal{T} = -p\,\mathbb{I}, \tag{1.1.4}$$

characterizes inviscid fluid. Here p is the pressure and $\mathbb{I}$ is the unit tensor:

$$\mathbb{I} = \begin{pmatrix} 1,\ 0,\ 0 \\ 0,\ 1,\ 0 \\ 0,\ 0,\ 1 \end{pmatrix} \quad \text{for } N = 3. \tag{1.1.5}$$

Besides the pressure forces, the friction shear forces also act in real fluids as a consequence of the *viscosity.* Therefore, in the case of viscous fluid, we add a contribution $\mathcal{T}'$ characterizing the shear stress to the term $-p\,\mathbb{I}$:

$$\mathcal{T} = -p\,\mathbb{I} + \mathcal{T}'. \tag{1.1.6}$$

On the basis of the so-called *Stokes' postulates*, it is possible to derive the dependence of the stress tensor on the thermodynamic variables and the *velocity deformation tensor* $\mathbb{D}(\boldsymbol{v}) = (d_{ij}(\boldsymbol{v}))_{i,j=1}^3$ with

$$d_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{1.1.7}$$

Theorem 1.2 *The stress tensor has the form*

$$\mathcal{T} = (-p + \lambda \operatorname{div} \boldsymbol{v})\,\mathbb{I} + 2\mu\mathbb{D}(\boldsymbol{v}), \tag{1.1.8}$$

where λ, μ *are constants or scalar functions of thermodynamical quantities.*

If the stress tensor depends linearly on the velocity deformation tensor as in (1.1.8), the fluid is called *Newtonian.*

We get the so-called *Navier–Stokes equations*

$$\begin{aligned} &\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) \\ &= \rho\boldsymbol{f} - \operatorname{grad} p + \operatorname{grad}(\lambda \operatorname{div}\boldsymbol{v}) + \operatorname{div}(2\mu\mathbb{D}(\boldsymbol{v})). \end{aligned} \tag{1.1.9}$$

1.1.6 *Properties of the viscosity coefficients*

Here μ and λ are called the first and the second *viscosity coefficients*, respectively, μ is also called *dynamical viscosity.* In the kinetic theory of gases the conditions

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0, \tag{1.1.10}$$

are derived. For monoatomic gases, $3\lambda + 2\mu = 0$. This condition is usually used even in the case of more complicated gases.

1.1.7 *The energy equation*

The law of conservation of energy is expressed as the energy equation. To this end we define the *total energy*

$$E = \rho(e + |\boldsymbol{v}|^2/2), \tag{1.1.11}$$

where e is the internal (specific) energy.

Energy equation has the form

$$\begin{aligned} &\frac{\partial E}{\partial t} + \operatorname{div}(E\boldsymbol{v}) \\ &= \rho\boldsymbol{f}\cdot\boldsymbol{v} + \operatorname{div}(\mathcal{T}\boldsymbol{v}) + \rho q - \operatorname{div}\mathbf{q}. \end{aligned} \tag{1.1.12}$$

For a *Newtonian fluid* we have

$$\frac{\partial E}{\partial t} + \operatorname{div}(E\boldsymbol{v}) = \rho\boldsymbol{f}\cdot\boldsymbol{v} - \operatorname{div}(p\boldsymbol{v}) + \operatorname{div}(\lambda\boldsymbol{v}\ \operatorname{div}\boldsymbol{v}) \tag{1.1.13}$$

$$+ \operatorname{div}(2\mu\mathbb{D}(\boldsymbol{v})\boldsymbol{v}) + \rho q - \operatorname{div}\mathbf{q}.$$

Here q is the *density of heat sources* and $\mathbf{q}$ is the *heat flux*, which depends on the temperature by *Fourier's law*:

$$\mathbf{q} = -k\nabla\theta. \tag{1.1.14}$$

Here $k \geq 0$ denotes the *heat conduction coefficient*.

1.2 Thermodynamical relations

In order to complete the conservation law system, additional equations derived in thermodynamics have to be included.

The absolute temperature θ, the density ρ and the pressure p are called the *state variables.* All these quantities are positive functions. The gas is characterized by the *equation of state*

$$p = p(\rho, \theta) \tag{1.2.1}$$

and the relation

$$e = e(\rho, \theta). \tag{1.2.2}$$

Here we shall consider the so-called *perfect gas* (also called ideal gas) whose state variables satisfy the equation of state in the form

$$p = R\,\theta\,\rho. \tag{1.2.3}$$

$R > 0$ is the *gas constant*, which can be expressed in the form

$$R = c_p - c_v, \tag{1.2.4}$$

where c_p and c_v denote the *specific heat at constant pressure* and the *specific heat at constant volume*, respectively. Experiments show that $c_p > c_v$, so that $R > 0$ and c_p, c_v are constant for a wide range of temperature. The quantity

$$\gamma = \frac{c_p}{c_v} > 1 \tag{1.2.5}$$

is called the *Poisson adiabatic constant.* For example, for air, $\gamma = 1.4$.

The internal energy of a perfect gas is given by

$$e = c_v\theta. \tag{1.2.6}$$

1.2.1 *Entropy*

One of the important thermodynamical quantities is the entropy S, defined by the relation

$$\theta\,\mathrm{d}S = \mathrm{d}e + p\,\mathrm{d}V, \tag{1.2.7}$$

where $V = 1/\rho$ is the so-called specific volume. This identity is derived in thermodynamics under the assumption that the internal energy is a function of S and $V : e = e(S, V)$, which explains the meaning of the differentials in (1.2.7).

Theorem 1.3 *For a perfect gas we have*

$$S = c_v \ln \frac{p/p_0}{(\rho/\rho_0)^\gamma} + \text{const} \tag{1.2.8}$$

where p_0 and ρ_0 are fixed (reference) values of pressure and density.

1.2.2 *Barotropic flow*

We say that the flow is barotropic if the pressure can be expressed as a function of the density:

$$p = p(\rho). \tag{1.2.9}$$

This means that $p(x,t) = p(\rho(x,t))$ or, more briefly, $p = p{\circ}\rho$. We assume that

$$p : (0, +\infty) \to (0, +\infty) \tag{1.2.10}$$

and there exists the continuous derivative

$p' > 0$ on $(0, +\infty)$.

In the special case of *adiabatic barotropic flow* of a perfect gas we have the relation

$$p = p_0 \left(\frac{\rho}{\rho_0} \right)^\gamma, \tag{1.2.11}$$

where the positive constants p_0 and ρ_0 are reference values of the pressure and the density.

1.2.3 *Complete system describing the flow of a heat-conductive perfect gas:*

$$\rho_t + \operatorname{div}(\rho \boldsymbol{v}) = 0, \tag{1.2.12}$$

$$(\rho \boldsymbol{v})_t + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho \boldsymbol{f} - \nabla p + \nabla(\lambda \operatorname{div} \boldsymbol{v}) + \operatorname{div}(2\mu \mathbb{D}(\boldsymbol{v})), \tag{1.2.13}$$

$$E_t + \operatorname{div}(E\boldsymbol{v}) = \rho \boldsymbol{f} \cdot \boldsymbol{v} - \operatorname{div}(p\boldsymbol{v}) + \operatorname{div}(\lambda \boldsymbol{v} \operatorname{div} \boldsymbol{v}) + \operatorname{div}(2\mu \mathbb{D}(\boldsymbol{v})\boldsymbol{v}) + \rho q + \operatorname{div}(k\nabla\theta), \tag{1.2.14}$$

$$p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2), \quad \theta = (E/\rho - |\boldsymbol{v}|^2/2)/c_v. \tag{1.2.15}$$

1.2.4 *Euler equations for a perfect gas*

If we set $\mu = \lambda = k = 0$, we obtain the model of inviscid compressible flow, described by the continuity equation, the Euler equations, the energy equation and thermodynamical relations. Since gases are light, usually it is possible to neglect the effect of the volume force. Neglecting heat sources also, we get the system

$$\rho_t + \operatorname{div}(\rho \boldsymbol{v}) = 0, \tag{1.2.16}$$

$$(\rho \boldsymbol{v})_t + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p = 0, \tag{1.2.17}$$

$$E_t + \operatorname{div}((E + p)\boldsymbol{v}) = 0, \tag{1.2.18}$$

$$p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2). \tag{1.2.19}$$

This system is simply called the *compressible Euler equations*.

In the case of a *barotropic flow*, equations (1.2.14), (1.2.15) or (1.2.18), (1.2.19) are replaced by(1.2.9).

1.2.5 *Speed of sound; Mach number*

It is shown in thermodynamics that the pressure can be expressed as a continuously differetiable function of the density and entropy: $p = p(\rho, S)$ with $\partial p/\partial \rho > 0$. For example, for a perfect gas, in view of Theorem 1.3, we have

$$p = p(\rho, S) = \kappa \left(\frac{\rho}{\rho_0}\right)^\gamma \exp(S/c_v), \qquad \kappa = \text{const} > 0. \tag{1.2.20}$$

(The adiabatic barotropic flow of an ideal perfect gas with $S = \text{const}$ is obviously a special case of this model.) Let us introduce the quantity

$$a = \sqrt{\frac{\partial p}{\partial \rho}} \tag{1.2.21}$$

which has the dimension m s^{-1} of velocity and is called the *speed of sound.* This terminology is based on the fact that a represents the speed of propagation of pressure waves of small intensity.

A further important characteristic of gas flow is the *Mach number*

$$M = \frac{|\boldsymbol{v}|}{a} \tag{1.2.22}$$

(which is obviously a dimensionless quantity). We say that the flow is *subsonic* or *sonic* or *supersonic* at a point x and time t, if

$$M(x,t) < 1 \qquad \text{or} \qquad M(x,t) = 1 \qquad \text{or} \qquad M(x,t) > 1, \tag{1.2.23}$$

respectively.

1.3 Incompressible flow

We divide fluids in liquids and gases. Gases, also called compressible fluids, have variable density, whereas liquids, called incompressible fluids, have a constant density $\rho =$ const. > 0. For incompressible fluids, equations (1.2.12) and (1.2.13) can be written in the form

$$\text{div}\, \boldsymbol{v} = 0, \tag{1.3.1}$$

$$\boldsymbol{v}_t + \sum_{i=1}^{N} v_j \frac{\partial \boldsymbol{v}}{\partial x_j} = \boldsymbol{f} - \frac{\nabla p}{\rho} + \text{ div } \left(\frac{2\mu}{\rho} \mathbb{D}(\boldsymbol{v})\right). \tag{1.3.2}$$

Moreover, assuming that $\mu =$ const. > 0 and denoting $\nu = \mu/\rho$ (=kinematic viscosity), equation (1.3.2) reads

$$\boldsymbol{v}_t + \sum_{i=1}^{N} v_j \frac{\partial \boldsymbol{v}}{\partial x_j} = \boldsymbol{f} - \frac{\nabla p}{\rho} + \nu \Delta \boldsymbol{v}. \tag{1.3.3}$$

2

MATHEMATICAL THEORY OF VISCOUS INCOMPRESSIBLE FLOW

In this chapter we shall be concerned with the theory of the incompressible Navier-Stokes system (1.3.1), (1.3.3):

$$\text{div } \boldsymbol{u} = 0, \tag{2.0.1}$$
$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{i=1}^{N} u_j \frac{\partial \boldsymbol{u}}{\partial x_j} = \boldsymbol{f} - \nabla p + \nu \Delta \boldsymbol{u},$$

considered in the space-time cylinder $Q_T = \Omega \times (0, T)$, where $T > 0$ and $\Omega \subset I\!R^N (N = 2, 3)$ is a bounded domain occupied by the fluid. We denote the velocity by $\boldsymbol{u}$ and for simplicity we set $p := p/\rho$ (it is called kinematic pressure). System (2.0.1) is equipped with the initial condition

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}^0(x), \quad x \in \Omega, \tag{2.0.2}$$

and boundary conditions. We assume that on the whole boundary $\partial\Omega \times (0, T)$, the velocity is prescribed:

$$\boldsymbol{u}|_{\partial\Omega \times (0,T)} = \underline{\varphi}. \tag{2.0.3}$$

In the case of stationary flow, when $\partial/\partial t = 0, \boldsymbol{u} = \boldsymbol{u}(x), p = p(x), x \in \Omega$, we get the system

$$\text{div } \boldsymbol{u} = 0, \quad \text{in } \Omega, \tag{2.0.4}$$
$$\sum_{i=1}^{N} u_j \frac{\partial \boldsymbol{u}}{\partial x_j} = \boldsymbol{f} - \nabla p + \nu \Delta \boldsymbol{u}, \quad \text{in } \Omega$$

with the boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = \underline{\varphi}. \tag{2.0.5}$$

2.1 Function spaces and auxiliary results

In the whole chapter we shall assume that $N = 2$ or $N = 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. We shall work with spaces of continuous and continuously differentiable functions $C(\overline{\Omega})$, $C^k(\overline{\Omega})$, $C_0^\infty(\Omega)$, Lebesgue spaces $L^p(\Omega)$, Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ and spaces $L^2(\partial\Omega)$, $H^{\frac{1}{2}}(\partial\Omega) = W^{1-\frac{1}{2},2}(\partial\Omega)$ of functions defined on $\partial\Omega$.

Let us recall that

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega);\ u|\partial\Omega = 0\right\}. \tag{2.1.6}$$

$L^2(\Omega)$ and $H^1(\Omega)$ are Hilbert spaces with the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega uv\, dx \tag{2.1.7}$$

and

$$(u, v)_{H^1(\Omega)} = \int_\Omega (uv + \text{ grad } u \cdot \text{ grad } v)\, dx, \tag{2.1.8}$$

respectively. In virtue of the Friedrichs inequality

$$\|u\|_{L^2(\Omega)} \le c_F \|u\|_{H_0^1(\Omega)} = c_F \left(\int_\Omega |\text{grad} u|^2\, dx\right)^{1/2}, \quad u \in H_0^1(\Omega), \tag{2.1.9}$$

besides the norm $\|\cdot\|_{H^1(\Omega)}$ induced in the space $H_0^1(\Omega)$ by the scalar product $(\cdot,\cdot)_{H^1(\Omega)}$ also the norm $\|\cdot\|_{H_0^1(\Omega)}$ determined by the scalar product

$$(u, v)_{H_0^1(\Omega)} = \int_\Omega \text{ grad } u \cdot \text{ grad } v\, dx \tag{2.1.10}$$

can be used. These two norms are equivalent in the space $H_0^1(\Omega)$.

In this chapter we shall work with N-component vector-valued functions, whose components are elements of some of the above spaces. We shall use the following notation:

$$\begin{aligned}
\mathbf{C}_0^\infty(\Omega) &= [\mathcal{D}(\Omega)]^N = \{\mathbf{u} = (u_1, \dots, u_N);\ u_i \in C_0^\infty(\Omega),\ i = 1, \dots, N\}, \\
\mathbf{L}^2(\Omega) &= [L^2(\Omega)]^N = \left\{\mathbf{u} = (u_1, \dots, u_N);\ u_i \in L^2(\Omega),\ i = 1, \dots, N\right\}, \\
\mathbf{H}^1(\Omega) &= [H^1(\Omega)]^N, \qquad \mathbf{H}_0^1(\Omega) = [H_0^1(\Omega)]^N, \quad \text{etc.}
\end{aligned} \tag{2.1.11}$$

In these spaces, the sum of elements and their product with a real number are defined in an obvious way:

$$\begin{aligned}
&(u_1, \dots, u_N) + (v_1, \dots\dots, v_N) = (u_1 + v_1, \dots, u_N + v_N), \\
&\lambda(u_1, \dots, u_N) = (\lambda u_1, \dots, \lambda u_N).
\end{aligned} \tag{2.1.12}$$

The scalar product in $\mathbf{L}^2(\Omega)$ is introduced by

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{L}^2(\Omega)} = \sum_{i=1}^N (u_i, v_i)_{L^2(\Omega)}, \tag{2.1.13}$$

$$\boldsymbol{u} = (u_1, \dots, u_N),\ \boldsymbol{v} = (v_1, \dots, v_N) \in \mathbf{L}^2(\Omega).$$

In a similar way the scalar product can be defined in $\mathbf{H}^1(\Omega)$. In $\mathbf{H}_0^1(\Omega)$, two scalar products

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}_0^1(\Omega)} = \int_\Omega \sum_{i=1}^N \operatorname{grad} u_i \cdot \operatorname{grad} v_i \, dx = \tag{2.1.14}$$
$$= \int_\Omega \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, dx$$

and

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}^1(\Omega)} = (\boldsymbol{u}, \boldsymbol{v})_{\mathbf{L}_2(\Omega)} + (\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}_0^1(\Omega)}, \tag{2.1.15}$$

can be used. They induce in $\mathbf{H}_0^1(\Omega)$ equivalent norms

$$\|\boldsymbol{u}\|_{\mathbf{H}_0^1(\Omega)} = (\boldsymbol{u}, \boldsymbol{u})_{\mathbf{H}_0^1(\Omega)}^{\frac{1}{2}}, \tag{2.1.16}$$
$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} = (\boldsymbol{u}, \boldsymbol{u})_{\mathbf{H}^1(\Omega)}^{\frac{1}{2}}.$$

It means that there exists a constant $c > 0$ such that

$$\|\boldsymbol{u}\|_{\mathbf{H}_0^1(\Omega)} \le \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \le c \, \|\boldsymbol{u}\|_{\mathbf{H}_0^1(\Omega)} \quad \forall\, \mathbf{u} \in \mathbf{H}_0^1(\Omega). \tag{2.1.17}$$

In what follows it will be convenient to work with the norm $\|\cdot\|_{\mathbf{H}_0^1(\Omega)}$ and the scalar product $(\cdot,\cdot)_{\mathbf{H}_0^1(\Omega)}$. We shall use the following simplified notation:

$$(u, v) = (u, v)_{L^2(\Omega)}, \tag{2.1.18}$$
$$(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v})_{\mathbf{L}^2(\Omega)},$$
$$((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}_0^1(\Omega)},$$
$$\|\boldsymbol{u}\| = \|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}, \quad |||\boldsymbol{u}||| = \|\boldsymbol{u}\|_{\mathbf{H}_0^1(\Omega)}.$$

In order to avoid a misunderstanding in some cases, a scalar product or a norm will be equipped with a subscript denoting the space considered.

Further, let us recall that $H^{\frac{1}{2}}(\partial\Omega)$ is the subspace of $L^2(\partial\Omega)$ formed by the traces on $\partial\Omega$ of all functions from $H^1(\Omega)$. Similarly we denote by $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ the subspace of $\mathbf{L}^2(\partial\Omega)$ formed by the traces on $\partial\Omega$ of all vector functions which are elements of $\mathbf{H}^1(\Omega)$. We use the notation $\boldsymbol{u}|\partial\Omega = (u_1|\partial\Omega, \dots, u_N|\partial\Omega)$ for the trace of a vector function $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathbf{H}^1(\Omega)$ on $\partial\Omega$.

By $X^\star$ we denote the *dual* of a normed linear space X. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality: if $f \in X^\star,\ \varphi \in X$, then $\langle f, \varphi\rangle = f(\varphi)$.

For $q \in L^2(\Omega)$ the symbol $\partial q/\partial x_j$ $(i = 1, \ldots, N)$ will denote the derivative in the sense of distributions, i. e., the continuous linear functional on $C_0^\infty(\Omega)$ defined by

$$\left\langle \frac{\partial q}{\partial x_i}, \varphi \right\rangle = -\int_\Omega q \frac{\partial \varphi}{\partial x_i}\, dx = -(q, \frac{\partial \varphi}{\partial x_i}) \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.1.19}$$

If $v \in H^1(\Omega)$, then

$$\operatorname{grad} v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_N} \right) \in \mathbf{L}^2(\Omega).$$

For $q \in L^2(\Omega)$ we understand by grad q the vector whose components are the distributions $\partial v/\partial x_j,\ j = 1, .., N$. Let $\boldsymbol{u} = (u_1, \ldots, u_N) \in \mathbf{H}^1(\Omega)$. Then

$$\operatorname{div}\, \boldsymbol{u} = \sum_{i=1}^N \frac{\partial u_i}{\partial x_i} \in L^2(\Omega).$$

If $u \in H^1(\Omega)$, then we put

$$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2},$$

where the second order derivatives are considered in the sense of distributions. For $\boldsymbol{u} \in H^1(\Omega)$ we denote by $\Delta \boldsymbol{u} = (\Delta u_1, \ldots, \Delta u_N)$ the vector formed by the distributions Δu_i.

In the theory of Navier–Stokes problems we shall work with spaces of *solenoidal functions* satisfying the condition div $\boldsymbol{u} = 0$. We put

$$\underline{\mathcal{V}} = \{\mathbf{u} \in \mathbf{C}_0^\infty(\Omega);\ \operatorname{div} \boldsymbol{u} = 0\} \tag{2.1.20}$$

and denote by $\boldsymbol{V}$ the closure of $\underline{\mathcal{V}}$ in the space $\mathbf{H}_0^1(\Omega)$:

$$\boldsymbol{V} = \overline{\underline{\mathcal{V}}}^{\mathbf{H}_0^1(\Omega)}. \tag{2.1.21}$$

The space $\boldsymbol{V}$ is clearly a Hilbert one with the scalar product $((\cdot, \cdot))$.

Further, we put

$$L_0^2(\Omega) = L^2(\Omega)/\mathbb{R}^1 = \left\{ q \in L^2(\Omega);\ \int_\Omega q\, dx = 0 \right\}. \tag{2.1.22}$$

Lemma 2.1 *Let* $\ell \in (\mathbf{H}_0^1(\Omega))^\star$ *and* $\langle \ell, \boldsymbol{v} \rangle = 0\ \forall \boldsymbol{v} \in \underline{\mathcal{V}}$. *Then there exists a function* $p \in L_0^2(\Omega)$ *such that*

$$\langle \ell, \boldsymbol{v} \rangle = -(p, \operatorname{div} \boldsymbol{v}) = -\int_\Omega p \operatorname{div} \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \mathbf{H}_0^1(\Omega).$$

It means that

$$\langle \ell, \boldsymbol{v} \rangle = \langle \operatorname{grad}\ p, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in \mathbf{C}_0^\infty(\Omega).$$

and $\ell = \operatorname{grad}\, p$ *in the sense of distributions.*

Proof is based on results from [De Rahm (1960)] and its sketch can be found in [Temam (1977), Chap. I, Proposition 1.1]. Another proof is given in [Girault – Raviart (1986), Chap. I, Lemma 2.1 and Theorem 2.3]. □

Lemma 2.2 *The operator* div *is a mapping of* $\mathbf{H}_0^1(\Omega)$ *onto* $L_0^2(\Omega)$. *More precisely, the operator* div *is an isomorphism of the orthogonal complement*

$$\boldsymbol{V}^\perp = \left\{\boldsymbol{u} \in \mathbf{H}_0^1(\Omega);\ ((\boldsymbol{u},\, \boldsymbol{v})) = 0\ \forall\, \boldsymbol{v} \in \boldsymbol{V}\right\}$$

of the subspace $\boldsymbol{V} \subset \mathbf{H}_0^1(\Omega)$ *onto* $L_0^2(\Omega)$.

Proof See [Girault – Raviart (1986), Chap. I, Corollary 2.4]. □

The following characterization of the space $\boldsymbol{V}$ can be established on the basis of Lemma 2.1:

Lemma 2.3 $\boldsymbol{V} = \left\{\boldsymbol{u} \in \mathbf{H}_0^1(\Omega);\ div\,\boldsymbol{u} = 0\right\}$.

Now let us set $\mathbf{H} = \mathbf{L}^2(\Omega)$. Clearly, $\boldsymbol{V} \subset \mathbf{H}$. Moreover, theorem on the compact imbedding of $H^1(\Omega)$ into $L^2(\Omega)$ implies that also the *imbedding of the space* $\boldsymbol{V}$ *into* $\mathbf{H}$ *is compact.*

2.1.1 *Inf-sup condition*

By Lemma 2.2, for each $q \in L_0^2(\Omega)$ there exists a unique function $\boldsymbol{v} \in \boldsymbol{V}^\perp$ such that

$$\operatorname{div} \boldsymbol{v} = q, \qquad |||\boldsymbol{v}||| \le c\|q\|,$$

where $c > 0$ is a constant independent of q. Hence, taking $q \neq 0$, we have $\boldsymbol{v} \neq 0$ and

$$(q, \operatorname{div} \boldsymbol{v})/|||\boldsymbol{v}||| = \|q\|^2 \,/\, |||\boldsymbol{v}||| \ge \|q\|/c.$$

This yields the inequality

$$\sup_{\substack{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) \\ \boldsymbol{v} \neq 0}} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|q\|\, |||\boldsymbol{v}|||} \ge \gamma, \quad q \in L_0^2(\Omega), \tag{2.1.23}$$

where $\gamma = 1/c > 0$. Condition (2.1.23) can be written in the equivalent form

$$\inf_{\substack{q \in L_0^2(\Omega) \\ q \neq 0}} \sup_{\substack{\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) \\ \boldsymbol{v} \neq 0}} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|q\|\, |||\boldsymbol{v}|||} \ge \gamma, \tag{2.1.24}$$

and is called the inf-sup condition. Its discrete version plays an important role in the analysis of numerical methods for Navier–Stokes problems.

2.2 The stationary Stokes problem

If the fluid viscosity is large ($\nu >> 1$), the viscous term $\nu\Delta\boldsymbol{u}$ dominates over the convective term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$. This is the reason for ignoring the convective term and obtaining a simplified linear system which together with the continuity equation and boundary conditions form the so-called Stokes problem.

2.2.1 *Stokes problem with homogeneous boundary conditions*

Let us consider the Stokes problem with homogeneous Dirichlet (i. e. no-slip) boundary conditions:

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2.2.1}$$

$$-\nu \Delta \boldsymbol{u} + \operatorname{grad} p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2.2.2}$$

$$\boldsymbol{u}|\partial\Omega = 0. \tag{2.2.3}$$

The velocity vector is denoted by $\boldsymbol{u} = (u_1, \ldots, u_N)$ here. We assume that the kinematic viscosity $\nu = \text{const} > 0$ and the density of external volume force $\boldsymbol{f} : \Omega \to \mathbb{R}^N$ are given.

Definition 2.4 *A couple $(\boldsymbol{u}, p)$ is called the classical solution of the Stokes problem with homogeneous boundary conditions, if $\boldsymbol{u} \in C^2(\overline{\Omega})$ and $p \in C^1(\overline{\Omega})$ satisfy equations (2.2.1), (2.2.2) and condition (2.2.3).*

Now the Stokes problem will be reformulated in a weak sense. Let $(\boldsymbol{u}, p)$ be a classical solution of the Stokes problem. Multiplying equation (2.2.2) by an arbitrary $\boldsymbol{v} \in \underline{\mathcal{V}}$ and integrating over Ω, we obtain

$$\int_\Omega -\nu \Delta \boldsymbol{u} \cdot \boldsymbol{v}\, dx + \int_\Omega \boldsymbol{v} \cdot \operatorname{grad} p\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx. \tag{2.2.4}$$

The integrals on the left-hand side can be transformed with the use of Green's theorem:

$$-\nu \int_\Omega \Delta \boldsymbol{u} \cdot \boldsymbol{v}\, dx = -\nu \int_{\partial\Omega} \frac{\partial \boldsymbol{u}}{\partial n} \boldsymbol{v}\, dS + \nu \int_\Omega \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx, \tag{2.2.5}$$

$$\int_\Omega \boldsymbol{v} \cdot \operatorname{grad} p\, dx = \int_{\partial\Omega} p \boldsymbol{v} \cdot \boldsymbol{n}\, dS - \int_\Omega p \operatorname{div} \boldsymbol{v}\, dx. \tag{2.2.6}$$

($\boldsymbol{n}$ denotes the unit outer normal to $\partial\Omega$ here and $\partial/\partial n$ is the derivative with respect to the direction $\boldsymbol{n}$.) Integrals along $\partial\Omega$ vanish, because $\boldsymbol{v}|\partial\Omega = 0$. We also have

$$\int_\Omega p \operatorname{div} \boldsymbol{v}\, dx = 0,$$

as div $\boldsymbol{v} = 0$ for $\boldsymbol{v} \in \underline{\mathcal{V}}$. Identity (2.2.4) can be rewritten in the form

$$\nu \int_\Omega \sum_{i,j=1}^N \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in \underline{\mathcal{V}}$$

or

$$\nu((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \underline{\mathcal{V}}. \tag{2.2.7}$$

On the basis of this result and the density of $\underline{\mathcal{V}}$ in the space $\boldsymbol{V}$ we introduce the following generalization of the concept of the solution of the Stokes problem.

Definition 2.5 *Let $\nu > 0$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$. We say that a vector function $\boldsymbol{u} : \Omega \to \mathbb{R}^N$ is the weak solution of the Stokes problem with homogeneous boundary conditions, if*

$$\boldsymbol{u} \in \boldsymbol{V} \quad \textit{and} \quad \nu((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2.2.8}$$

Let us notice that conditions (2.2.1) and (2.2.3) are already hidden in the assumption $\boldsymbol{u} \in \boldsymbol{V}$. Conditions (2.2.8) form the *weak formulation* of the Stokes problem.

Lemma 2.6 *The mapping "$\boldsymbol{v} \in H_0^1(\Omega) \to (\boldsymbol{f}, \boldsymbol{v})$" is a continuous linear functional on $\mathbf{H}_0^1(\Omega)$, and $a(\boldsymbol{u}, \boldsymbol{v}) = \nu((\boldsymbol{u}, \boldsymbol{v}))$ is a continuous $\mathbf{H}_0^1(\Omega)$-elliptic bilinear form on $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$.*

Proof The continuity of the functional $(\boldsymbol{f}, \cdot)$ is a consequence of the inequalities

$$|(\boldsymbol{f}, \boldsymbol{v})| \le \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \le \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \le \tag{2.2.9}$$

$$\le c\, \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} \|\boldsymbol{v}\|_{\mathbf{H}_0^1(\Omega)} = c\, \|\boldsymbol{f}\|\; |||\boldsymbol{v}|||, \quad \boldsymbol{v} \in \mathbf{H}_0^1(\Omega). \tag{2.2.10}$$

(c is the constant from (2.1.17).) Its linearity is obvious. The properties of the form $a(\cdot,\cdot)$ follow from the fact that $a(\cdot,\cdot)$ is a positive multiple of a scalar product in the space $\mathbf{H}_0^1(\Omega)$. □

Corollary 2.7 *$(\boldsymbol{f}, \cdot)$ is a continuous linear functional on $\boldsymbol{V}$ and $a(\cdot,\cdot)$ is a continuous $\boldsymbol{V}$-elliptic bilinear form on $\boldsymbol{V} \times \boldsymbol{V}$.*

Theorem 2.8 *There exists exactly one weak solution of the Stokes problem with homogeneous boundary conditions.*

Proof Equation (2.2.8) can be written in the form

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

In virtue of 2.7, the existence and uniqueness of this problem is an immediate consequence of the Lax–Milgram lemma. □

There is a question how to introduce the pressure to the velocity satisfying (2.2.8).

Theorem 2.9 *Let $\boldsymbol{u}$ be a weak solution of the Stokes problem with homogeneous boundary conditions. Then there exists a function $p \in L_0^2(\Omega)$ such that*

$$\nu((\boldsymbol{u}, \boldsymbol{v})) - (p,\ \mathit{div}\ \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega). \tag{2.2.11}$$

The couple $(\boldsymbol{u}, p)$ satisfies (2.2.2) in the sense of distributions.

Proof In virtue of Lemma 2.6 and condition (2.2.8), the mapping "$\boldsymbol{v} \in \mathbf{H}_0^1(\Omega) \to \nu((\boldsymbol{u}, \boldsymbol{v})) - (\boldsymbol{f}, \boldsymbol{v})$" is a continuous linear functional vanishing on $\boldsymbol{V}$ and, thus, also on the set $\underline{\mathcal{V}}$. By Lemma 2.1, there exists $p \in L_0^2(\Omega)$ such that

$$\nu((\boldsymbol{u}, \boldsymbol{v})) - (\boldsymbol{f}, \boldsymbol{v}) = (p,\ \mathrm{div}\ \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{H}_0^1(\Omega).$$

This proves identity (2.2.11). Further, we have

$$\nu((\boldsymbol{u}, \boldsymbol{v})) = -\nu\langle \Delta \boldsymbol{u}, \boldsymbol{v}\rangle, \tag{2.2.12}$$

$$(\boldsymbol{f}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \tag{2.2.13}$$
$$(p,\ \mathrm{div}\ \boldsymbol{v}) = -\langle \mathrm{grad}\ p, \boldsymbol{v} \rangle, \quad \boldsymbol{v} \in \mathbf{C}_0^\infty(\Omega). \tag{2.2.14}$$

This and (2.2.11) already imply that $\boldsymbol{u}$ and p satisfy (2.2.2) in the sense of distributions. □

2.2.2 *Stokes problem with nonhomogeneous boundary conditions*

For a given constant $\nu > 0$ and given functions $\boldsymbol{f} : \ \Omega \to \mathbb{R}^N$ and $\underline{\varphi} : \ \partial\Omega \to \mathbb{R}^N$ we consider the problem

$$\mathrm{div}\, \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2.2.15}$$
$$-\nu\Delta\boldsymbol{u} + \ \mathrm{grad}\ p = \boldsymbol{f} \qquad \text{in } \ \Omega, \tag{2.2.16}$$
$$\boldsymbol{u}|\partial\Omega = \underline{\varphi}. \tag{2.2.17}$$

The classical solution of this problem is defined analogously as in Definition 2.4.

Let us assume that $\boldsymbol{f} \in L^2(\Omega)$, $\underline{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ and

$$\int_{\partial\Omega} \underline{\varphi} \cdot \boldsymbol{n}\, dS = 0. \tag{2.2.18}$$

Notice that provided a function $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ satisfies conditions (2.2.15) and (2.2.17) (in the sense of traces), then relation (2.2.18) is fulfilled. It means that (2.2.18) is a *necessary condition* for the solvability of problem (2.2.15) – (2.2.17).

Lemma 2.10 *Let the function* $\underline{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ *satisfy (2.2.18). Then there exists* $\boldsymbol{g} \in \mathbf{H}^1(\Omega)$ *such that*

$$\begin{aligned} &\mathrm{div}\, \boldsymbol{g} = 0 \quad \textit{in } \Omega \qquad \textit{and} \\ &\boldsymbol{g}|\partial\Omega = \underline{\varphi} \quad \textit{(in the sense of traces).} \end{aligned} \tag{2.2.19}$$

Proof First, it is clear that there exists a function $\boldsymbol{g}_1 \in \mathbf{H}^1(\Omega)$ such that $\boldsymbol{g}_1|\partial\Omega = \underline{\varphi}$. Further,

$$\int_\Omega \mathrm{div}\ \boldsymbol{g}_1\, dx = \int_{\partial\Omega} \boldsymbol{g}_1 \cdot \boldsymbol{n}\, dS = \int_{\partial\Omega} \underline{\varphi} \cdot \boldsymbol{n}\, dS = 0.$$

It means that div $\boldsymbol{g}_1 \in L_0^2(\Omega)$. In virtue of Lemma 2.2, there exists $\boldsymbol{g}_2 \in \mathbf{H}_0^1(\Omega)$ such that $\mathrm{div}\, \boldsymbol{g}_2 = \mathrm{div}\, \boldsymbol{g}_1$. Now it suffices to put $\boldsymbol{g} = \boldsymbol{g}_1 - \boldsymbol{g}_2$. It is obvious that $\boldsymbol{g} \in \mathbf{H}^1(\Omega)$, $\mathrm{div}\, \boldsymbol{g} = \mathrm{div}\, \boldsymbol{g}_1 - \mathrm{div}\, \boldsymbol{g}_2 = 0$ and $\boldsymbol{g}|\partial\Omega = \underline{\varphi}$. Hence, $\boldsymbol{g}$ satisfies conditions (2.2.19). □

The weak formulation of the Stokes problem with nonhomogeneous boundary conditions can be obtained similarly as in 2.2.1 with the use of Green's theorem. We again introduce the concept of a weak solution:

Definition 2.11 *Let* $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\underline{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ *and let (2.2.18) hold. Supposing that* $\boldsymbol{g}$ *is a function from Lemma 2.10, we call* $\boldsymbol{u}$ *a weak solution of the Stokes problem (2.2.15) – (2.2.17), if*

$$\begin{aligned} &\text{a) } \boldsymbol{u} \in \mathbf{H}^1(\Omega), \\ &\text{b) } \boldsymbol{u} - \boldsymbol{g} \in \mathbf{V}, \\ &\text{c) } \nu((\boldsymbol{u}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{2.2.20}$$

Condition (2.2.20), c) represents a weak version of equation (2.2.16) and (2.2.20), a–b) guarantee that $\operatorname{div} \boldsymbol{u} = 0$ in Ω and that (2.2.17) is fulfilled in the sense of traces.

Theorem 2.12 *Problem (2.2.20), a)–c) has a unique solution which does not depend on the choice of the function* $\boldsymbol{g}$ *from Lemma 2.10.*

Proof In view of (2.2.20), b), the weak solution can be sought in the form $\boldsymbol{u} = \boldsymbol{g} + \boldsymbol{z}$ where $\boldsymbol{z} \in \boldsymbol{V}$ is a solution of the problem

$$\begin{aligned} &\text{a) } \boldsymbol{z} \in \boldsymbol{V}, \\ &\text{b) } \nu((\boldsymbol{z}, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}) - \nu((\boldsymbol{g}, \boldsymbol{v})) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{2.2.21}$$

We easily find that the right-hand side of (2.2.21), b) defines a continuous linear functional on the space $\boldsymbol{V}$. The function $a(\boldsymbol{z}, \boldsymbol{v}) = \nu((\boldsymbol{z}, \boldsymbol{v}))$ is a continuous $\boldsymbol{V}$-elliptic bilinear form on $\boldsymbol{V} \times \boldsymbol{V}$. The Lax–Milgram lemma implies that problem (2.2.21) has a unique solution $\boldsymbol{z} \in \mathbf{V}$. It is obvious that $\boldsymbol{u} = \boldsymbol{g} + \boldsymbol{z}$ is a weak solution of the Stokes problem. Now we show that $\boldsymbol{u}$ does not depend on the choice of the function $\boldsymbol{g}$. Let $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ be two functions associated with the given $\underline{\varphi}$ by Lemma 2.10 and let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be the corresponding weak solutions. Then, of course,

$$\nu((\boldsymbol{u}_i, \boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ i = 1, 2.$$

By subtracting,

$$((\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v})) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Let us substitute $\boldsymbol{v} := \boldsymbol{u}_1 - \boldsymbol{u}_2$ ($\in \boldsymbol{V}$). Then

$$0 = ((\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)) = |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2,$$

which immediately implies that $\boldsymbol{u}_1 = \boldsymbol{u}_2$. □

Exercise 2.1 Similarly as in 2.9, prove the existence of a pressure function $p \in L^2_0(\Omega)$ to a weak solution of the Stokes problem with nonhomogeneos boundary conditions.

2.3 The stationary Navier–Stokes equations

Now let us consider the boundary value problem for the stationary nonlinear Navier–Stokes equations with homogeneous boundary conditions: we seek $\boldsymbol{u}, p$ such that

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2.3.1}$$

$$-\nu \Delta \boldsymbol{u} + \sum_{j=1}^{N} u_j \frac{\partial \boldsymbol{u}}{\partial x_j} + \operatorname{grad} p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2.3.2}$$

$$\boldsymbol{u}|\partial\Omega = 0. \tag{2.3.3}$$

The viscosity coefficient $\nu = \text{const} > 0$ and the density $\boldsymbol{f}$ of volume force are given; p denotes the kinematic pressure.

Definition 2.13 *A couple $(\boldsymbol{u}, p)$ is called a classical solution of the Navier–Stokes problem with homogeneous boundary conditions, if $\boldsymbol{u} \in \mathbf{C}^2(\overline{\Omega})$ and $p \in C^1(\overline{\Omega})$ satisfy (8.4.1) – (8.4.3).*

We again pass to the weak formulation of problem (8.4.1) – (8.4.3). Let $(\boldsymbol{u}, p)$ be its classical solution. Multiplying (8.4.2) by an arbitrary vector function $\boldsymbol{v} = (v_1, \ldots, v_N) \in \underline{\mathcal{V}}$, integrating over Ω and using Green's theorem, we get the identity

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \underline{\mathcal{V}}, \tag{2.3.4}$$

where

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \sum_{i,j=1}^{N} u_j \frac{\partial v_i}{\partial x_j} w_i \, dx \tag{2.3.5}$$

for $\boldsymbol{u} = (u_1, \ldots, u_N), \boldsymbol{v} = (v_1, \ldots, v_N)$,
$\boldsymbol{w} = (w_1, \ldots, w_N)$ "sufficiently smooth" in $\overline{\Omega}$.

Lemma 2.14 *The mapping "$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \to b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$" is a continuous trilinear form on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$.*

Proof Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{u} = (u_1, \ldots, u_N)$, $\boldsymbol{v} = (v_1, \ldots, v_N)$, $\boldsymbol{w} = (w_1, \ldots \ldots, w_N)$. Then $u_i, v_i, w_i \in H^1(\Omega)$. Hence, $\partial v_i / \partial x_j \in L^2(\Omega)$. In virtue of the continuous imbedding of $H^1(\Omega)$ into $L^4(\Omega)$ (take into account that $N = 2$ or $N = 3$), we have $u_j, w_i \in L^4(\Omega)$. This implies that $u_j w_i \partial v_i / \partial x_j \in L^1(\Omega)$. It means that the integral in (2.3.5) exists and is finite. The form b is thus defined on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. Its linearity with respect to the arguments $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ is obvious.

Let us prove the continuity of b. Due to the continuous imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exists a constant c_4 such that

$$\|u\|_{L^4(\Omega)} \leq c_4 \|u\|_{H^1(\Omega)} \quad \forall\, u \in H^1(\Omega). \tag{2.3.6}$$

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbf{H}^1(\Omega)$. The Cauchy inequality and (2.3.6) yield

$$\left| \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, dx \right| \leq \int_\Omega \left| u_j \frac{\partial v_i}{\partial x_j} w_i \right| \, dx \leq \left(\int_\Omega (u_j w_i)^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx \right)^{\frac{1}{2}} \leq \tag{2.3.7}$$

$$\leq \left(\int_\Omega u_j^4 \, dx \right)^{\frac{1}{4}} \left(\int_\Omega w_i^4 \, dx \right)^{\frac{1}{4}} \left(\int_\Omega \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx \right)^{\frac{1}{2}} \leq \tag{2.3.8}$$

$$\leq \|u_j\|_{L^4(\Omega)} \|w_i\|_{L^4(\Omega)} \|v_i\|_{H^1(\Omega)} \leq \tag{2.3.9}$$

$$\leq c_4^2 \|u_j\|_{H^1(\Omega)} \|w_i\|_{H^1(\Omega)} \|v_i\|_{H^1(\Omega)}. \tag{2.3.10}$$

Summing these inequalities for i, $j = 1, \ldots, N$, we obtain

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq \tilde{c} \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \|\boldsymbol{w}\|_{\mathbf{H}^1(\Omega)} \quad \forall\, \boldsymbol{u}, \boldsymbol{v}, \mathbf{w} \in \mathbf{H}^1(\Omega), \tag{2.3.11}$$

which we wanted to prove. □

Corollary 2.15 *The function b is a continuous trilinear form on $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$. There exists a constant $c^\star > 0$ such that*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq c^\star |||\boldsymbol{u}|||\, |||\boldsymbol{v}|||\, |||\boldsymbol{w}||| \quad \forall\, \boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in \boldsymbol{V}. \tag{2.3.12}$$

Lemma 2.16 *Let $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$, $\operatorname{div} \boldsymbol{u} = 0$ and $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbf{H}_0^1(\Omega)$. Then*

$$\text{(i)} \quad b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \qquad \text{(ii)} \quad b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$$

Proof Since b is a continuous trilinear form and $\mathbf{C}_0^\infty(\Omega)$ is dense in $\mathbf{H}_0^1(\Omega)$, it is sufficient to prove assertion (i) for v, $w \in \mathbf{C}_0^\infty(\Omega)$. By Green's theorem

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = \sum_{i,j=1}^N \int_\Omega u_j \frac{\partial v_i}{\partial x_j} v_i \, dx = \sum_{i,j=1} \int_\Omega u_j \frac{1}{2} \frac{\partial}{\partial x_j} (v_i^2) \, dx = \tag{2.3.13}$$

$$= -\sum_{i,j=1}^N \frac{1}{2} \int_\Omega \frac{\partial u_j}{\partial x_j} v_i^2 \, dx = -\sum_{i=1}^N \frac{1}{2} \int_\Omega v_i^2 \operatorname{div} \ \boldsymbol{u} \, dx = 0. \tag{2.3.14}$$

Assertion (ii) is obtained from (i) by substituting $\boldsymbol{v} + \boldsymbol{w}$ for $\boldsymbol{v}$:

$$0 = b(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}, \boldsymbol{v} + \boldsymbol{w}) = \tag{2.3.15}$$

$$= b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{w}) = \tag{2.3.16}$$

$$= b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}). \tag{2.3.17}$$

□

The above considerations lead us to the following definition:

Definition 2.17 *Let $\nu > 0$ and $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ be given. We say that $\boldsymbol{u}$ is a weak solution of the Navier–Stokes problem with homogeneous boundary conditions, if*

$$\mathbf{u} \in \boldsymbol{V} \text{ and } \nu((\boldsymbol{u},\boldsymbol{v})) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3.18}$$

It is obvious that any classical solution is a weak one. Similarly as in 2.9 we can prove that to the weak solution $\boldsymbol{u}$ there exists the pressure $p \in L^2_0(\Omega)$ satisfying the identity

$$\nu((\boldsymbol{u},\boldsymbol{v})) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - (p,\ \mathrm{div}\ \boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{H}^1_0(\Omega). \tag{2.3.19}$$

Then the couple $(\boldsymbol{u}, p)$ satisfies equation (2.3.2) in the sense of distributions.

The following result will be helpful in the proof of the existence of a weak solution of the Navier–Stokes problem.

Lemma 2.18 *Let $\boldsymbol{u}^\alpha,\ \boldsymbol{u} \in \boldsymbol{V},\ \alpha = 1,2,\ldots,$ and $\boldsymbol{u}^\alpha \to \boldsymbol{u}$ in $\mathbf{L}^2(\Omega)$ as $\alpha \to +\infty$. Then $b(\boldsymbol{u}^\alpha,\boldsymbol{u}^\alpha,\boldsymbol{v}) \to b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})$ for each $\boldsymbol{v} \in \underline{\mathcal{V}}$.*

Proof By assertion (ii) of Lemma 2.16,

$$b(\boldsymbol{u}^\alpha,\boldsymbol{u}^\alpha,\boldsymbol{v}) = -b(\boldsymbol{u}^\alpha,\boldsymbol{v},\boldsymbol{u}^\alpha) = -\sum_{i,j=1}^{N} \int_\Omega u_i^\alpha u_j^\alpha \frac{\partial v_i}{\partial x_j}\, dx.$$

The assumption that $\boldsymbol{u}^\alpha \to \boldsymbol{u}$ in $L^2(\Omega)$ implies that

$$\int_\Omega \left| u_i^\alpha u_j^\alpha - u_i u_j \right| \, dx \to 0 \tag{2.3.20}$$

for all $i,\ j = 1,\ldots,N$. Since $\boldsymbol{v} \in \underline{\mathcal{V}}$, there exists $\tilde{c} > 0$ such that

$$\left| \frac{\partial v_i}{\partial x_j}(x) \right| \le \tilde{c} \quad \forall\, x \in \overline{\Omega},\ \forall\, i,\ j = 1,\ldots,N. \tag{2.3.21}$$

From this and (2.3.20) for $i,\ j = 1,\ldots,N$ we have

$$\left| \int_\Omega \left(u_i^\alpha u_j^\alpha \frac{\partial v_i}{\partial x_j} - u_i u_j \frac{\partial v_i}{\partial x_j} \right) dx \right| \le \tilde{c} \int_\Omega \left| u_i^\alpha u_j^\alpha - u_i u_j \right| \, dx \longrightarrow 0$$

and thus

$$\int_\Omega u_i^\alpha u_j^\alpha \frac{\partial v_i}{\partial x_j}\, dx \longrightarrow \int_\Omega u_i u_j \frac{\partial v_i}{\partial x_j}\, dx. \tag{2.3.22}$$

Summing (2.3.22), we obtain

$$b(\boldsymbol{u}^\alpha,\boldsymbol{u}^\alpha,\boldsymbol{v}) = -\sum_{i,j=1}^{N} \int_\Omega u_i^\alpha u_j^\alpha \frac{\partial v_i}{\partial x_j}\, dx \to -\sum_{i,j=1}^{N} \int_\Omega u_i u_j \frac{\partial v_i}{\partial x_j}\, dx = \tag{2.3.23}$$

$$= b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}), \tag{2.3.24}$$

which concludes the proof. □

Theorem 2.19 . *Theorem on the existence of a solution. There exists at least one weak solution of the Navier–Stokes problem with homogeneous boundary conditions.*

Proof We shall use the Galerkin method. Since $\boldsymbol{V}$ is a separable Hilbert space, there exists a sequence $\{\boldsymbol{v}^i\}_{i=1}^{\infty}$ dense in $\boldsymbol{V}$. By the definition of $\boldsymbol{V}$, for each i there exists a sequence $\{\boldsymbol{w}^{i,\alpha}\}_{\alpha=1}^{\infty} \subset \underline{\mathcal{V}}$ such that $\boldsymbol{w}^{i,\alpha} \to \boldsymbol{v}^i$ as $\alpha \to \infty$. If we order all elements $\boldsymbol{w}^{i,\alpha}$, i, $\alpha = 1, 2, \ldots$, into a sequence (and omit, if they occur, the elements which can be written as linear combinations of the preceding ones), we obtain a sequence $\{\boldsymbol{w}^i\}_{i=1}^{\infty} \subset \underline{\mathcal{V}}$ of linearly independent elements such that

$$\boldsymbol{V} = \overline{\bigcup_{k=1}^{\infty} \mathbf{X}_k}^{\mathbf{H}_0^1(\Omega)}, \tag{2.3.25}$$

where

$$\mathbf{X}_k = [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^k] \tag{2.3.26}$$

is the linear space spanned over the set $\{\boldsymbol{w}^1, \ldots, \boldsymbol{w}^k\}$. $\mathbf{X}_k$ is a finite-dimensional Hilbert space equipped with the scalar product $((\cdot, \cdot))$.

For any $k = 1, 2, \ldots$, let $\boldsymbol{u}^k \in \mathbf{X}_k$ satisfy

$$\nu((\boldsymbol{u}^k, \boldsymbol{w}^i)) + b(\boldsymbol{u}^k, \boldsymbol{u}^k, \boldsymbol{w}^i) = (\boldsymbol{f}, \boldsymbol{w}^i) \quad \forall\, i = 1, \ldots, k. \tag{2.3.27}$$

Since

$$\boldsymbol{u}^k = \sum_{j=1}^{k} \xi_j^k \boldsymbol{w}^j, \quad \xi_j^k \in \mathbb{R}^1, \tag{2.3.28}$$

conditions (2.3.27) represent a system of k nonlinear algebraic equations with respect to k unknowns $\xi_1^k, \ldots, \xi_k^k$.

Let us prove the existence of the solution $\boldsymbol{u}^k$. By 2.15, the mapping "$\boldsymbol{v} \in \mathbf{X}_k \to \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})$" is a continuous linear functional on $\mathbf{X}_k$ for any $\boldsymbol{u} \in \mathbf{X}_k$. In virtue of the Riesz theorem, there exists $P_k(\boldsymbol{u}) \in \mathbf{X}_k$ such that

$$((P_k(\boldsymbol{u}), \boldsymbol{v})) = \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{X}_k.$$

Hence, $P_k : \mathbf{X}_k \to \mathbf{X}_k$. Since the spaces $\mathbf{X}_k$ and $\mathbb{R}^k$ are isomorphic and the quadratic functions "$(\xi_1, \ldots, \xi_k) \in \mathbb{R}^k \to \nu((\boldsymbol{u}, \boldsymbol{w}^i)) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}^i) - (\boldsymbol{f}, \boldsymbol{w}^i)$ where $\boldsymbol{u} = \sum_{j=1}^{k} \xi_j \boldsymbol{w}^j$" are obviously continuous for all $i = 1, \ldots, k$, the mapping P_k is also continuous.

By (2.1.17) and 2.16,

$$((P_k(\boldsymbol{u}), \boldsymbol{u})) = \nu|||\boldsymbol{u}|||^2 + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) - (\boldsymbol{f}, \boldsymbol{u}) = \tag{2.3.29}$$

$$= \nu|||\boldsymbol{u}|||^2 - (\boldsymbol{f}, \boldsymbol{u}) \geq \tag{2.3.30}$$

$$\geq \nu|||\boldsymbol{u}|||^2 - c\|\boldsymbol{f}\|\,|||\boldsymbol{u}||| \tag{2.3.31}$$

for any $\mathbf{u} \in \mathbf{X}_k$. ($c > 0$ is the constant from (2.1.17).) If $\boldsymbol{u} \in \mathbf{X}_k$ and $|||\boldsymbol{u}||| = K > 0$ with K sufficiently large, then $\nu|||\boldsymbol{u}||| - c\,\|\boldsymbol{f}\| > 0$ and, thus, $((P_k(\boldsymbol{u}), \boldsymbol{u})) > 0$.

This implies that for each $k = 1, 2, \ldots$ there exists at least one solution $\boldsymbol{u}^k$ of the equation $P_k(\boldsymbol{u}^k) = 0$, equivalent to system (2.3.27).

Further, we show that the sequence $\{\boldsymbol{u}^k\}_{k=1}^{\infty}$ is bounded in the space $\boldsymbol{V}$. By (2.3.27),

$$\nu((\boldsymbol{u}^k, \boldsymbol{v})) + b(\boldsymbol{u}^k, \boldsymbol{u}^k, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{X}_k.$$

Substituting $\boldsymbol{v} := \boldsymbol{u}^k$, we have

$$\nu|||\boldsymbol{u}^k|||^2 = \nu|||\boldsymbol{u}^k|||^2 + b(\boldsymbol{u}^k, \boldsymbol{u}^k, \boldsymbol{u}^k) = (\boldsymbol{f}, \boldsymbol{u}^k) \le \|\boldsymbol{f}\|\,\|\boldsymbol{u}^k\| \le c\|\boldsymbol{f}\|\,|||\boldsymbol{u}^k|||,$$

which immediately implies that

$$|||\boldsymbol{u}^k||| \le \frac{c\,\|\boldsymbol{f}\|}{\nu} \qquad \forall\, k = 1, 2, \ldots . \tag{2.3.32}$$

From the bounded sequence $\{\boldsymbol{u}^k\}_{k=1}^{\infty}$ a weakly convergent subsequence $\{\boldsymbol{u}^{k_\alpha}\}_{\alpha=1}^{\infty}$ can be subtracted:

$$\boldsymbol{u}^{k_\alpha} \to \boldsymbol{u} \quad \text{weakly in } \boldsymbol{V} \text{ as } \alpha \to \infty. \tag{2.3.33}$$

In virtue of the inclusion $\boldsymbol{V} \subset \mathbf{H}_0^1(\Omega)$ and the compact imbedding $\mathbf{H}_0^1(\Omega) \subset \mathbf{L}^2(\Omega)$,

$$\boldsymbol{u}^{k_\alpha} \to \boldsymbol{u} \text{ strongly in } \mathbf{L}^2(\Omega).$$

From (2.3.33) we obtain

$$((\boldsymbol{u}^{k_\alpha}, \boldsymbol{w}^i)) \to ((\boldsymbol{u}, \mathbf{w}^i)) \quad \text{as } \alpha \to \infty \quad \forall\, i = 1, 2, \ldots .$$

Further, by Lemma 2.18, we have

$$b(\boldsymbol{u}^{k_\alpha}, \boldsymbol{u}^{k_\alpha}, \boldsymbol{w}^i) \to b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}^i) \quad \text{as } \alpha \to \infty \quad \forall\, i = 1, 2, \ldots .$$

In view of (2.3.27),

$$\nu((\boldsymbol{u}^{k_\alpha}, \boldsymbol{w}^i)) + b(\boldsymbol{u}^{k_\alpha}, \boldsymbol{u}^{k_\alpha}, \boldsymbol{w}^i) = (\boldsymbol{f}, \boldsymbol{w}^i) \quad \forall\, i = 1, \ldots, k_\alpha,\ \forall\, \alpha = 1, 2, \ldots .$$

Passing to the limit as $\alpha \to \infty$, we find from the above relations that

$$\nu((\boldsymbol{u}, \boldsymbol{w}^i)) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}^i) = (\boldsymbol{f}, \boldsymbol{w}^i) \quad \forall\, i = 1, 2, \ldots$$

Hence, by (2.3.25), we have

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},$$

which means that $\boldsymbol{u}$ is a weak solution of the Navier–Stokes problem. □

In the following we shall investigate the uniqueness of a weak solution of the Navier–Stokes problem.

Theorem 2.20 *Let the condition*

$$\nu^2 > c^\star c \|\boldsymbol{f}\| \tag{2.3.34}$$

be fulfilled with the constants $c^\star$ and c from (2.3.12) and (2.1.17), respectively. Then there exists exactly one weak solution of the Navier–Stokes problem with homogeneous boundary conditions.

Proof Let $\boldsymbol{u}^1$, $\boldsymbol{u}^2$ be two solutions of (2.3.18). It means that for $i = 1, 2$,

$$\boldsymbol{u}_i \in \boldsymbol{V},\ \nu((\boldsymbol{u}^i, \boldsymbol{v})) + b(\boldsymbol{u}^i, \boldsymbol{u}^i, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3.35}$$

Substituting here $\boldsymbol{v} := \boldsymbol{u}^i$, we easily find that

$$\nu |||\boldsymbol{u}^i|||^2 = (\boldsymbol{f}, \boldsymbol{u}^i) \le \|\boldsymbol{f}\|\ \|\boldsymbol{u}^i\| \le c\,\|\boldsymbol{f}\|\ |||\boldsymbol{u}^i|||$$

and, hence,

$$|||\boldsymbol{u}^i||| \le \frac{c\,\|\boldsymbol{f}\|}{\nu}. \tag{2.3.36}$$

Let us set $\boldsymbol{u} = \boldsymbol{u}^1 - \boldsymbol{u}^2$. Subtracting equations (2.3.35), $i = 1, 2$, and using the properties of the form b, we obtain

$$0 = \nu((\boldsymbol{u}^1, \boldsymbol{v})) + b(\boldsymbol{u}^1, \mathbf{u}^1, \boldsymbol{v}) - \nu((\boldsymbol{u}^2, \boldsymbol{v})) - b(\boldsymbol{u}^2, \boldsymbol{u}^2, \boldsymbol{v}) = \tag{2.3.37}$$

$$= \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}^1, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}^2, \boldsymbol{v}) \tag{2.3.38}$$

for each $\boldsymbol{v} \in \boldsymbol{V}$. If we choose $\boldsymbol{v} = \boldsymbol{u}$, then, in view of Lemma 2.16,

$$\nu((\boldsymbol{u}, \boldsymbol{u})) = -b(\boldsymbol{u}, \boldsymbol{u}^2, \boldsymbol{u}),$$

from which, due to inequalities (2.3.12) and (2.3.36), we derive the estimate

$$\nu |||\boldsymbol{u}|||^2 \le c^\star |||\boldsymbol{u}|||^2\, |||\boldsymbol{u}^2||| \le c c^\star \nu^{-1}\, \|\boldsymbol{f}\|\ |||\boldsymbol{u}|||^2.$$

Thus,

$$|||\boldsymbol{u}|||^2 \left(\nu - c c^\star \nu^{-1} \|\boldsymbol{f}\|\right) \le 0.$$

This and (2.3.34) immediately imply that $|||\boldsymbol{u}||| = 0$, which means that $\boldsymbol{u}^1 = \boldsymbol{u}^2$. □

2.3.1 *The Navier–Stokes problem with nonhomogeneous boundary conditions.* We seek $\boldsymbol{u}$ and p satisfying

$$\text{a) } \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.3.39}$$

$$\text{b) } -\nu\Delta \boldsymbol{u} + \sum_{j=1}^{n} u_j \frac{\partial \boldsymbol{u}}{\partial x_j} + \operatorname{grad} p = \boldsymbol{f} \quad \text{in } \Omega,$$

$$\text{c) } \boldsymbol{u}|\partial\Omega = \underline{\varphi}.$$

The constant $\nu > 0$ and the functions $\boldsymbol{f}$ and $\underline{\varphi}$ are given.

Similarly as above we introduce the concept of a *weak solution* of problem (2.3.39), a)–c): We assume that $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\underline{\varphi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ satisfies condition (2.2.18) and that $\boldsymbol{g}$ is a function from Lemma 2.10 with the properties

$$\text{a)} \quad \boldsymbol{g} \in \mathbf{H}^1(\Omega), \qquad \text{b)} \quad \operatorname{div} \boldsymbol{g} = 0 \text{ in } \Omega, \qquad \text{c)} \quad \boldsymbol{g}|\partial\Omega = \underline{\varphi}. \tag{2.3.40}$$

Then $\boldsymbol{u}$ is called a weak solution of problem (2.3.39), a)–c), if

$$\begin{aligned} &\text{a) } \mathbf{u} \in \mathbf{H}^1(\Omega), \\ &\text{b) } \boldsymbol{u} - \boldsymbol{g} \in \boldsymbol{V}, \\ &\text{c) } \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \mathbf{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{2.3.41}$$

It is obvious that the behaviour of the function $\boldsymbol{g}$ in Ω is not important in formulation (2.3.41), a)–c). It is essential that $\boldsymbol{g}$ satisfies conditions (2.3.40), a)-c).

Let us assume that

$$\int_\Gamma \underline{\varphi} \cdot \boldsymbol{n} \, dS = 0 \tag{2.3.42}$$

for each component Γ of $\partial\Omega$.

Theorem 2.21 *Let $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{g}$ satisfy (2.3.40), (2.3.42). Then problem (2.3.41), a)-c) has at least one solution.*

Proof The solution will be sought in the form $\boldsymbol{u} = \hat{\mathbf{u}} + \boldsymbol{g}$ where $\hat{\mathbf{u}} \in \boldsymbol{V}$ and $\boldsymbol{g}$ is a suitable function with properties (2.3.40), a)-c). Substituting this representation into (2.3.41), we see that the unknown $\hat{\mathbf{u}}$ is a solution of the problem

$$\begin{aligned} &\hat{\mathbf{u}} \in \boldsymbol{V}, \\ &\nu((\hat{\mathbf{u}}, \boldsymbol{v})) + b(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \boldsymbol{v}) + b(\hat{\mathbf{u}}, \boldsymbol{g}, \boldsymbol{v}) + b(\boldsymbol{g}, \hat{\mathbf{u}}, \boldsymbol{v}) = \langle \hat{\mathbf{f}}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{2.3.43}$$

$\hat{\mathbf{f}}$ is a continuous linear functional on $\boldsymbol{V}$ defined by the relation

$$\langle \hat{\mathbf{f}}, \boldsymbol{v} \rangle = (\boldsymbol{f}, \boldsymbol{v}) - \nu((\boldsymbol{g}, \boldsymbol{v})) - b(\boldsymbol{g}, \boldsymbol{g}, \boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3.44}$$

The existence of the solution $\hat{\mathbf{u}}$ of problem (2.3.43) will be proved in the same way as in the case of the problem with homogeneous boundary conditions:
1. Consider such a sequence $\{\boldsymbol{w}^k\}_{k=1}^\infty \subset \mathcal{V}$ that

$$\boldsymbol{V} = \overline{\bigcup_{k=1}^\infty \mathbf{X}_k}^{\mathbf{H}_0^1(\Omega)},$$

where $\mathbf{X}_k = [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^k]$.
2. Prove the existence of a solution $\hat{\mathbf{u}}^k \in \mathbf{X}_k$ of the problem

$$\nu((\hat{\mathbf{u}}^k, \boldsymbol{w}^j)) + b(\hat{\mathbf{u}}^k, \hat{\mathbf{u}}^k, \boldsymbol{w}^j) + b(\hat{\mathbf{u}}^k, \boldsymbol{g}, \boldsymbol{w}^j) + b(\boldsymbol{g}, \hat{\mathbf{u}}^k, \boldsymbol{w}^j) = \tag{2.3.45}$$

$$= \langle \hat{\mathbf{f}}, \boldsymbol{w}^j \rangle, \quad j = 1, \ldots, k\,.$$

3. On the basis of Lemma 2.16 derive from (2.3.45) the estimate

$$|||\hat{\mathbf{u}}^k||| \leq \tilde{c}\|\hat{\mathbf{f}}\|_{\boldsymbol{V}^\star} \quad \forall\, k = 1, 2, \ldots. \tag{2.3.46}$$

4. From $\{\hat{\mathbf{u}}^k\}$, subtract a subsequence $\{\hat{\mathbf{u}}^{k_j}\}$ converging weakly in $\boldsymbol{V}$ and strongly in $\mathbf{L}^2(\Omega)$ to some $\hat{\mathbf{u}} \in \boldsymbol{V}$. Use the obvious limit passages

$$\text{a) } b(\hat{\mathbf{u}}^{k_j}, \boldsymbol{g}, \boldsymbol{v}) \to b(\hat{\mathbf{u}}, \boldsymbol{g}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.3.47}$$
$$\text{b) } b(\boldsymbol{g}, \hat{\mathbf{u}}^{k_j}, \boldsymbol{v}) \to b(\boldsymbol{g}, \hat{\mathbf{u}}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}$$

and Lemma 2.18 to find that $\hat{\mathbf{u}}$ is a solution of (2.3.41), a)–c).

In the above process the most difficult is the realization of the second and third steps.

To prove that system (2.3.45) has a solution, it is sufficient to verify the existence of a constant $\beta > 0$ such that

$$\nu((\boldsymbol{v}, \boldsymbol{v})) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{v}) + b(\boldsymbol{g}, \boldsymbol{v}, \boldsymbol{v}) \geq \beta|||\boldsymbol{v}|||^2 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.3.48}$$

or, in virtue of Lemma 2.16,

$$\nu|||v|||^2 + b(\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{v}) \geq \beta|||\boldsymbol{v}|||^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

This estimate will be true, if we succeed to find $\boldsymbol{g}$ (satisfying (2.3.40)) such that

$$|b(\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{v})| \leq \frac{\nu}{2}|||\boldsymbol{v}|||^2 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3.49}$$

Provided (2.3.49) holds, condition (2.3.48) is satisfied with $\beta = \nu/2$. Then the existence of a solution of problem (2.3.45) (for $k = 1, 2, \ldots$) can be proved analogously as in the proof of Theorem 2.19. Further, by (2.3.45), (2.3.49) and 2.16, we have

$$\frac{\nu}{2}|||\hat{\mathbf{u}}^k|||^2 \leq \nu((\hat{\mathbf{u}}^k, \hat{\mathbf{u}}^k)) - |b(\hat{\mathbf{u}}^k, \boldsymbol{g}, \hat{\mathbf{u}}^k)| \leq \tag{2.3.50}$$
$$\leq \nu((\hat{\mathbf{u}}^k, \hat{\mathbf{u}}^k)) + b(\hat{\mathbf{u}}^k, \boldsymbol{g}, \hat{\mathbf{u}}^k) = \tag{2.3.51}$$
$$= \langle \hat{\mathbf{f}}, \hat{\mathbf{u}}^k \rangle \leq \|\hat{\mathbf{f}}\|_{\boldsymbol{V}^\star}|||\hat{\mathbf{u}}^k|||, \tag{2.3.52}$$

which proves (2.3.46).

The validity of (2.3.49) is a consequence of the following result:

Lemma 2.22 *Let (2.3.42) be satisfied. Then for any $\gamma > 0$ there exists $\boldsymbol{g}$ satisfying the conditions*

$$\text{a)} \quad \boldsymbol{g} \in \mathbf{H}^1(\Omega), \qquad \text{b)} \quad \operatorname{div} \boldsymbol{g} = 0 \textit{ in } \Omega, \qquad \text{c)} \quad \boldsymbol{g}|\partial\Omega = \underline{\varphi} \tag{2.3.53}$$

and

$$|b(\boldsymbol{v}, \boldsymbol{g}, \boldsymbol{v})| \leq \gamma|||\boldsymbol{v}|||^2 \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{2.3.54}$$

Remark 2.23 In the case that the condition (2.3.42) is not satisfied (only the weaker condition (2.2.18) holds), the existence of a weak solution can be established only for "small data", i.e. for small $\|\boldsymbol{f}\|, \|\underline{\varphi}\|_{\mathbf{H}^{1/2}(\partial\Omega)}$ and large ν.

2.4 The Oseen problem

The solution of the Navier–Stokes problem is sometimes approximated by a sequence of linear Stokes problems (see, e. g., [Glowinski (1984)]). Another often more adequate linearization of the Navier–Stokes equations is the Oseen problem (cf., e. g., [Crouzeix (1974)]), which is formulated in the following way:

Definition 2.24 *Let $\boldsymbol{w} \in \mathbf{C}^1(\overline{\Omega})$, $\operatorname{div} \boldsymbol{w} = 0$ in Ω, $\boldsymbol{f} : \overline{\Omega} \to \mathbb{R}^N$ and let $\underline{\varphi} : \partial\Omega \to \mathbb{R}^N$ fulfils (2.2.18). Then a classical solution of the Oseen problem is defined as a couple $(\boldsymbol{u}, p) \in \mathbf{C}^2(\overline{\Omega}) \times C^1(\overline{\Omega})$ satisfying*

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.4.1}$$

$$-\nu\Delta\boldsymbol{u} + \sum_{j=1}^{N} w_j \frac{\partial \mathbf{u}}{\partial x_j} + \operatorname{grad} p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.4.2}$$

and

$$\boldsymbol{u}|\partial\Omega = \underline{\varphi}. \tag{2.4.3}$$

Quite analogously as in preceding paragraphs, the Oseen problem can be reformulated in a weak sense.

Definition 2.25 *Let $\boldsymbol{w} \in \mathbf{H}^1(\Omega)$, $\operatorname{div} \boldsymbol{w} = 0, \boldsymbol{f} \in \mathbf{L}^2(\Omega)$ and let $\boldsymbol{g}$ be a function with properties (2.3.40), a)–c). We call $\boldsymbol{u}$ a weak solution of the Oseen problem (2.4.1) – (2.4.3), if*

$$\begin{aligned} &\text{a) } \boldsymbol{u} \in \mathbf{H}^1(\Omega), \qquad \text{b) } \quad \boldsymbol{u} - \boldsymbol{g} \in \boldsymbol{V}, \\ &\text{c) } \nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{w}, \boldsymbol{u}, \mathbf{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{2.4.4}$$

Remark 2.26 On the basis of the properties of the form b and Lemma 2.1 it is possible to prove the existence of a pressure function $p \in L_0^2(\Omega)$ associated with a weak solution $\boldsymbol{u}$ of the Oseen problem, so that

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{H}_0^1(\Omega). \tag{2.4.5}$$

(Cf. Theorem 2.9 and (2.3.19).)

The solvability of the Oseen problem can be easily established on the basis of the Lax–Milgram lemma:

Theorem 2.27 *Problem (2.4.4), a)–c) has a unique solution.*

Proof Put

$$a_{\boldsymbol{w}}(\hat{\mathbf{u}}, \boldsymbol{v}) = \nu((\hat{\mathbf{u}}, \boldsymbol{v})) + \mathbf{b}(\boldsymbol{w}, \hat{\mathbf{u}}, \boldsymbol{v}) \tag{2.4.6}$$

defined for all $\hat{\mathbf{u}}$, $\boldsymbol{v} \in \mathbf{V}$. The properties of b imply that $a_{\boldsymbol{w}}$ is a continuous bilinear form defined on $\boldsymbol{V} \times \boldsymbol{V}$. Using Lemma 2.16, we can find that $a_{\boldsymbol{w}}$ is $\boldsymbol{V}$-elliptic:

$$a_{\boldsymbol{w}}(\hat{\mathbf{u}}, \hat{\mathbf{u}}) = \nu((\hat{\mathbf{u}}, \hat{\mathbf{u}})) + b(\boldsymbol{w}, \hat{\mathbf{u}}, \hat{\mathbf{u}}) = \tag{2.4.7}$$

$$= \nu((\hat{\mathbf{u}}, \hat{\mathbf{u}})) = \nu |||\hat{\mathbf{u}}|||^2 \quad \forall\, \hat{\mathbf{u}} \in \boldsymbol{V}. \tag{2.4.8}$$

Let us seek a solution of problem (2.4.4), a)–c) in the form $\boldsymbol{u} = \hat{\mathbf{u}} + \boldsymbol{g}$ where $\hat{\mathbf{u}} \in \boldsymbol{V}$ is unknown. Then (2.4.4), c) is equivalent to the problem

$$\hat{\boldsymbol{u}} \in \boldsymbol{V}, \quad a_{\boldsymbol{w}}(\hat{\mathbf{u}}, \boldsymbol{w}) = \langle \underline{\ell}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.4.9}$$

where $\underline{\ell} \in \boldsymbol{V}^\star$ is given by

$$\langle \underline{\ell}, \boldsymbol{v} \rangle = (\boldsymbol{f}, \boldsymbol{v}) - \nu((\boldsymbol{g}, \boldsymbol{v})) - b(\boldsymbol{w}, \boldsymbol{g}, \boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}.$$

From the Lax–Milgram lemma we immediately obtain the unique solvability of problem (2.4.9) with respect to $\hat{\mathbf{u}} \in \boldsymbol{V}$ and, thus, the unique solvability of (2.4.4), a)–c). □

The Oseen problem offers the following iterative process for the approximate solution of the Navier–Stokes equations:

$$\begin{aligned} &\text{a) } \boldsymbol{u}^0 \in \mathbf{H}^1(\Omega), \operatorname{div} \boldsymbol{u}^0 = 0, \\ &\text{b) } \boldsymbol{u}^{k+1} \in \mathbf{H}^1(\Omega),\ \mathbf{u}^{k+1} - \boldsymbol{g} \in \boldsymbol{V}, \\ &\quad \nu((\boldsymbol{u}^{k+1}, \boldsymbol{v})) + b(\boldsymbol{u}^k, \mathbf{u}^{k+1}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ k \geq 0. \end{aligned} \tag{2.4.10}$$

2.5 Functions with values in Banach spaces

In the investigation of nonstationary problems we shall work with functions which depend on time and have values in Banach spaces. If $u(x,t)$ is a function of the space variable x and time t, then it is sometimes suitable to separate these variables and consider u as a function $u(t) = u(\cdot, t)$ which for each t in consideration attains a value $u(t)$ that is a function of x and belongs to a suitable space of functions depending on x. It means that $u(t)$ represents the mapping "$x \to [u(t)]\,(x) = u(x,t)$".

Let a, $b \in \mathbb{R}^1$, $a < b$, and let X be a Banach space with a norm $\|\cdot\|$. By a *function defined on the interval* $[a,b]$ *with its values in the space* X we understand any mapping $u: \ [a,b] \to X$.

By the symbol $C([a,b];\, X)$ we shall denote the space of all functions continuous on the interval $[a,b]$ (i. e., continuous at each $t \in [a,b]$) with values in X. The space $C([a,b];\, X)$ equipped with the norm

$$\|u\|_{C([a,b];\, X)} = \max_{t \in [a,b]} \|u(t)\| \tag{2.5.11}$$

is a Banach space.

2.5.1 *Lebesgue Spaces of Functions with Values in a Banach Space.*

Let X be a Banach space. For $p \in [1, \infty]$ we denote by $L^p(a, b; X)$ the space of (equivalent classes of) strongly measurable functions $u: (a, b) \to X$ such that

$$\|u\|_{L^p(a,b;X)} := \left[\int_a^b \|u(t)\|_X^p \, dt \right]^{1/p} < \infty, \quad \text{if } 1 \le p < \infty, \tag{2.5.12}$$

and

$$\|u\|_{L^\infty(a,b;X)} := \operatorname{ess} \sup_{t \in (a,b)} \|u(t)\|_X = \inf_{\text{meas}(N)=0} \sup_{t \in (a,b)-N} \|u(t)\|_X < +\infty, \qquad \text{if } p = \infty. \tag{2.5.13}$$

It can be proved that $L^p(a, b; X)$ is a Banach space.

2.6 The nonstationary Navier–Stokes equations

In what follows we shall be concerned with the investigation of an initial-boundary value problem describing nonstationary viscous incompressible flow. This problem consists in finding the velocity $\boldsymbol{u}(x, t)$ and pressure $p(x, t)$ defined on the set $\overline{\Omega} \times [0, T]$ $(T > 0)$ and satisfying the system of continuity equation and Navier–Stokes equations to which we add boundary and initial conditions. For simplicity we shall confine our consideration to the case of homogeneous Dirichlet boundary conditions. We thus have the following problem:

Find $\boldsymbol{u}(x, t): \overline{\Omega} \times [0, T] \to \mathbb{R}^N$ and $p(x, t): \overline{\Omega} \times [0, T] \to \mathbb{R}^1$ such that

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q = \Omega \times (0, T), \tag{2.6.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + \sum_{j=1}^n u_j \frac{\partial \boldsymbol{u}}{\partial x_j} + \operatorname{grad} p = \boldsymbol{f} \quad \text{in } Q, \tag{2.6.2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega \times [0, T], \tag{2.6.3}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), \quad \text{in } x \in \Omega. \tag{2.6.4}$$

We assume that the constants ν, $T > 0$ and the functions $\boldsymbol{u}_0: \Omega \to \mathbb{R}^N$ and $\boldsymbol{f}: Q \to \mathbb{R}^N$ are prescribed.

Definition 2.28 *We define a classical solution of the nonstationary Navier–Stokes problem with homogeneous boundary conditions as functions* $\boldsymbol{u} \in \mathbf{C}^2(\overline{Q})$ *and* $p \in C^1(\overline{Q})$ *satisfying (2.6.1) – (2.6.4).*

In the study of problem (2.6.1) – (2.6.4) we shall need some fundamental results concerning the imbedding of the spaces $L^p(a, b; X)$. Let us consider three Banach spaces X_0, X and X_1 such that

a) $X_0 \subset X \subset X_1$, (2.6.5)

b) X_0 and X_1 are reflexive,

c) the imbeddings of X into X_1 and of X_0 into X are continuous,

d) the imbedding of X_0 into X is compact.

Further, let $1 < \alpha_0,\ \alpha_1 < +\infty,\ a,\ b \in \mathbb{R}^1,\ a < b$. By v' we denote the generalized derivative of a function $v \in L^1(a,b;X)$. We put

$$W = W(a,b;\alpha_0,\alpha_1;X_0,X_1) = \\ = \{v \in L^{\alpha_0}(a,b;X_0);\ v' \in L^{\alpha_1}(a,b;X_1)\} \tag{2.6.6}$$

and define the norm in the space W by

$$\|v\|_W = \|v\|_{L^{\alpha_0}(a,b;X_0)} + \|v'\|_{L^{\alpha_1}(a,b;X_1)}.$$

The space W equipped with this norm is a Banach space.

Theorem 2.29 *(Lions) Under the assumptions (2.6.5, a)–d) and* $1 < \alpha_0,\ \alpha_1 < +\infty,\ a,\ b \in \mathbb{R}^1,\ a < b$, *the imbedding of the space* W *into* $L^{\alpha_0}(a,b;X)$ *is compact.*

Now we come to the *weak formulation of the nonstationary Navier–Stokes problem.* Let $\boldsymbol{u}$, p form a classical solution of problem (2.6.1) - (2.6.4) in the sense of Definition 2.28. Multiplying (2.6.2) by an arbitrary $\boldsymbol{v} \in \underline{\mathcal{V}}$, integrating over Ω and using Green's theorem, we find similarly as in the preceding paragraphs that the identity

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}(\cdot,t),\boldsymbol{v}\right) + \nu((\boldsymbol{u}(\cdot,t),\boldsymbol{v})) + \tag{2.6.7}$$

$$+b(\boldsymbol{u}(\cdot,t),\boldsymbol{u}(\cdot,t),\boldsymbol{v}) = \langle \boldsymbol{f}(\cdot,t),\boldsymbol{v}\rangle\ (:= (\boldsymbol{f}(\cdot,t),\boldsymbol{v})), \quad \forall\, \boldsymbol{v} \in \underline{\mathcal{V}}. \tag{2.6.8}$$

is satisfied for each $t \in (0,T)$. It is obvious that $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ and

$$\left(\frac{\partial \boldsymbol{u}}{\partial t},\boldsymbol{v}\right) = \frac{d}{dt}(\boldsymbol{u},\boldsymbol{v}).$$

(For simplicity we omit the argument t.) This leads us to the definition of a weak solution.

Definition 2.30 *Let* $\nu,\ T > 0$,

$$\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}^\star) \tag{2.6.9}$$

and

$$\boldsymbol{u}_0 \in \mathbf{H} \tag{2.6.10}$$

be given. We say that a function $\boldsymbol{u}$ *is a weak solution of problem (2.6.1) – (2.6.4), if*

$$\boldsymbol{u} \in L^2(0,T,\boldsymbol{V}) \cap L^\infty(0,T;\mathbf{H}), \tag{2.6.11}$$

$$\frac{d}{dt}(\boldsymbol{u},\boldsymbol{v}) + \nu((\boldsymbol{u},\boldsymbol{v})) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \mathbf{v} \in \boldsymbol{V} \tag{2.6.12}$$

and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{2.6.13}$$

Let us remind that we use the notation $\mathbf{H} = \mathbf{L}^2(\Omega)$. Equation (2.6.12) is meant in the sense of scalar distributions on the interval $(0,T)$; i. e., (2.6.12) is equivalent to the condition

$$-\int_0^T (\boldsymbol{u}(t),\boldsymbol{v})\varphi'(t)\,dt + \int_0^T \{\nu((\boldsymbol{u}(t),\boldsymbol{v})) + b(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v})\}\,\varphi(t)\,dt = \tag{2.6.14}$$

$$= \int_0^T \langle \boldsymbol{f}(t),\boldsymbol{v}\rangle\varphi(t)\,dt \quad \forall\,\varphi \in C_0^\infty(0,T).$$

The assumption $\boldsymbol{u} \in L^\infty(0,T;\mathbf{H})$ could seem unnatural, but under the mere assumption $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ condition (2.6.13) we would have difficulties.

The classical solution is obviously a weak one. On the other hand, by Lemma 2.1, a function $p: \ Q \to \mathbb{R}^1$ can be associated with a weak solution $\boldsymbol{u}$, so that $(\boldsymbol{u},p)$ form a solution in the sense of distributions.

Let us define the mappings $\mathcal{A}: \ \boldsymbol{V} \to \boldsymbol{V}^\star, \ \mathcal{B}: \ \boldsymbol{V} \to \boldsymbol{V}^\star$ by

$$\langle \mathcal{A}\boldsymbol{u},\boldsymbol{v}\rangle = ((\boldsymbol{u},\boldsymbol{v})), \quad \boldsymbol{u},\,\boldsymbol{v} \in \boldsymbol{V}, \tag{2.6.15}$$

$$\langle \mathcal{B}\boldsymbol{u},\boldsymbol{v}\rangle = b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}), \quad \boldsymbol{u},\,\boldsymbol{v} \in \boldsymbol{V}. \tag{2.6.16}$$

It is clear that $\mathcal{A}$ is a linear operator.

Lemma 2.31 *Let $\boldsymbol{u} \in L^2(0,T;\mathbf{V})$. Then the functions $\mathcal{A}\,\boldsymbol{u}$ and $\mathcal{B}\,\boldsymbol{u}$ are defined for a. e. $t \in (0,T), \mathcal{A}\,\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}^\star)$ and $\mathcal{B}\,\boldsymbol{u} \in L^1(0,T;\boldsymbol{V}^\star)$.*

Proof 1. For each $\boldsymbol{z} \in \boldsymbol{V}$ the mapping

$$\boldsymbol{v} \in \boldsymbol{V} \to ((\boldsymbol{z},\boldsymbol{v})) \tag{2.6.17}$$

defines a continuous linear functional $\mathcal{A}\,\boldsymbol{z} \in \boldsymbol{V}^\star$ as follows from the inequality $|((\boldsymbol{z},\boldsymbol{v}))| \le |||\boldsymbol{z}|||\,|||\boldsymbol{v}|||$. Hence, we write

$$\langle \mathcal{A}\,\boldsymbol{z},\boldsymbol{v}\rangle = ((\boldsymbol{z},\boldsymbol{v})), \quad \boldsymbol{v} \in \boldsymbol{V}.$$

If $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$, then for a. e. $t \in (0,T)$

$$\langle \mathcal{A}\,\boldsymbol{u}(t),\,\boldsymbol{v}\rangle = ((\boldsymbol{u}(t),\boldsymbol{v})), \quad \boldsymbol{u},\,\boldsymbol{v} \in \boldsymbol{V},$$

and, hence,

$$\|\mathcal{A}\,\boldsymbol{u}(t)\|_{\boldsymbol{V}^\star} \le |||\boldsymbol{u}(t)|||.$$

This implies that

$$\int_0^T \|\mathcal{A}\,\boldsymbol{u}(t)\|^2_{\boldsymbol{V}^\star}\,dt \le \int_0^T |||\boldsymbol{u}(t)|||^2\,dt < +\infty,$$

which means that $\mathcal{A}\,\boldsymbol{u} \in L^2(0,T;\mathbf{V}^\star)$.

2. In virtue of 2.15, the relation

$$\langle \mathcal{B}\, \boldsymbol{z}, \boldsymbol{v}\rangle = b(\boldsymbol{z},\, \boldsymbol{z},\, \boldsymbol{v}), \quad \boldsymbol{z},\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.6.18}$$

defines the mapping $\mathcal{B}:\ \boldsymbol{V} \to \boldsymbol{V}^\star$. We have

$$|\langle \mathcal{B}\, \boldsymbol{z},\, \boldsymbol{v}\rangle| \le c^\star |||\boldsymbol{z}|||^2 |||\boldsymbol{v}|||,$$

so that

$$\|\mathcal{B}\boldsymbol{z}\|_{\boldsymbol{V}^\star} \le c^\star |||\boldsymbol{z}|||^2, \quad \boldsymbol{z} \in \boldsymbol{V}. \tag{2.6.19}$$

Now, if $\boldsymbol{u} \in \mathbf{L}^2(0,T;\boldsymbol{V})$, then

$$\|\mathcal{B}\underline{\boldsymbol{u}}(t)\|_{\boldsymbol{V}^\star} \le c^\star |||\boldsymbol{u}(t)|||^2 \quad \text{for a. e. } t \in (0,T),$$

which yields

$$\int_0^T \|\mathcal{B}\underline{\boldsymbol{u}}(t)\|_{\mathbf{V}^\star}\, dt \le c \int_0^T |||\boldsymbol{u}(t)|||^2\, dt < +\infty.$$

Hence, $\mathcal{B}\mathbf{u} \in \mathbf{L}^1(0,T;\boldsymbol{V}^\star)$. □

Lemma 2.32 *If $\boldsymbol{u} \in L^2(0,T;\mathbf{V})$ satisfies (2.6.12), then $\boldsymbol{u}$ has the derivative $\boldsymbol{u}'(t)$ at a. e. $t \in (0,T)$ and $\boldsymbol{u}' \in L^1(0,T;\boldsymbol{V}^\star)$.*

Proof Let us put

$$\boldsymbol{g} = \boldsymbol{f} - \nu \mathcal{A}\, \boldsymbol{u} - \mathcal{B}\, \boldsymbol{u}. \tag{2.6.20}$$

We see that $\boldsymbol{g} \in L^1(0,T;\boldsymbol{V}^\star)$. Further, we have

$$\frac{d}{dt}\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{g}, \boldsymbol{v}\rangle \quad \forall\, \mathbf{v} \in \boldsymbol{V}. \tag{2.6.21}$$

This implies that $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ has the derivative $\boldsymbol{u}' = \boldsymbol{g} \in L^1(0,T;\boldsymbol{V}^\star)$. □

Corollary 2.33 *If $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ satisfies (2.6.12), then $\boldsymbol{u} \in C([0,T];\boldsymbol{V}^\star)$. Hence, condition (2.6.13) makes sense.*

The above results can still be strengthened, if we take into account the assumption that $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\mathbf{H})$.

Lemma 2.34 *Let $N = 2$. Then*

$$\|v\|_{L^4(\Omega)} \le 2^{\frac{1}{4}} \|v\|^{\frac{1}{2}}_{L^2(\Omega)} \|v\|^{\frac{1}{2}}_{H^1_0(\Omega)} \quad \forall\, v \in H^1_0(\Omega). \tag{2.6.22}$$

Lemma 2.35 *If $N = 2$, then there exists $c_1 > 0$ such that the inequality*

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le c_1 \|\boldsymbol{u}\|^{\frac{1}{2}}\, |||\boldsymbol{u}|||^{\frac{1}{2}}\, |||\boldsymbol{v}|||\, \|\boldsymbol{w}\|^{\frac{1}{2}} |||\boldsymbol{w}|||^{\frac{1}{2}} \tag{2.6.23}$$

holds for any $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbf{H}^1_0(\Omega)$. If $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\mathbf{H})$, then $\mathcal{B}\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}^\star)$ and

$$\|\mathcal{B}\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{V}^\star)} \le c_1 \|\boldsymbol{u}\|_{L^\infty(0,T;\mathbf{H})} \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{V})}. \tag{2.6.24}$$

Theorem 2.36 *Let* $N = 2$. *If* $\boldsymbol{u}$ *is a solution of problem (2.6.11) – (2.6.13) with* $\boldsymbol{f}$ *satisfying (2.6.9), then the derivative* $\boldsymbol{u}'(t)$ *exists for a. e.* $t \in (0,T)$ *and* $\boldsymbol{u}' \in L^2(0,T;\boldsymbol{V}^\star)$.

Proof is an immediate consequence of relations (2.6.20), (2.6.21) which imply that $\boldsymbol{u}' = \boldsymbol{f} - \nu\mathcal{A}\boldsymbol{u} - \mathcal{B}\boldsymbol{u}$ and of Lemmas 2.35 and 2.31. □

Similar results hold also in the case $N = 3$:

Lemma 2.37 *If* $N = 3$, *then*

$$\|v\|_{L^4(\Omega)} \le 2^{\frac{1}{2}}3^{\frac{3}{4}}\|v\|^{\frac{1}{4}}_{L^2(\Omega)}\|v\|^{\frac{3}{4}}_{H^1_0(\Omega)} \quad \forall\, v \in H^1_0(\Omega). \tag{2.6.25}$$

Lemma 2.38 *Let* $N = 3$. *Then there exists* $c_1 > 0$ *such that*

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le c_1\,|||\boldsymbol{v}|||\,\|\boldsymbol{u}\|^{\frac{1}{4}}\,|||\boldsymbol{u}|||^{\frac{3}{4}}\,\|\boldsymbol{w}\|^{\frac{1}{4}}\,|||\boldsymbol{w}|||^{\frac{3}{4}} \quad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbf{H}^1_0(\Omega). \tag{2.6.26}$$

For $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\mathbf{H})$ *we have* $\mathcal{B}\boldsymbol{u} \in L^{4/3}(0,T;\boldsymbol{V}^\star)$.

As a consequence of the above results we obtain

Theorem 2.39 *Let* $N = 3$. *If* $\boldsymbol{u}$ *is a solution of (2.6.11) – (2.6.13) with* $\boldsymbol{f}$ *satisfying (2.6.9), then the derivative* $\boldsymbol{u}'(t)$ *exists for a. e.* $t \in (0,T)$ *and* $\mathbf{u}' \in L^{4/3}(0,T;\boldsymbol{V}^\star)$.

Remark 2.40 The above results, particularly Theorems 2.36 and 2.39, are of fundamental importance in the investigation of the existence of a weak solution of the nonstationary Navier–Stokes problem. Let us put $a = 0,\ b = T,\ X_0 = \boldsymbol{V},\ X = \mathbf{H},\ X_1 = \boldsymbol{V}^\star,\ \alpha_0 = 2$. Further, we set $\alpha_1 = 2$ or $\alpha_1 = \frac{4}{3}$ for $N = 2$ or $N = 3$, respectively. We can see that assumptions (2.6.4), a)–d) are satisfied. In virtue of Theorem 2.29, the *imbedding of the space* $W = W(0,T;\alpha_0,\alpha_1;\boldsymbol{V},\boldsymbol{V}^\star)$, defined by (2.6.6), *into* $L^2(0,T;\mathbf{H})$ *is compact*:

$$W \hookrightarrow\hookrightarrow L^2(0,T;\mathbf{H}). \tag{2.6.27}$$

If $\boldsymbol{u}$ is a weak solution of problem (2.6.1) – (2.6.4), then $\boldsymbol{u} \in W$, as follows from Theorems 2.36 and 2.39.

2.7 Solvability of the nonstationary problem

In this paragraph we shall be concerned with the study of the existence, uniqueness and regularity of a solution of the nonstationary Navier–Stokes problem.

Our attention will be first paid to the existence of a weak solution defined in 2.30. It means we want to find

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\mathbf{H}) \tag{2.7.1}$$

satisfying the conditions

$$\frac{d}{dt}(\boldsymbol{u},\boldsymbol{v}) + \nu((\boldsymbol{u},\boldsymbol{v})) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \mathbf{v} \in \boldsymbol{V}, \tag{2.7.2}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{2.7.3}$$

with given

$$\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}^\star), \tag{2.7.4}$$

$$\boldsymbol{u}_0 \in \mathbf{H} \tag{2.7.5}$$

and ν, $T > 0$. In Section 2.6 we showed that such a solution should be an element of the space $W = W(0,T;\alpha_0,\alpha_1;\boldsymbol{V},\boldsymbol{V}^\star)$, where $\alpha_0 = 2 = \alpha_1$ for $N = 2$ and $\alpha_0 = 2,\ \alpha_1 = \frac{4}{3}$ for $N = 3$, and, thus, $\boldsymbol{u}' \in L^{\alpha_1}(0,T;\boldsymbol{V}^\star)$. Condition (2.7.2) can be written in the form

$$\boldsymbol{u}' + \nu\mathcal{A}\boldsymbol{u} + \mathcal{B}\boldsymbol{u} = \boldsymbol{f}. \tag{2.7.6}$$

We shall now prove the following fundamental existence theorem for the nonstationary Navier–Stokes problem.

Theorem 2.41 *Let ν, $T > 0$ and let conditions (2.7.4) and (2.7.5) be satisfied. Then there exists at least one solution of problem (2.7.1) – (2.7.3).*

Proof will be carried out in several steps with the use of the *semidiscretization in time.* This method (also known as the Rothe method) is extensively studied in the monographs [Rektorys (1982)] and [Kačur (1985)], but the Navier–Stokes problem is not treated there. The semidiscretization in time converts the nonstationary equation into a sequence of stationary problems and allows us to construct a sequence of approximate solutions from which we can extract a subsequence weakly convergent to a weak solution of the original nonstationary problem.

I) The construction of approximate solutions. For any integer $n > 0$ let us set $\tau = \tau_n = T/n$ and consider the partition of the interval $[0,T]$ formed by the points $t_k = k\tau,\ k = 0,\dots,n$. Let us find $\boldsymbol{u}^0, \boldsymbol{u}^1, \dots, \boldsymbol{u}^n$ satisfying the following conditions:

$$\boldsymbol{u}^0 = \boldsymbol{u}_0 \in \mathbf{H}, \tag{2.7.7}$$

$$\begin{aligned} &\text{a) } \boldsymbol{u}^k \in \boldsymbol{V}, \\ &\text{b) } \left(\frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\tau}, \boldsymbol{v}\right) + \nu((\boldsymbol{u}^k,\boldsymbol{v})) + b(\boldsymbol{u}^k,\boldsymbol{u}^k,\boldsymbol{v}) = \\ &\quad = \langle \boldsymbol{f}^k, \boldsymbol{v}\rangle \quad \forall\, \mathbf{v} \in \boldsymbol{V}, \end{aligned} \tag{2.7.8}$$

$$\text{c) } \boldsymbol{f}^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \boldsymbol{f}(t)\,dt \in \boldsymbol{V}^\star, \quad k = 1,\dots,n.$$

Writing (2.7.8), b) in the form

$$\frac{1}{\tau}(\boldsymbol{u}^k, \boldsymbol{v}) + \nu((\boldsymbol{u}^k, \boldsymbol{v})) + b(\boldsymbol{u}^k, \boldsymbol{u}^k, \boldsymbol{v}) \\ = \langle \boldsymbol{f}^k, \boldsymbol{v}\rangle + \frac{1}{\tau}(\boldsymbol{u}^{k-1}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.7.9}$$

we see that we have a modified stationary Navier–Stokes problem. Similarly as in Section 2.3 we can prove that (2.7.9) has at least one solution $\boldsymbol{u}^k \in \boldsymbol{V}$.

Now, using $\boldsymbol{u}^k$, $k = 0,\dots,n$, we construct sequences of functions $\boldsymbol{u}_\tau :$ $[0,T] \to \boldsymbol{V}$ and $\boldsymbol{w}_\tau :\ [0,T] \to \mathbf{H}$ ($\tau = \tau_n,\ n = 1,2,\dots$):

$$\boldsymbol{u}_\tau(0) = \boldsymbol{u}^1,\ \boldsymbol{u}_\tau(t) = \boldsymbol{u}^k \text{ for } t \in (t_{k-1}, t_k],\ k = 1,\dots,n, \tag{2.7.10}$$

and

$$\boldsymbol{w}_\tau \text{ is continuous in } [0,T], \text{ linear on each interval } [t_{k-1}, t_k], \\ k = 1,\dots,n, \text{ and } \boldsymbol{w}_\tau(t_k) = \boldsymbol{u}^k \text{ for } k = 0,\dots,n. \tag{2.7.11}$$

Notice that $\boldsymbol{w}_\tau :\ [\tau, T] \to \boldsymbol{V}$. As $n \to +\infty$, we get the sequences $\{\boldsymbol{u}_\tau\}_{\tau\to 0+}$ and $\{\boldsymbol{w}_\tau\}_{\tau\to 0+}$.

II) A priori estimates.

Lemma 2.42 *Let $\boldsymbol{f}^k$, $k = 1,\dots,n$, be defined by (2.7.8), c). Then*

$$\tau \sum_{k=1}^n \|\boldsymbol{f}^k\|^2_{\boldsymbol{V}^\star} \le \int_0^T \|\boldsymbol{f}(t)\|^2_{\boldsymbol{V}^\star}\,dt.$$

Lemma 2.43 *The following discrete estimates are valid:*

$$\max_{k=0,\dots,n} \|\boldsymbol{u}^k\| \le c_1, \tag{2.7.12}$$

$$\tau \sum_{k=1}^n |||\boldsymbol{u}^k|||^2 \le c_2, \tag{2.7.13}$$

$$\sum_{k=1}^n \|\boldsymbol{u}^k - \boldsymbol{u}^{k-1}\|^2 \le c_3, \tag{2.7.14}$$

$$\tau \sum_{k=1}^n \left\| \frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\tau} \right\|^{\alpha_1}_{\boldsymbol{V}^\star} \le c_4, \tag{2.7.15}$$

where $c_1,\dots,c_4$ are constants independent of τ.

This implies:

Lemma 2.44 *The sequences* $\boldsymbol{u}_\tau$ *and* $\boldsymbol{w}_\tau$ ($\tau = \tau_n = T/n,\ n = 1, 2, \ldots$) *are bounded in the space* $L^\infty(0,T;\mathbf{H})$,

the sequences $\boldsymbol{u}_\tau$ *and* $\tilde{\boldsymbol{w}}_\tau$ *are bounded in* $L^2(0,T;\boldsymbol{V})$,

the sequence $d\boldsymbol{w}_\tau/dt$ *is bounded in* $L^{\alpha_1}(0,T;\mathbf{V}^\star)$ ($\alpha_1 = 2$ *for* $N = 2$ *and* $\alpha_1 = \frac{4}{3}$ *for* $N = 3$).

Further,

$$\boldsymbol{u}_\tau - \boldsymbol{w}_\tau \to 0 \quad \text{in } L^2(0,T;\mathbf{H}) \text{ as } \tau \to 0+ \,. \tag{2.7.16}$$

III) Passage to the limit as $\tau \to 0+$ (i. e., $n \to +\infty$).

It is possible to show that there exist sequences $\boldsymbol{u}_\tau, \boldsymbol{w}_\tau (\tau > 0)$ and a function $\boldsymbol{u}$ such that

$$\boldsymbol{u}_\tau \to \boldsymbol{u} \ \text{ weakly in } \ L^2(0,T;\boldsymbol{V}), \tag{2.7.17}$$

$$\boldsymbol{u}_\tau \to \boldsymbol{u} \ \text{ weak-}\star \text{ in } L^\infty(0,T;\mathbf{H}), \tag{2.7.18}$$

$$\boldsymbol{w}_\tau \to \boldsymbol{u} \ \text{ weak-}\star \text{ in } L^\infty(0,T;\mathbf{H}), \tag{2.7.19}$$

$$\frac{d\boldsymbol{w}_\tau}{dt} \to \frac{d\boldsymbol{u}}{dt} \ \text{ weakly in } L^{\alpha_1}(0,T;\boldsymbol{V}^\star), \tag{2.7.20}$$

$$\boldsymbol{w}_\tau \to \boldsymbol{u} \ \text{ weakly in } L^2(\varepsilon,T;\boldsymbol{V}) \text{ for each } \varepsilon > 0. \tag{2.7.21}$$

Using Theorem 2.29 and (2.7.18), we can show that

$$\boldsymbol{w}_\tau \to \boldsymbol{u} \text{ strongly in } L^2(0,T;\mathbf{H}). \tag{2.7.22}$$

Taking into account (2.7.10) and (2.7.11), we can interptret equation (2.7.8), b) as

$$(\frac{d\boldsymbol{w}_\tau}{dt}, \boldsymbol{v}) + \nu((\boldsymbol{u}_\tau, \boldsymbol{v})) + b(\boldsymbol{u}_\tau, \boldsymbol{u}_\tau, \boldsymbol{v}) = \langle \boldsymbol{f}_\tau, \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{2.7.23}$$

or

$$\frac{d\boldsymbol{w}_\tau}{dt} + \nu\mathcal{A}\boldsymbol{u}_\tau + \mathcal{B}\boldsymbol{u}_\tau = \boldsymbol{f}_\tau, \tag{2.7.24}$$

where

$$\boldsymbol{f}_\tau : \ (0,T] \to \boldsymbol{V}^\star, \tag{2.7.25}$$
$$\boldsymbol{f}_\tau\, |(t_{k-1}, t_k] = \boldsymbol{f}_k, \quad k = 1, \ldots, n.$$

It is possible to show that $\boldsymbol{f}_\tau \to \boldsymbol{f}$ in $L^2(0,T;\boldsymbol{V}^*)$. Using the compactness of the imbedding (2.6.27), we get the strong convergence

$$\boldsymbol{u}_\tau,\ \boldsymbol{w}_\tau \to \boldsymbol{u} \text{ in } L^2(0,T;\mathbf{H}).$$

This and (2.7.17) – (2.7.21) allow us to pass now to the limit in (2.7.24). Using the above results, we obtain the identity

$$\frac{d\boldsymbol{u}}{dt} + \nu\mathcal{A}\boldsymbol{u} + \mathcal{B}\boldsymbol{u} = \boldsymbol{f}. \tag{2.7.26}$$

It means that $\boldsymbol{u}$ satisfies (2.7.1) and (2.7.2).

Finally one shows that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, which concludes the proof of Theorem 2.41.

2.7.1 *Uniqueness*

Further, we shall briefly touch the problem of the *uniqueness of the weak solution* of the nonstationary Navier–Stokes problem. First we shall deal with the case $N = 2$. Then the weak solution $\boldsymbol{u}$ satisfies the conditions $\boldsymbol{u} \in L^2(0, T; \boldsymbol{V})$, $\boldsymbol{u}' \in L^2(0, T; \boldsymbol{V}^\star)$, which implies, as can be shown, that

$$\boldsymbol{u} \in C([0, T]; \mathbf{H}) \tag{2.7.27}$$

and

$$\frac{d}{dt}(\boldsymbol{u}(t), \boldsymbol{u}(t)) = 2\langle \boldsymbol{u}'(t), \boldsymbol{u}(t)\rangle \tag{2.7.28}$$

in the sense of distributions on $(0, T)$. (For the proof, see [Temam (1977), Chap. III, Lemma 1.2] or [Girault–Raviart (1979)].)

Theorem 2.45 *(Lions – Prodi (1959)). Let us assume that $N = 2$ and that conditions (2.7.4), (2.7.5) are satisfied. Then problem (2.7.1) – (2.7.3) has exactly one solution.*

The situation is essentially more complicated in the case of three-dimensional flow. If $N = 3$, then the solution $\boldsymbol{u}$ of problem (2.7.1) – (2.7.3) satisfies $\boldsymbol{u}' \in L^{4/3}(0, T; \boldsymbol{V}^\star)$ (see Theorem 2.39). We find that

$$\boldsymbol{u} \in L^{\frac{8}{3}}(0, T; \mathbf{L}^4(\Omega)). \tag{2.7.29}$$

Actually, by Lemma 2.37,

$$\|\boldsymbol{u}(t)\|_{\mathbf{L}^4(\Omega)} \leq c\|\boldsymbol{u}(t)\|^{\frac{1}{4}} |||\boldsymbol{u}(t)|||^{\frac{3}{4}}, \quad t \in (0, T). \tag{2.7.30}$$

In virtue of (2.7.1), the right-hand side is an element of the space $L^{\frac{8}{3}}(0, T)$ so that the same is true for the left-hand side, which proves (2.7.29).

In the three-dimensional case the attempts to prove the uniqueness of the solution in the class of functions satisfying (2.7.1), in which the existence was established, have not been successful. The uniqueness was obtained in a smaller class:

Theorem 2.46 *If $N = 3$ and (2.7.4) – (2.7.5) hold, then problem (2.7.1) – (2.7.3) has at most one solution satisfying the condition*

$$\boldsymbol{u} \in L^8(0, T; \mathbf{L}^4(\Omega)). \tag{2.7.31}$$

From the above results we conclude that in the case $N = 3$ there is a close relation between the uniqueness and regularity of a weak solution. There are two fundamental problems:

— the uniqueness of weak solutions (i. e., solutions of (2.7.1) – (2.7.3)) whose existence is guaranteed by Theorem 2.41,

— the existence of the *strong solutions* of (2.7.1) – (2.7.3) satisfying condition (2.7.31), for which the uniqueness holds.

UP TO NOW, THE ANSWER TO THESE QUESTIONS REMAINS OPEN. **There exists the Prize of the Clay Institute** – 10^6 **US $ for the solution of this problem**.

3

FINITE ELEMENT SOLUTION OF INCOMPRESSIBLE VISCOUS FLOW

3.1 Continuous problem

3.1.1 *Stokes problem*

Let $\varphi \in \boldsymbol{H}^{1/2}(\partial\Omega)$, $\displaystyle\int_{\partial\Omega} \varphi \cdot \boldsymbol{n} \,\mathrm{d}s = 0$, $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega)$, $\operatorname{div} \boldsymbol{g} = 0$ in Ω, $\boldsymbol{g}|_{\partial\Omega} = \varphi$, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega), \boldsymbol{u}^* \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u}^*|_{\partial\Omega} = \varphi$ and consider the Stokes problem to find $\boldsymbol{u}$ such that

$$\boldsymbol{u} - \boldsymbol{g} \in \boldsymbol{V}, \tag{3.1.1}$$

$$\nu(\!(\boldsymbol{u}, \boldsymbol{v})\!) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \tag{3.1.2}$$

This can be reformulated with the aid of the pressure: Find $\boldsymbol{u}, p$ such that

$$\boldsymbol{u} - \boldsymbol{u}^* \in \boldsymbol{H}_0^1(\Omega), \qquad p \in L_0^2 = \left\{ q \in L^2(\Omega);\, \int_\Omega q \,\mathrm{d}x = 0 \right\}, \tag{3.1.3}$$

$$\nu(\!(\boldsymbol{u}, \boldsymbol{v})\!) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{3.1.4}$$

$$-(q, \operatorname{div} \boldsymbol{u}) = 0 \qquad\qquad \forall\, q \in L_0^2(\Omega). \tag{3.1.5}$$

Exercise 3.1 Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{u}|_{\partial\Omega} = \varphi$, $\displaystyle\int_{\partial\Omega} \varphi \cdot \boldsymbol{n} \,\mathrm{d}S = 0$. Prove that $\operatorname{div} \boldsymbol{u} = 0$ $\Leftrightarrow$

— $(q, \operatorname{div} \boldsymbol{u}) = 0 \quad \forall\, q \in L^2(\Omega) \quad (+) \quad \Leftrightarrow$
— $(q, \operatorname{div} \boldsymbol{u}) = 0 \quad \forall\, q \in L_0^2(\Omega). \quad (*)$

Proof The implication $\Rightarrow$ is obvious.
The implication $\Leftarrow$: Let $(*)$ hold. Then we can write

$$q \in L^2(\Omega) \to \tilde{q} = q - \frac{1}{|\Omega|} \int_\Omega q \,\mathrm{d}x \in L_0^2(\Omega) \Rightarrow \tag{3.1.6}$$

$$\Rightarrow 0 = (\tilde{q}, \operatorname{div} \boldsymbol{u}) = (q, \operatorname{div} \boldsymbol{u}) - \frac{1}{|\Omega|} \int_\Omega q \,\mathrm{d}x (1, \operatorname{div} \boldsymbol{u}) \tag{3.1.7}$$

$$(1, \operatorname{div} \boldsymbol{u}) = \int_\Omega \operatorname{div} \boldsymbol{u} \,\mathrm{d}x = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}S = \int_{\partial\Omega} \varphi \cdot \boldsymbol{n} \,\mathrm{d}S = 0$$

$(+) \;\Rightarrow\; \operatorname{div} \boldsymbol{u} = 0$ — clear. ∎

3.1.2 *Navier-Stokes problem*

Find $\boldsymbol{u}$ such that

$$\boldsymbol{u} - \boldsymbol{g} \in \boldsymbol{V} \tag{3.1.8}$$

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.1.9}$$

The formulation with pressure reads

$$\boldsymbol{u} - \boldsymbol{u}^* \in \boldsymbol{H}_0^1(\Omega), \tag{3.1.10}$$

$$\nu((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, v \in \boldsymbol{H}_0^1(\Omega) \tag{3.1.11}$$

$$-(q, \operatorname{div} \boldsymbol{u}) = 0 \quad \forall\, q \in L_0^2(\Omega). \tag{3.1.12}$$

3.2 Discrete problem

For simplicity we assume that $N = 2$, Ω is a polygonal domain, $\mathcal{T}_h$ is a triangulation of Ω with standard properties. This means that $K \in \mathcal{T}_h$ are closed triangles,

$$\overline{\Omega} = \cup K \in \mathcal{T}_h K \tag{3.2.13}$$

and two elements $K_1, K_2 \in \mathcal{T}_h, K_1 \neq K_2$ are either disjoint or $K_1 \cap K_2$ is formed by a vertex of K_1 and K_2 or a side of K_1 and K_2.

Over $\mathcal{T}_h$ we construct finite dimensional spaces and consider approximations

$$X_h \approx H^1(\Omega), \quad X_{h0} \approx H_0^1(\Omega), \quad \boldsymbol{V}_h \approx \boldsymbol{V}, \tag{3.2.14}$$

$$\boldsymbol{X}_h \approx \boldsymbol{H}^1(\Omega), \quad \boldsymbol{X}_{h0} \approx \boldsymbol{H}_0^1(\Omega) \tag{3.2.15}$$

$$M_h \approx L^2(\Omega), \quad M_{h0} \approx L_0^2(\Omega), \tag{3.2.16}$$

$$\boldsymbol{X}_h, \boldsymbol{V}_h, \ldots \subset \boldsymbol{L}^2(\Omega),\ M_h \subset L^2(\Omega), M_{h0} \subset L_0^2(\Omega), \tag{3.2.17}$$

$$((\cdot,\cdot))_h \approx ((\cdot,\cdot)), \quad |||\cdot|||_h \approx |||\cdot|||, \quad \boldsymbol{g}_h \approx \boldsymbol{g}, \quad \boldsymbol{u}_h^* \approx \boldsymbol{u}^*, \tag{3.2.18}$$

$$\operatorname{div}_h \approx \operatorname{div}, b_h(\cdot,\cdot,\cdot) \approx b(\cdot,\cdot,\cdot) \tag{3.2.19}$$

We assume that $|||\cdot|||_h = ((\cdot,\cdot))^{1/2}$ is a norm in $\boldsymbol{X}_{h0}$, $|||\boldsymbol{v}|||_h = |||\boldsymbol{v}|||$ for $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, $\operatorname{div}_h : \boldsymbol{X}_{h0} \to M_{h0}$ is a linear continuous operator and b_h is a trilinear continuous form on $\boldsymbol{X}_{h0}$ such that $b_h(\boldsymbol{u}, \boldsymbol{v}\boldsymbol{w}) = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ for $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega), \operatorname{div}\boldsymbol{u} = 0, \boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega)$.

3.2.1 *Discrete Stokes problem*

Let us set

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{X}_{h0}; (q_h, \operatorname{div}_h \boldsymbol{v}_h) = 0 \quad \forall\, q_h \in M_{h0}\} \tag{3.2.20}$$

Then we define an approximate velocity $\boldsymbol{u}_h$ as a function satisfying

$$\boldsymbol{u}_h - \boldsymbol{g}_h \in \boldsymbol{V}_h \tag{3.2.21}$$

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{3.2.22}$$

The discrete problem with the pressure reads: Find $\boldsymbol{u}_h, p_h$ such that

$$\boldsymbol{u}_h - \boldsymbol{u}_h^* \in \boldsymbol{X}_{h0}, \quad p_h \in M_{h0} \tag{3.2.23}$$

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p_h, \operatorname{div}_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0} \tag{3.2.24}$$

$$-(q_h, \operatorname{div}_h \boldsymbol{u}_h) = 0 \quad \forall q_h \in M_{h0} \tag{3.2.25}$$

Similarly we can introduce the discrete versions of the Navier-Stokes problems.

3.3 Choice of the finite element spaces

We define $\boldsymbol{X}_h$, M_h as spaces of piecewise polynomial functions. However, they **cannot be chosen in an arbitrary way**.

Example 3.1 Let

$$X_h = \left\{v_h \in C(\overline{\Omega}); v_h|K \in P^1(K) \;\forall K \in \mathcal{T}_h\right\}, \tag{3.3.26}$$

$$X_{h0} = \{v_h \in X_h; v_h|_{\partial\Omega} = 0\} \tag{3.3.27}$$

$$\boldsymbol{X}_h = X_h \times X_h \tag{3.3.28}$$

$$\boldsymbol{X}_{h0} = X_{h0} \times X_{h0} \tag{3.3.29}$$

$$\boldsymbol{V}_h = \{v_h \in \boldsymbol{X}_{h0}; \operatorname{div}(v_h|_K) = 0 \;\forall K \in \mathcal{T}_h\} \tag{3.3.30}$$

This is in agreement with the above definition of $\boldsymbol{V}_h$, provided

$$M_h = \left\{q \in L^2(\Omega); q \text{ is constant on each } K \in \mathcal{T}_h\right\}, \tag{3.3.31}$$

$$M_{h0} = \left\{q \in M_h; \int_\Omega q\,dx = 0\right\}.$$

In this case we have

$$((\cdot,\cdot))_h = ((\cdot,\cdot)), \quad |||\cdot|||_h = |||\cdot||| \tag{3.3.32}$$

$$b_h = b, \quad \operatorname{div}_h = \operatorname{div} \tag{3.3.33}$$

Let $\Omega = (0,1)^2$, $\boldsymbol{u}_h \in \boldsymbol{V}_h$. Then

$$\boldsymbol{u}_h|_{\partial\Omega} = 0, \quad \operatorname{div}(\boldsymbol{v}_h|_K) = 0 \quad \forall K \in \mathcal{T}_h.$$

Let us consider a uniform triangulation of Ω obtained in such a way that Ω is split in n^2 squares with sides of the length $h = 1/n$ and then each square is divided in two triangles by the diagonal going right up. Then any function $\boldsymbol{u}_h \in \boldsymbol{V}_h$ is determined by $2(n-1)^2$ degrees of freedom. On the other hand, the number of conditions $\operatorname{div}(v_h|_K) = 0 \quad \forall K \in \mathcal{T}_h$ is $2n^2$. It is possible to show that in this case $V_h = \{0\}$. Thus, the above choice of the finite element spaces does not make sense.

3.4 Babuška–Brezzi condition

It appears that the spaces $\boldsymbol{X}_{h0}$ and M_{h0} have to satisfy the discrete inf-sup condition, called the *Babuška-Brezzi condition* (BB condition): There exists a constant $\beta > 0$ independent of h such that

$$\inf_{0 \neq q_h \in M_{h0}} \sup_{0 \neq \boldsymbol{v}_h \in \boldsymbol{X}_{h0}} \frac{(q_h, \operatorname{div}_h \boldsymbol{v}_h)}{\|q_h\|\, |||\boldsymbol{v}_h|||_h} \geq \beta. \tag{3.4.34}$$

3.4.1 *Verification of the validity of the BB condition*

Theorem 3.2 *Let there exist an interpolation operator* $I_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{X}_{h0}$ *with the following properties:*

a) I_h *is a linear mapping,*

b) I_h *is uniformly continuous: there exists a constant* $C > 0$ *independent of* h *such that*

$$|||I_h\, \boldsymbol{v}|||_h \leq C|||\boldsymbol{v}||| \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),$$

c) $(q_h, \operatorname{div}_h(\boldsymbol{v} - I_h\, \boldsymbol{v})) = 0 \quad \forall\, q_h \in M_{h0},\ \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$.

Then there exists $\beta > 0$ *independent of* h *such that the BB condition is satisfied.*

Proof We start from the inf-sup condition: there exists $\gamma > 0$ such that

$$\sup_{0 \neq \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{v})}{|||\boldsymbol{v}|||} \geq \gamma \|q\| \quad \forall\, q \in L_0^2(\Omega).$$

Let $q_h \in M_{h0}$. Then $q_h \in L_0^2(\Omega)$. Thus, in view of assumption c), we can write

$$\gamma\|q_h\| \leq \sup_{\partial \neq \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{(q_h, \operatorname{div} \boldsymbol{v})}{|||\boldsymbol{v}|||} = \tag{3.4.35}$$

$$= \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{(q_h, \operatorname{div}_h(I_h \boldsymbol{v}))}{|||\boldsymbol{v}|||} =: (**) \tag{3.4.36}$$

If $I_h \boldsymbol{v} = 0$, then the expression behind sup vanishes and need not be considered. Hence,

$$(**) = \sup_{\substack{0 \neq \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \\ I_h \boldsymbol{v} \neq 0}} \frac{(q_h, \operatorname{div}_h(I_h \boldsymbol{v}))}{|||\boldsymbol{v}|||} \overset{\text{b)}}{\leq} \tag{3.4.37}$$

$$\leq C \sup_{0 \neq v \in \boldsymbol{H}_0^1(\Omega)} \frac{(q_h, \operatorname{div}_h(I_h \boldsymbol{v}))}{|||I_h\, \boldsymbol{v}|||_h} \overset{I_h(\boldsymbol{H}_0^1(\Omega)) \subset \boldsymbol{X}_{H_0}}{\leq} \tag{3.4.38}$$

$$\leq C \sup_{0 \neq \boldsymbol{v}_h \in \boldsymbol{X}_{h0}} \frac{(q_h, \operatorname{div}_h \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_h} \tag{3.4.39}$$

$$\Rightarrow \sup_{0 \neq \boldsymbol{v}_h \in \boldsymbol{X}_{h0}} \frac{(q_h, \operatorname{div}_h \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_h} \geq \beta\|q_h\| \quad \forall\, q_h \in M_{h0} \tag{3.4.40}$$

$$\text{with } \beta = \gamma / C \tag{3.4.41}$$

$$\Rightarrow \text{BB condition.} \tag{3.4.42}$$

□

3.4.2 *Examples*

3.4.2.1 *Conforming finite elements:* We set

$$X_h = \left\{\varphi_h \in C(\overline{\Omega}); \varphi_h|_K \in P^2(K)\ \forall K \in \mathcal{T}_h\right\}\ (\subset H^1(\Omega)) \tag{3.4.43}$$

$$M_h = \left\{q_h \in C(\overline{\Omega}); q_h|_K \in P^1(K)\ \forall K \in \mathcal{T}_h\right\} \tag{3.4.44}$$

We speak about the *Taylor-Hood* P^2/P^1 elements. It is a special case of the conforming P^{k+1}/P^k elements, where $k \geq 1$.

Another possibility is to use the so-called $4P^1/P^1$, i.e. isoP^2/P^1 elements. In this case the pressure is approximated by continuous piecewise linear functions over the mesh $\mathcal{T}_h$, whereas the velocity is approximated by continuous piecewise linear vector functions on the mesh $\mathcal{T}_{h/2}$, obtained by dividing any element $K \in \mathcal{T}_h$ in 4 equal triangles.

In the case of conforming finite elements we have

$$\boldsymbol{X}_h \subset \boldsymbol{H}^1(\Omega), \quad \boldsymbol{X}_{h0} \subset \boldsymbol{H}_0^1(\Omega), \quad M_h \subset L^2(\Omega), \quad M_{h0} \subset L_0^2(\Omega),$$

$$\mathrm{div}_h = \mathrm{div}, ((\cdot,\cdot))_h = ((\cdot,\cdot)), \quad |||\cdot|||_h = |||\cdot|||, \quad b_h = b,$$

$\boldsymbol{u}_h^* = I_h \boldsymbol{u}^* =$ Lagrange interpolation, $\boldsymbol{V}_h$ is defined by (3.2.20).

3.4.2.2 *Nonconforming finite elements:* We speak about nonconforming finite elements, if X_h is not a subspace of $H^1(\Omega)$. A typical example of nonconforming finite elements for the solution of viscous incompressible flow are the Crouzeix-Raviart elements:

$$\begin{aligned} X_h = \quad & \{\boldsymbol{v}_h \in L^2(\Omega); \boldsymbol{v}_h|_K \in P^1(K) \forall K \in \mathcal{T}_h, \\ & \boldsymbol{v}_h \text{ is continuous at all midpoints of sides of all } K \in \mathcal{T}_h\}, \\ M_h = & \{q_h \in L^2(\Omega); q_h \text{ is constant on each } K \in \mathcal{T}_h\}. \end{aligned}$$

In this case we define

$$\mathrm{div}_h \boldsymbol{u}_h \in L^2(\Omega) : \ (\mathrm{div}_h \boldsymbol{u}_h)|_K = \mathrm{div}(\boldsymbol{u}_h|_K), \quad K \in \mathcal{T}_h, \tag{3.4.45}$$

$$|||\boldsymbol{u}_h|||_h^2 = \sum_{K \in \mathcal{T}_h} \sum_{i,j=1}^{2} \int_K \left(\frac{\partial u_{hi}}{\partial x_j}\right)^2 dx, \tag{3.4.46}$$

$$b_h(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h) = \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{i,j=1}^{2} \int_K \left(u_{hj} \frac{\partial v_{hi}}{\partial x_j} w_{hi} - u_{hj} \frac{\partial w_{hi}}{\partial x_j} v_{hi}\right) dx \tag{3.4.47}$$

3.5 Existence of an approximate solution

Now we shall be concerned with the existence of a solution to the discrete Stokes problem. For simplicity we consider the case with zero boundary condition.

Theorem 3.3 *Let us assume that*
1. $((\boldsymbol{v}_h, \boldsymbol{v}_h))_h^{1/2} = |||\boldsymbol{v}_h|||_h$ *is a norm in* $\boldsymbol{X}_{h0}$,

2. BB condition holds:

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{X}_{h0}} \frac{(q_h, \operatorname{div}_h \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_h} \geq \gamma \|q_h\| \quad \forall\, q_h \in M_{h0}$$

Then the discrete Stokes problem has a unique solution $\boldsymbol{u}_h, p_h$.

Proof We want to find $\boldsymbol{u}_h \in \boldsymbol{X}_{h0}$, $p_h \in M_{h0}$ satisfying

$(+)\quad \nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p_h, \operatorname{div}_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0}$

$(*)\quad -q(h, \operatorname{div}_h \boldsymbol{u}_h) = 0 \quad \forall\, q_h \in M_{h0}.$

We define

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \boldsymbol{X}_{h0}; (q_h, \operatorname{div}_h \boldsymbol{v}_h) = 0\ \forall\, q_h \in M_{h0}\}$$

a) Obviously, $(*) \Rightarrow \boldsymbol{u}_h \in \boldsymbol{V}_h$
and
$(+) \Rightarrow \nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h = (\boldsymbol{f}, \boldsymbol{v}_h)\ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h$

Using the Lax-Milgram lemma, we see that there exists a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$ of this problem.

b) Now we shall prove the existence of the pressure p_h. Let us set
$F(\boldsymbol{v}_h) = -\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h + (\boldsymbol{f}, \boldsymbol{v}_h), \quad \boldsymbol{v}_h \in \boldsymbol{X}_{h0}.$
Thus, $F \in (\boldsymbol{X}_{h0})^*$.

By the Riesz theorem, there exists exactly one $\tilde{F} \in \boldsymbol{X}_{h0}$ such that
$((\tilde{F}, \boldsymbol{v}_h))_h = F(\boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0}.$

We see that $F(\boldsymbol{v}_h) = 0 \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h \;\Leftrightarrow\; ((\tilde{F}, \boldsymbol{v}_h))_h = 0 \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h$.
Hence, $\tilde{F} \in \boldsymbol{V}_h^{\perp}$.

Let $q_h \in M_{h0}$. Then the mapping

$$\text{“}\boldsymbol{v}_h \in \boldsymbol{X}_{h0} \to (q_h, \operatorname{div}_h \boldsymbol{v}_h) \in \mathbb{R}\text{”}$$

is a continuous linear functional on $\boldsymbol{X}_{h0}$ and there exists $\tilde{B}q_h \in \boldsymbol{X}_{h0}$ such that

$$((\tilde{B}q_h, \boldsymbol{v}_h))_h = (q_h, \operatorname{div}_h \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0}. \tag{3.5.48}$$

By the BB condition,

$$\sup_{0 \neq \boldsymbol{v}_h \in \boldsymbol{X}_{h0}} \frac{((\tilde{B}q_h, \boldsymbol{v}_h))_h}{|||\boldsymbol{v}_h|||_h} \geq \gamma \|q_h\| \quad \forall\, q_h \in M_{h0}.$$

We see that

$$\tilde{B}: \ M_{h0} \to \boldsymbol{X}_{h0} \quad \text{is a linear operator and}$$
$$|||\tilde{B}q_h|||_h \geq \gamma \|q_h\| \quad \forall\, q_h \in M_{h0}.$$

This implies that $\tilde{B}$ is a one-to-one mapping of M_{h0} onto the range of $\tilde{B}$, $\mathcal{R}(\tilde{B}) \subset \boldsymbol{X}_{h0}$. It is possible to show that $\mathcal{R}(\tilde{B}) = \boldsymbol{V}_h^{\perp}$, which follows from (3.5.48). Hence, for $\tilde{F}$ there exists a unique $p_h \in M_{h0}$ such that $\tilde{B}p_h = \tilde{F}$. This is equivalent to the relation

$$((\tilde{B}p_h, \boldsymbol{v}_h))_h = ((\tilde{F}, \boldsymbol{v}_h))_h \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0},$$

which means that

$$(p_h, \operatorname{div}_h \boldsymbol{v}_h) = -F(\boldsymbol{v}_h) = +\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0},$$

and, thus,

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p_h, \operatorname{div}_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0},$$

which we wanted to prove. □

3.6 Error estimates

3.6.1 *Abstract error estimate for the velocity*

Let us again consider zero boundary conditions. We shall be first concerned with the estimate of the error $|||\boldsymbol{u} - \boldsymbol{u}_h|||_h$.

We have

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{3.6.49}$$

$$\nu((\boldsymbol{u}, \boldsymbol{v}_h))_h = (\boldsymbol{f}, \boldsymbol{v}_h) + \ell_h(\boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.6.50}$$

The second identity was obtained by substituting the exact solution $\boldsymbol{u}$ in the discrete problem. The functional ℓ_h represents here the fact that $\boldsymbol{V}_h$ is not in general a subspace of $\boldsymbol{V}$ from the continuous problem. Then we get

$$\nu((\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h))_h = \ell_h(\boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

For any $\varphi_h \in \boldsymbol{V}_h$ we have

$$\begin{aligned}
&|||\boldsymbol{u} - \boldsymbol{u}_h|||_h \le |||\boldsymbol{u} - \varphi_h|||_h + |||\boldsymbol{u}_h - \varphi_h|||_h,\\
&\nu((\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \varphi_h))_h = \ell_h(\boldsymbol{u}_h - \varphi_h),\\
&\nu|||\boldsymbol{u}_h - \varphi_h|||^2 = \nu((\boldsymbol{u}_h - \varphi_h, \boldsymbol{u}_h - \varphi_h))_h\\
&= \nu((\boldsymbol{u} - \varphi_h, \boldsymbol{u}_h - \varphi_h))_h + \underbrace{\nu((\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}_h - \varphi_h))_h}_{=-\ell_h(\boldsymbol{u}_h - \varphi_h)}\\
&\le \nu|||\boldsymbol{u} - \varphi_h|||_h\ |||\boldsymbol{u}_h - \varphi_h|||_h + \|\ell_h\|^*\ |||\boldsymbol{u}_h - \varphi_h|||_h.
\end{aligned}$$

This implies that

$$|||\boldsymbol{u}_h - \varphi_h|||_h \le |||\boldsymbol{u} - \varphi_h|||_h + \frac{1}{\nu}\|\ell_h\|^*,$$

where

$$\|\ell_h\|^* = \sup_{\boldsymbol{v}_h \in\, \boldsymbol{X}_{h0}} \frac{|\ell_h(\ \boldsymbol{v}_h)|}{|||\ \boldsymbol{v}_h|||_h}$$

and, hence,

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||_h \le 2|||\boldsymbol{u}-\varphi_h|||_h + \frac{1}{\nu}\|\ell_h\|^* \quad \forall \varphi_h \in \boldsymbol{V}_h,$$

which is the *abstract error estimate of the velocity.* The first term represents the error of the approximation of functions from the space $\boldsymbol{V}$ by functions from $\boldsymbol{V}_h$. The expression $\|\ell_h\|^*$ represents the error caused by the fact that $\boldsymbol{V}_h$ is not a subspace of $\boldsymbol{V}$.

3.6.2 *Abstract error estimate for the pressure*

Theorem 3.4 *Let the functions $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, $p \in L^2(\Omega)$, $\boldsymbol{u}_h \in \boldsymbol{X}_{h0}$ and $p_h \in M_{h0}$ satisfy the identities*

$$\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p_h, \operatorname{div}_h(\boldsymbol{v}_h)) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0} \tag{3.6.51}$$

$$\nu((\boldsymbol{u}, \boldsymbol{v}_h))_h - (p, \operatorname{div}_h(\boldsymbol{v}_h)) = (\boldsymbol{f}, \boldsymbol{v}_h) + \ell_h^*(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_{h0}, \tag{3.6.52}$$

and let the BB condition be satisfied. Then

$$\|p-p_h\|_{L^2(\Omega)} \le c\left(|||\boldsymbol{u}-\boldsymbol{u}_h|||_h + \inf_{q_h\in M_{h0}} \|p-q_h\|_{L^2} + \sup_{\boldsymbol{v}_h\in \boldsymbol{X}_{h0}} \frac{|\ell_h^*(\boldsymbol{v}_h)|}{|||\boldsymbol{v}_h|||_h}\right). \tag{3.6.53}$$

Proof Subtracting equations (3.6.51) and (3.6.52), we get

$$\nu((\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p-p_h, \operatorname{div}_h(\boldsymbol{v}_h)) = \ell_h^*(\boldsymbol{v}_h).$$

Adding and subtracting a function $q_h \in M_{h0}$ in the second term yields

$$(q_h-p_h, \operatorname{div}_h(\boldsymbol{v}_h)) = \nu((\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h))_h - (p-q_h, \operatorname{div}_h(\boldsymbol{v}_h)) - \ell_h^*(\boldsymbol{v}_h). \tag{3.6.54}$$

By the (BB) condition,

$$\beta\|q_h-p_h\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}_h\in \boldsymbol{X}_{h0}} \frac{(q_h-p_h, \operatorname{div}_h(\boldsymbol{v}_h))}{|||\boldsymbol{v}_h|||_h}.$$

Using (3.6.54) and the Cauchyho inequality, we get

$$\beta\|q_h-p_h\|_{L^2(\Omega)} \le \nu|||\boldsymbol{u}-\boldsymbol{u}_h|||_h + c\|p-q_h\|_{L^2(\Omega)} + \sup_{\boldsymbol{v}_h\in \boldsymbol{X}_{h0}} \frac{\ell_h^*(\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_h}.$$

The triangle inequality implies that

$$\begin{aligned}
&\|p-p_h\|_{L^2(\Omega)} \\
&\le \|p-q_h\|_{L^2(\Omega)} + \|q_h-p_h\|_{L^2(\Omega)} \\
&\le \frac{\nu}{\beta}|||\boldsymbol{u}-\boldsymbol{u}_h|||_h + \left(1+\frac{c}{\beta}\right)\|p-q_h\|_{L^2(\Omega)} + \frac{1}{\beta}\sup_{\boldsymbol{v}_h\in \boldsymbol{X}_{h0}} \frac{\ell_h^*(\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_h}.
\end{aligned}$$

This inequality holds for all $q_h \in M_{h0}$. Passing to infimum, we obtain estimate (3.6.53), which we wanted to prove. □

3.7 Numerical realization of the discrete problem

Let us consider the discrete Stokes problem for the velocity and pressure. Since $\boldsymbol{X}_{h0}$ and M_{h0} are finite dimensional spaces, we can choose bases in these spaces.

In the space $\boldsymbol{X}_{h0}$ we proceed in such a way that we construct a simple basis

$$w_k, \quad k = 1, \dots, m, \quad \text{in } X_{h0}$$

and then use the system $(w_k, 0)$, $(0, w_k)$, $\quad k = 1, \dots, m$, as a basis in $\boldsymbol{X}_{h0}$. We denote these basis functions by $\boldsymbol{w}_i^*, \quad i = 1, \dots, \ell = 2m$.

In the space M_{h0} we shall consider a basis formed by functions $q_j^*,\ j = 1, \dots, n$.

Then we can express the approximate solution in the form

$$\boldsymbol{u}_h = \boldsymbol{u}_h^* + \sum_{j=1}^{\ell} U_j\, \boldsymbol{w}_j^*, \tag{3.7.55}$$

$$p_h = \sum_{j=1}^{n} P_j\, q_j^*, \tag{3.7.56}$$

where $\boldsymbol{u}_h^* \in \boldsymbol{X}_h$ is a function satisfying (approximately) Dirichlet boundary conditions. Substituting this representation to the discrete Stokes problem

$$\nu((\boldsymbol{u}_h, p_h))_h - (p_h, \operatorname{div}_h v_h) = (\boldsymbol{f}, v_h) \quad \forall\, v_h \in \boldsymbol{X}_{h0} \tag{3.7.57}$$

$$-(q_h, \operatorname{div}_h \boldsymbol{u}_h) = 0 \quad \forall\, q_h \in M_{h0}, \tag{3.7.58}$$

and using test functions $v_h := \boldsymbol{w}_i^*, \quad i = 1, \dots, \ell, \quad q_h := q_i^*, \quad i = 1, \dots, n$, we get

$$\nu \sum_{j=1}^{\ell} U_j \underbrace{((\boldsymbol{w}_j^*, \boldsymbol{w}_i^*))_h}_{a_{ij}} - \sum_{j=1}^{m} P_j \underbrace{(q_j^*, \operatorname{div}_h \boldsymbol{w}_i^*)}_{-b_{ij}} = \underbrace{(\boldsymbol{f}, w_i^*) - \nu((u_h^*, \boldsymbol{w}_i^*))_h}_{F_i} \tag{3.7.59}$$

$$i = 1, \dots, \ell,$$

$$-\sum_{j=1}^{\ell} U_j \underbrace{(q_i^*, \operatorname{div}_h \boldsymbol{w}_j^*)}_{-b_{ji}} = \underbrace{(q_i^*, \operatorname{div}_h \boldsymbol{u}_h^*)}_{G_i} \quad i = 1, \dots, n.$$

These equations can be rewritten in the matrix form. We set

$$U = (U_1, \dots, U_\ell)^T, \quad P = (P_1, \dots, P_n)^T,$$
$$\mathbb{A} = (a_{ij})_{i,j=1}^{\ell}, \quad \mathbb{B} = (b_{ij})_{\substack{i=1,\dots,\ell \\ j=1,\dots,n}},$$
$$F = (F_1, \dots, F_\ell)^T, \quad G = (G_1, \dots, G_n)^T,$$

and have the system

$$\begin{pmatrix} \mathbb{A}, & \mathbb{B} \\ \mathbb{B}^T, & \emptyset \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}.$$

Here $\mathbb{A}$ is a symmetric positive definite matrix and

$$\mathcal{A} = \begin{pmatrix} \mathbb{A}, & \mathbb{B} \\ \mathbb{B}^T, & \emptyset \end{pmatrix}$$

is a symmetric, nonsingular matrix. The above system is a *saddle-point system.*

3.7.1 *Solution of the saddle point system*

There are various possibilities how to solve the algebraic system equivalent to the discrete Stokes problem:

1. Direct method: multifrontal elimination - possible to use the package UMF-PACK)
2. Iterative solvers:
   a) CG with preconditioning.
   b) Uzawa algorithm:
   Choose P^0; then compute

$$\begin{aligned} \mathbb{A}\, U^{k+1} &= G - \mathbb{B}\, P^k && (3.7.60) \\ P^{k+1} &= P^k + \rho(F - \mathbb{B}^T U^{k+1}), \quad k \geq 0, \text{ with } 0 < \rho < 2\nu. \end{aligned}$$

3.8 Discrete Navier-Stokes problem

We want to find $\boldsymbol{u}_h, p_h$ such that

$$\begin{aligned} &\boldsymbol{u}_h = \boldsymbol{u}_h^* + \boldsymbol{z}_h, \quad \boldsymbol{z}_h \in \boldsymbol{X}_{h0},\ p_h \in M_{h0}, && (3.8.61) \\ &\nu((\boldsymbol{u}_h, \boldsymbol{v}_h))_h + b_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - (p_h, \operatorname{div}_h \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) && (3.8.62) \\ &\qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h_0}, \\ &-(q_h, \operatorname{div}_h \boldsymbol{u}_h) = 0 \quad \forall\, q_h \in M_{h0} && (3.8.63) \end{aligned}$$

3.8.1 *Iterative processes*

3.8.1.1 *Stokes iterations*

(i) Choose $\boldsymbol{u}_h^0$, p_h^0.
(ii) Find $\boldsymbol{u}_h^{k+1} = \boldsymbol{u}_h^* + \boldsymbol{z}_h^{k+1}$, $\boldsymbol{z}_h^{k+1} \in \boldsymbol{X}_{h0}$, $p_h^{k+1} \in M_{h0}$:

$$\begin{aligned} &\nu((\boldsymbol{u}_h^{k+1}, \boldsymbol{v}_h))_h + b_h(\boldsymbol{u}_h^k, \boldsymbol{u}_h^k, \boldsymbol{v}_h) - (p_h^{k+1}, \operatorname{div}_h \boldsymbol{v}_h) = (\boldsymbol{f}, v_h) && (3.8.64) \\ &\qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0} \\ &-(q_h, \operatorname{div}_h \boldsymbol{u}_h^{k+1}) = 0 \qquad \forall\, q_h \in M_{h0}, && (3.8.65) \\ &\qquad k = 0, 1, \ldots\ . \end{aligned}$$

3.8.1.2 *Oseen iterations* We use the iterative process given by the formula

$$\nu((\boldsymbol{u}_h^{k+1}, \boldsymbol{v}_h))_h + b_h(\boldsymbol{u}_h^k, \boldsymbol{u}_h^{k+1}, \boldsymbol{v}_h) - (p_h^{k+1}, \mathrm{div}_h\, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad (3.8.66)$$
$$\forall\, \boldsymbol{v}_h \in \boldsymbol{X}_{h0},$$
$$-(q_h, \mathrm{div}_h\, \boldsymbol{u}_h^{k+1}) = 0 \quad \forall\, q_h \in M_{h0}.$$

This process requires the solution of a nonsymmetric system in each iteration:

$$\begin{pmatrix} \mathbb{A} + I\!N(\boldsymbol{u}_h^k), & \mathbb{B} \\ \mathbb{B}^T, & \emptyset \end{pmatrix} \begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix} = \begin{pmatrix} F(\boldsymbol{u}_h^k) \\ G \end{pmatrix}.$$

The matrix $I\!N$ is nonsymmetric. This system can be solved by UMFPACK or iteratively with the aid of *Krylov subspace methods*, e.g. GMRES, BiCGStab, ORTODIR, ORTOMIN,...

3.8.1.3 *The Newton method* The nonlinear discrete Navier-Stokes problem can be written in the form

$$\phi \begin{pmatrix} U \\ P \end{pmatrix} := \begin{pmatrix} \mathbb{A} + I\!N(U), & \mathbb{B} \\ \mathbb{B}^T & \emptyset \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} - \begin{pmatrix} F \\ G \end{pmatrix} = 0.$$

(For simplicity we consider zero boundary conditions for the velocity.) The Newton method allows us to compute the vector $\begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix}$ provided the vector $\begin{pmatrix} U^k \\ P^k \end{pmatrix}$ is known:

$$\frac{D\phi \begin{pmatrix} U^k \\ P^k \end{pmatrix}}{D \begin{pmatrix} U \\ P \end{pmatrix}} \left(\begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix} - \begin{pmatrix} U^k \\ P^k \end{pmatrix} \right) = -\phi \begin{pmatrix} U^k \\ P^k \end{pmatrix},$$

where

$$\frac{D\phi \begin{pmatrix} U \\ P \end{pmatrix}}{D \begin{pmatrix} U \\ P \end{pmatrix}} = \begin{pmatrix} \mathbb{A} + \frac{D(I\!N(U)\,U)}{DU}, & \mathbb{B} \\ \mathbb{B}^T, & \emptyset \end{pmatrix},$$

$$\frac{D(I\!N(U)\,U)}{DU} = (\beta_{ij}(U)) \ \text{ nonsymmetric matrix},$$

$$\beta_{ij}(U) = b_h(\boldsymbol{w}_j^*, \boldsymbol{u}_h, \boldsymbol{w}_i^*) + b_h(\boldsymbol{u}_h, \boldsymbol{w}_j^*, \boldsymbol{w}_i^*)$$

All these methods converge only provided $\nu >> 1$, i.e. the Reynolds number $Re = \frac{UL}{\nu}$ is sufficiently small. (Here U and L denote the characteristic velocity and the characteristic length, respectively.) This is caused by the fact that stationary solutions are unstable for large Reynolds numbers Re. In this case, it is necessary to use the *nonstationary Navier-Stokes problem*.

3.8.2 *Discretization of the nonstationary Navier-Stokes problem*

We proceed in such a way that we construct a partition of the time interval $(0, T)$: $0 = t_0 < t_1 < \ldots,\ \tau_k = t_{k+1} - t_k$, and the exact solution $\boldsymbol{u}(t_k), p(t_k)$ is approximated by the approximate solution $\boldsymbol{u}^k, p^k$, obtained by the semidiscretization in time:

$$\operatorname{div} \boldsymbol{u}^{k+1} = 0, \tag{3.8.67}$$

$$\frac{\boldsymbol{u}^{k+1} - \boldsymbol{u}^k}{\tau_k} + (\boldsymbol{u}^{k+1} \cdot \nabla)\, \boldsymbol{u}^{k+1} - \nu \nabla \boldsymbol{u}^{k+1} = \boldsymbol{f}^{k+1}, \tag{3.8.68}$$

& with a given initial condition $\boldsymbol{u}^0$ and prescribed boundary conditions.

This fully implicit nonlinear Navier-Stokes problem is linearized with the aid of the *Oseen method*. In this way we get the modified Oseen problem on each time level: Find $\boldsymbol{u}_h^{k+1}, p_h^{k+1}$ such that

$$\operatorname{div} \boldsymbol{u}^{k+1} = 0, \tag{3.8.69}$$

$$\frac{\boldsymbol{u}^{k+1} - \boldsymbol{u}^k}{\tau_k} + (\boldsymbol{u}^k \cdot \nabla)\, \boldsymbol{u}^{k+1} - \nu \nabla \boldsymbol{u}^{k+1} + \nabla p^{k+1} = f^{k+1}$$

& with a given initial condition $\boldsymbol{u}^0$ and prescribed boundary conditions.

Finally, problem (3.8.69) is discretized by the finite element method and converted to the solution of a linear algebraic system on each time level.

Remark 3.5 In the case of large Reynolds numbers, the Navier-Stokes problem is singularly perturbed with dominating convection. Then the described numerical methods produce approximate solutions with nonphysical spurious oscillations. In order to avoid this effect, called *Gibbs phenomenon*, it is necessary to apply a suitable stabilization technique, as, e.g. *streamline diffusion method* also called streamline upwind Petrov-Galerkin (SUPG) method. We refer the reader, for example, to the works (Franca *et al.*, 1986), (Lube and Weiss, 1995).

4

COMPRESSIBLE FLOW

4.1 Results for the full system of compressible Navier–Stokes equations

The full system of compressible Navier–Stokes equations for a heat-conductive gas in a bounded domain Ω with Dirichlet boundary conditions and initial data reads

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho\boldsymbol{v}) = 0, \quad x \in \Omega \subset I\!R^N,\ t \in (0,T)\ (N = 1,2,3),\\
&(\rho\boldsymbol{v})_t + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}(\mu\nabla\boldsymbol{v}) - \nabla\big((\mu+\lambda)\operatorname{div}\boldsymbol{v}\big) + \nabla p(\rho,\theta) = \rho\boldsymbol{f}(x,t),\\
&c_v\rho(\theta_t + \boldsymbol{v}\cdot\nabla\theta) - k\Delta\theta + p(\rho,\theta)\operatorname{div}\boldsymbol{v} - 2\mu(\mathbb{D}(\boldsymbol{v})\cdot\mathbb{D}(\boldsymbol{v})) - \lambda(\operatorname{div}\boldsymbol{v})^2 = 0,\\
&\mathbb{D}(\boldsymbol{v}) = \big(d_{ij}(\boldsymbol{v})\big)_{i,j=1}^N,\ d_{ij}(\boldsymbol{v}) := \frac{1}{2}\big(\partial_i v_j + \partial_j v_i\big), \qquad (4.1.1)\\
&p(\rho,\theta) = \theta q(\rho),\ q = q(\rho) > 0\ (\rho > 0, \theta > 0),\\
&(\rho,\boldsymbol{v},\theta)(x,0) = (\rho^0,\boldsymbol{v}^0,\theta^0)(x),\ x \in \Omega,\ \inf_x\{\rho^0,\theta^0\} > 0,\\
&\boldsymbol{v}(x,t) = 0,\ \theta(x,t) = \overline{\theta}(x,t),\ x \in \partial\Omega,\ t \geq 0.
\end{aligned}$$

We make the following assumptions.

(i) $3\lambda + 2\mu \geq 0, \mu > 0, k > 0$ and λ are constants;

(ii) $e = c_v\theta$, where $c_v = \text{const} > 0$ is the specific heat at constant volume;

(iii) $q'(\rho) > 0$ for $\rho > 0$.

Let us now formulate the following global existence result.

Theorem 4.1 *Let $\Omega \subset I\!R^3$ be a bounded domain of class $C^{2,\alpha}$ $(\alpha \in (0,1])$ and let assumptions (i),(ii),(iii) hold. Let $\boldsymbol{f} = \nabla F$, $F \in H^4(\Omega)$, $(\rho^0,\boldsymbol{v}^0,\theta^0) \in H^3(\Omega)^5$, $\boldsymbol{v}^0|_{\partial\Omega} = 0$, $\theta^0|_{\partial\Omega} = \overline{\theta}$, $\boldsymbol{v}_t(0)|_{\partial\Omega} = 0$, $\theta_t(0)|_{\partial\Omega} = 0$, where $\boldsymbol{v}_t(x,0)$ and $\theta_t(x,0)$ are computed from the partial differential equations in* (4.1.1)*, in which we put $\rho := \rho^0, \theta := \theta^0, \boldsymbol{v} := \boldsymbol{v}^0$. In addition, we assume that there is a rest state $(\widehat{\rho}, 0, \overline{\theta})$ with $\overline{\theta} =$ const > 0.*

Then there exist positive constants ε_0, β and $C_0 = C_0(\overline{\rho}, \overline{\theta}, \|F\|_{H^4(\Omega)})$ (where $\overline{\rho} := \frac{1}{|\Omega|}\int_\Omega \rho^0\,dx$) such that if

$$\|(\rho^0 - \widehat{\rho}, \boldsymbol{v}^0, \theta^0 - \overline{\theta})\|_{H^3(\Omega)^5} \leq \varepsilon_0, \qquad (4.1.2)$$

then the initial-boundary value problem (4.1.1) *has a unique solution* $(\rho,\boldsymbol{v},\theta)$ *global in time satisfying*

$$(\rho,\boldsymbol{v},\theta) \in \big(C^0 \cap L^\infty\big)([0,\infty); H^3(\Omega))^5 \cap C^1(0,\infty; H^2(\Omega)\times H^1(\Omega)^4),$$

$$\inf_{x,t} \rho(x,t) > 0, \inf_{x,t} \theta(x,t) > 0,$$

and

$$\sup_{x\in\Omega} |(\rho(x,t) - \widehat{\rho}(x), \boldsymbol{v}(x,t), \theta(x,t) - \overline{\theta})| \leq C_0 \exp(-\beta t).$$

The proof of Theorem 4.1 can be found in (Matsumura and Padula, 1992).

The solution from Theorem 4.1 is a strong solution with smooth first order partial derivatives because of the Sobolev imbedding $H^3(\Omega) \hookrightarrow C^1(\Omega)$.

Note that the existence of a regular solution on some maximal time interval $(0, T_{\max})$ can be proven even without the assumption that ε_0 in (4.1.2) is small enough. However, we do not a priori know how large or small $T_{\max}$ can be and the lower estimate of $T_{\max}$ in terms of the data is very pesimistic.

4.2 Results for equations of barotropic flow

Since no results on the global solvability of problem (4.1.1) with completely large data have been obtained up to now, we restrict ourselves to equations of barotropic flow.

$$\begin{aligned} &(\rho\boldsymbol{v})_t + \operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \mu\Delta\boldsymbol{v} - (\lambda+\mu)\nabla\operatorname{div}\boldsymbol{v} + \nabla p(\rho) = \rho\boldsymbol{f}, \\ &\rho_t + \operatorname{div}(\rho\boldsymbol{v}) = 0, \quad x\in\Omega\subset I\!R^N,\ t\in(0,T)\ (T>0). \end{aligned} \tag{4.2.3}$$

We impose the initial conditions

$$\rho|_{t=0} = \rho^0, \quad (\rho\boldsymbol{v})|_{t=0} = \boldsymbol{m}^0, \tag{4.2.4}$$

where

$$\begin{aligned} &\rho^0 \geq 0 \text{ a.e. in } \Omega, \quad \rho^0 \in L^\infty(\Omega), \\ &\frac{|\boldsymbol{m}^0|^2}{\rho^0} \in L^1(\Omega) \text{ (we set } |\boldsymbol{m}^0(x)|^2/\rho^0(x) = 0, \text{ if } \rho^0(x) = 0), \end{aligned}$$

and the Dirichlet boundary condition

$$\boldsymbol{v}(x,t) = 0, \quad x\in\partial\Omega, \quad t>0. \tag{4.2.5}$$

The unknown functions are $\boldsymbol{v} = \boldsymbol{v}(x,t) = (v_1(x,t), \ldots, v_N(x,t))^{\mathrm{T}}$ (or $\boldsymbol{m} = \rho\boldsymbol{v}$) and $\rho = \rho(x,t)$.

For large data the existence of a global smooth solution is not known. Nevertheless, when we restrict ourselves to weak solutions, recent developments yield important global existence results.

Definition 4.2 *By a weak solution of* (4.2.3)–(4.2.5) *we call a couple* $(\boldsymbol{v},\rho)$ *such that*

$$\rho,\ p(\rho),\ \rho|\boldsymbol{v}|^2,\ |\nabla\boldsymbol{v}|\ \in\ L^1_{\mathrm{loc}}(\Omega\times(0,\infty))$$

and, putting $Q_T=\Omega\times(0,T)$, *for any* $T>0$ *and any* $\boldsymbol{\varphi}\in C^1(0,T;C_0^\infty(\Omega))^N$, $\psi\in C^1(0,T;C^\infty(\Omega))$ *such that* $\boldsymbol{\varphi}(x,T)\equiv 0,\ \psi(x,T)\equiv 0$, *the following integral identities hold:*

$$\int_{Q_T}\Big(\rho\boldsymbol{v}\cdot\boldsymbol{\varphi}_t+\rho\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{\varphi}\cdot\boldsymbol{v}\big)-\mu\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}-(\lambda+\mu)\mathrm{div}\,\boldsymbol{v}\,\mathrm{div}\,\boldsymbol{\varphi}$$

$$+\,p(\rho)\mathrm{div}\,\boldsymbol{\varphi}+\rho\boldsymbol{f}\cdot\boldsymbol{\varphi}\Big)(x,t)\,dx\,dt+\int_\Omega\rho^0(x)\boldsymbol{v}^0(x)\boldsymbol{\varphi}(x,0)\,dx=0,\quad(4.2.6)$$

$$\int_{Q_T}\big(\rho\psi_t+\rho(\boldsymbol{v}\cdot\nabla)\psi\big)\,dx\,dt+\int_\Omega\rho^0(x)\psi(x,0)\,dx=0.$$

Theorem 4.3 *Let* $\mu>0,\lambda\ge-\frac{2}{3}\mu$ *be constants,* $\Omega\subset I\!R^N$ $(N\le 3)$ *a bounded domain with* $\partial\Omega\in C^{2,\alpha}$ $(\alpha>0)$, $p(\rho)=\kappa\rho^\gamma$, $\kappa>0$, $\gamma>3/2$.

Then, for any

$$\rho^0\in L^\gamma(\Omega),\ \rho^0\ge 0,\ \frac{|\boldsymbol{m}^0|^2}{\rho^0}\in L^1(\Omega),\ \boldsymbol{f}\in L^\infty(\Omega\times(0,\infty))^N,$$

there exists a weak solution to (4.2.3)–(4.2.5) *such that*

$$\rho\in L^\infty(0,\infty;L^\gamma(\Omega))\cap L^{\frac{5}{3}\gamma-1}(\Omega\times(0,T))$$
$$\cap\, C([0,T];L^\gamma_{\mathrm{weak}}(\Omega))\cap C([0,T];L^\beta(\Omega)),$$

for any $T>0$ *and* $\beta\in[1,\gamma)$,

$$\boldsymbol{v}\in L^2(0,\infty;W_0^{2,1}(\Omega))^N,\ \rho|\boldsymbol{v}|^2\in L^\infty(0,\infty;L^1(\Omega)),$$
$$\rho\boldsymbol{v}\in C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{weak}}(\Omega))^N,$$

and, in the sense of distributions on $(0,T)$, *the energy inequality*

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\rho|\boldsymbol{v}|^2+\frac{\kappa}{\gamma-1}\rho^\gamma\right)dx+\int_\Omega\left(\mu|\nabla\boldsymbol{v}|^2+(\lambda+\mu)|\mathrm{div}\,\boldsymbol{v}|^2\right)dx\le 0$$

holds true. In addition, the so-called renormalized continuity equation

$$b(\rho)_t+\mathrm{div}\,(b(\rho)\boldsymbol{v})+\big(\rho b'(\rho)-b(\rho)\big)\mathrm{div}\,\boldsymbol{v}=0$$

holds in the sense of distributions, i.e. in $\mathcal{D}'(Q_T)$, *for any function* $b\in C^1(I\!R)$.

Here $L^\delta_{\mathrm{weak}}(\Omega)$ *means the linear space* $L^\delta(\Omega)$ *endowed with the weak topology of* $L^\delta(\Omega)$, *induced by the usual* L^δ*-norm.*

A strategy for the proof of Theorem 4.3, in a slightly weaker form, was first given in (Lions, 1993). Complete proof was then published in the monograph (Lions, 1998). The result of the present Theorem 4.3 is proven in (Feireisl *et al.*, 2001). A detailed proof elaborated for a wider readership is presented in the recent monograph (Novotný and Straškraba, 2003).

4.3 Basic properties of the Euler equations

Neglecting viscosity, heat conduction and outer volume force, we get a system of the Euler equations. Let us consider *adiabatic flow of an inviscid perfect gas* in a bounded domain $\Omega \subset I\!R^N$ and time interval $(0,T)$ with $T > 0$. Here $N = 2$ or 3 for 2D or 3D flow, respectively. Then the Euler equations can be written in the form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{N} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = 0 \quad \text{in } Q_T = \Omega \times (0,T). \tag{4.3.1}$$

They are equipped with the initial condition

$$\boldsymbol{w}(x,0) = \boldsymbol{w}^0(x), \quad x \in \Omega, \tag{4.3.2}$$

with a given vector function $\boldsymbol{w}^0$ and suitable boundary conditions.

The state vector $\boldsymbol{w} = (\rho, \rho v_1, \dots, \rho v_N, E)^{\mathrm{T}} \in I\!R^m$, $m = N+2$ (i.e. $m = 4$ or 5 for 2D or 3D flow, respectively), the fluxes $\boldsymbol{f}_s$, $s = 1, \dots, N$, are m-dimensional mappings defined by

$$\begin{aligned} \boldsymbol{f}_s(\boldsymbol{w}) &= (f_{s1}(\boldsymbol{w}), \dots, f_{sm}(\boldsymbol{w}))^{\mathrm{T}} \\ &= (\rho v_s, \rho v_1 v_s + \delta_{1s} p, \dots, \rho v_N v_s + \delta_{Ns} p, (E+p) v_s)^{\mathrm{T}} \end{aligned} \tag{4.3.3}$$

is the *flux* of the quantity $\boldsymbol{w}$ in the direction x_s. The domain of definition of the vector-valued functions $\boldsymbol{f}_s$ is the open set $D \subset I\!R^m$ of vectors $\boldsymbol{w} = (w_1, \dots, w_m)^{\mathrm{T}}$ such that the corresponding density and pressure are positive. Let us denote

$$\mathbb{A}_s(\boldsymbol{w}) = D\boldsymbol{f}_s(\boldsymbol{w})/D\boldsymbol{w} = \text{ Jacobi matrix of } \boldsymbol{f}_s \quad s = 1, ldots, N, \quad \boldsymbol{w} \in D. \tag{4.3.4}$$

Then for each $\boldsymbol{w} \in D$ and $\boldsymbol{n} = (n_1, \dots, n_N)^{\mathrm{T}} \in I\!R^N$ with $|\boldsymbol{n}| = 1$ the mapping

$$\boldsymbol{\mathcal{P}}(\boldsymbol{w}, \boldsymbol{n}) = \sum_{s=1}^{N} n_s \boldsymbol{f}_s(\boldsymbol{w}) \tag{4.3.5}$$

has the Jacobi matrix

$$\mathbb{P}(\boldsymbol{w}, \boldsymbol{n}) = D\boldsymbol{\mathcal{P}}(\boldsymbol{w}, \boldsymbol{n})/D\boldsymbol{w} = \sum_{s=1}^{N} n_s \mathbb{A}_s(\boldsymbol{w}), \tag{4.3.6}$$

with eigenvalues $\lambda_i = \lambda_i(\boldsymbol{w}, \boldsymbol{n})$:

$$\lambda_1 = \boldsymbol{v} \cdot \boldsymbol{n} - a, \quad \lambda_2 = \dots = \lambda_{m-1} = \boldsymbol{v} \cdot \boldsymbol{n}, \quad \lambda_m = \boldsymbol{v} \cdot \boldsymbol{n} + a, \tag{4.3.7}$$

where $\boldsymbol{v} = (v_1, \dots, v_N)^{\mathrm{T}}$ is the velocity and $a = \sqrt{\gamma p / \rho}$ is the local speed of sound. The matrix $\mathbb{P}(\boldsymbol{w}, \boldsymbol{n})$ is diagonalizable with the aid of the matrices $\mathrm{T} = \mathrm{T}(\boldsymbol{w}, \boldsymbol{n})$ and $\mathrm{T}^{-1} = \mathrm{T}^{-1}(\boldsymbol{w}, \boldsymbol{n})$:

$$\mathbb{P}(\boldsymbol{w}, \boldsymbol{n}) = \mathrm{T}\, \Lambda\!\!\!\backslash\, \mathrm{T}^{-1}, \quad \Lambda\!\!\!\backslash = \mathrm{diag}(\lambda_1, \dots, \lambda_m). \tag{4.3.8}$$

The mapping $\boldsymbol{\mathcal{P}}(\boldsymbol{w}, \boldsymbol{n})$ is called the *flux of the quantity* $\boldsymbol{w}$ *in the direction* $\boldsymbol{n}$. The above results imply that the Euler equations form a *diagonally hyperbolic system.*

In the sequel, for simplicity we shall consider two-dimensional flow (i.e. $N = 2,\ m = 4$).

A further interesting property is the *rotational invariance* of the Euler equations, represented by the relations

$$\boldsymbol{\mathcal{P}}(\boldsymbol{w},\boldsymbol{n}) = \sum_{s=1}^{2} \boldsymbol{f}_s(\boldsymbol{w}) n_s = \boldsymbol{Q}^{-1}(\boldsymbol{n}) \boldsymbol{f}_1(\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{w}), \tag{4.3.9}$$

$$\mathbb{P}(\boldsymbol{w},\boldsymbol{n}) = \sum_{s=1}^{2} \mathbb{A}_s(\boldsymbol{w}) n_s = \boldsymbol{Q}^{-1}(\boldsymbol{n}) \mathbb{A}_1(\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{w})\boldsymbol{Q}(\boldsymbol{n}),$$

$$\boldsymbol{n} = (n_1, n_2) \in I\!R^2, |\boldsymbol{n}| = 1, \boldsymbol{w} \in D,$$

where

$$\boldsymbol{Q}(\boldsymbol{n}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 & n_2 & 0 \\ 0 & -n_2 & n_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.3.10}$$

This allows us to transform the Euler equations to rotated coordinate system $\tilde{x}_1, \tilde{x}_2$ by

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \boldsymbol{Q}_0(\boldsymbol{n}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \tilde{\sigma}, \tag{4.3.11}$$

where $\tilde{\sigma} \in I\!R^2$ and

$$\boldsymbol{Q}_0(\boldsymbol{n}) = \begin{pmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{pmatrix}. \tag{4.3.12}$$

Then the transformation of the state vector $\boldsymbol{w}$ yields the state vector

$$\boldsymbol{q} = \boldsymbol{Q}(\boldsymbol{n})\boldsymbol{w}. \tag{4.3.13}$$

We consider the transformed state vector $\boldsymbol{q}$ as a function of $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ and time t:

$$\boldsymbol{q} = \boldsymbol{q}(\tilde{x}, t) = \boldsymbol{Q}(\boldsymbol{n})\boldsymbol{w}(\boldsymbol{Q}_0^{-1}(\boldsymbol{n})(\tilde{x} - \tilde{\sigma}), t). \tag{4.3.14}$$

Then the function $\boldsymbol{q} = \boldsymbol{q}(\tilde{x}, t)$ satisfies the transformed system of the Euler equations

$$\frac{\partial \boldsymbol{q}}{\partial t} + \sum_{s=1}^{N} \frac{\partial \boldsymbol{f}_s(\boldsymbol{q})}{\partial \tilde{x}_s} = 0. \tag{4.3.15}$$

Finally, let us note that fluxes $\boldsymbol{f}_s$ and $\boldsymbol{\mathcal{P}}$ *homogeneous mappings of order one*: e.g.,

$$\boldsymbol{f}_s(\alpha \boldsymbol{w}) = \alpha \boldsymbol{f}_s(\boldsymbol{w}), \quad \alpha > 0. \tag{4.3.16}$$

This implies that

$$\boldsymbol{f}_s(\boldsymbol{w}) = \mathbb{A}_s(\boldsymbol{w})\boldsymbol{w}. \tag{4.3.17}$$

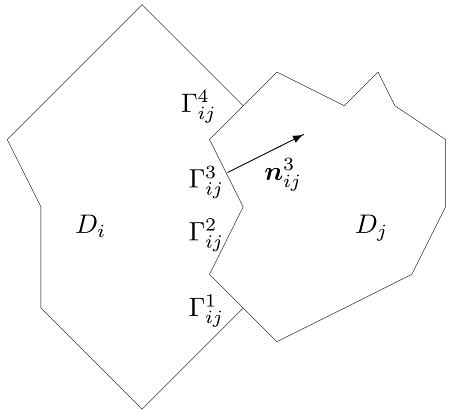


FIG. 4.1. Neighbouring finite volumes in 2D, $\Gamma_{ij} = \bigcup_{\alpha=1}^{4} \Gamma_{ij}^{\alpha}$

4.4 The finite volume method for the Euler equations

Now let us deal with the finite volume (FV) discretization of system (4.3.1). The finite volume method is very popular in computational fluid dynamics. For a survey of various techniques and results from the FV method, we refer the reader to the excellent monograph (Eymard *et al.*, 2000).

4.4.1 *Finite volume mesh*

Let $\Omega \subset \mathbb{R}^2$ be a domain occupied by the fluid. By Ω_h we denote a polygonal approximation of Ω. This means that the boundary $\partial\Omega_h$ of Ω_h consists of a finite number of closed simple piecewise linear curves. The system $\mathcal{D}_h = \{D_i\}_{i \in J}$, where $J \subset Z^+ = \{0, 1, \ldots\}$ is an index set and $h > 0$, will be called a *finite volume mesh* in Ω_h, if D_i, $i \in J$, are *closed polygons* with mutually disjoint interiors such that

$$\overline{\Omega}_h = \bigcup_{i \in J} D_i. \tag{4.4.18}$$

The elements $D_i \in \mathcal{D}_h$ are called *finite volumes*. Two finite volumes D_i, $D_j \in \mathcal{D}_h$ are either disjoint or their intersection is formed by a common part of their boundaries ∂D_i and ∂D_j. If ∂D_i and ∂D_j contains a nonempty straight segment, we call D_i and D_j neighbours and set

$$\Gamma_{ij} = \Gamma_{ji} = \partial D_i \cap \partial D_j. \tag{4.4.19}$$

Obviously, we can write

$$\Gamma_{ij} = \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^{\alpha}, \tag{4.4.20}$$

where Γ_{ij}^{α} are straight segments. See Fig. 4.1. We will call Γ_{ij}^{α} *faces* of D_i.

Further, we introduce the following *notation*:

$$
\begin{aligned}
&|D_i| = \text{ area of } D_i \\
&|\Gamma_{ij}^\alpha| = \text{length of } \Gamma_{ij}^\alpha \\
&\boldsymbol{n}_{ij}^\alpha = ((n_{ij}^\alpha)_1, \dots, (n_{ij}^\alpha)_N)^{\mathrm{T}} = \text{unit outer normal to } \partial D_i \text{ on } \Gamma_{ij}^\alpha, \\
&h_i = \operatorname{diam}(D_i), \\
&h = \sup\nolimits_{i \in J} h_i, \\
&|\partial D_i| = (N-1)\text{length of } \partial D_i, \\
&s(i) = \{j \in J; j \neq i, D_j \text{ is a neighbour of } D_i\}.
\end{aligned} \tag{4.4.21}
$$

Clearly, $\boldsymbol{n}_{ij}^\alpha = -\boldsymbol{n}_{ji}^\alpha$.

The straight segments that form the intersections of $\partial\Omega_h$ with finite volumes D_i adjacent to $\partial\Omega_h$ will be denoted by S_j and numbered by negative indexes j forming an index set $J_B \subset Z^- = \{-1, -2, \dots\}$. Hence, $J \cap J_B = \emptyset$ and $\partial\Omega_h = \bigcup_{j \in J_B} S_j$. For a finite volume D_i adjacent to the boundary $\partial\Omega_h$, i.e. if $S_j \subset \partial\Omega_h \cap \partial D_i$ for some $j \in J_B$, we set

$$
\begin{aligned}
&\gamma(i) = \{j \in J_B; S_j \subset \partial D_i \cap \partial\Omega_h\}, \\
&\Gamma_{ij} = \Gamma_{ij}^1 = S_j, \quad \beta_{ij} = 1 \quad \text{for } j \in \gamma(i).
\end{aligned} \tag{4.4.22}
$$

If D_i is not adjacent to $\partial\Omega_h$, then we put $\gamma(i) = \emptyset$. By $\boldsymbol{n}_{ij}^\alpha$ we again denote the unit outer normal to ∂D_i on Γ_{ij}^α. Then, putting

$$
S(i) = s(i) \cup \gamma(i), \tag{4.4.23}
$$

we have

$$
\begin{aligned}
&\partial D_i = \bigcup_{j \in S(i)} \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^\alpha, \\
&\partial D_i \cap \partial\Omega_h = \bigcup_{j \in \gamma(i)} \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^\alpha, \\
&|\partial D_i| = \sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} |\Gamma_{ij}^\alpha|.
\end{aligned} \tag{4.4.24}
$$

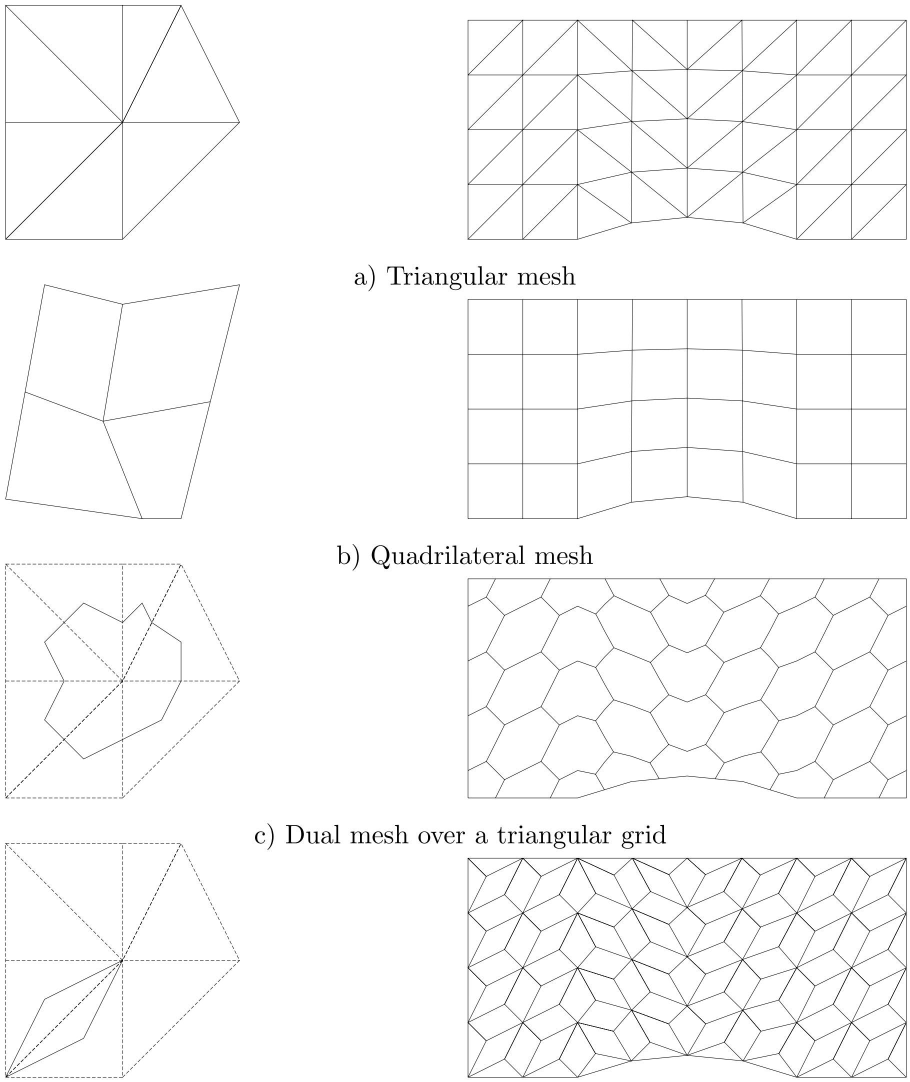
4.4.1.1 *Examples of finite volumes in 2D* In practical computations one uses several types of finite volume meshes:

a) *Triangular mesh* In this case $\mathcal{D}_h$ is a triangulation of the domain Ω_h with the usual properties from the finite element method Then, under the above notation, Γ_{ij} consists of only one straight segment and, thus, we have $\beta_{ij} = 1$ and simply write $\partial D_i = \bigcup_{j \in S(i)} \Gamma_{ij}$.

b) *Quadrilateral mesh* Now $\mathcal{D}_h$ consists of closed convex quadrilaterals D_i.

c) *Dual finite volume mesh over a triangular grid*

d) *Barycentric finite volumes over a triangular grid*



a) Triangular mesh

b) Quadrilateral mesh

c) Dual mesh over a triangular grid

d) Barycentric mesh over a triangular grid

FIG. 4.2. Finite volume meshes in 2D

4.4.2 *Derivation of a general finite volume scheme*

In order to derive a finite volume scheme, we can proceed in the following way. Let us assume that $\boldsymbol{w} : \overline{\Omega} \times [0, T] \to I\!R^m$ is a classical (i.e. C^1-) solution of system (4.3.1), $\mathcal{D}_h = \{D_i\}_{i \in J}$ is a finite volume mesh in a polygonal approximation Ω_h of Ω. Let us construct a partition $0 = t_0 < t_1 < \dots$ of the time interval $[0, T]$

and denote by $\tau_k = t_{k+1} - t_k$ the time step between t_k and t_{k+1}. Integrating equation (4.3.1) over the set $D_i \times (t_k, t_{k+1})$ and using Green's theorem on D_i, we get the identity

$$\int_{D_i} \boldsymbol{w}(x,t)\,\mathrm{d}x\Big|_{t=t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} \left(\int_{\partial D_i} \sum_{s=1}^{N} \boldsymbol{f}_s(\boldsymbol{w}) n_s \,\mathrm{d}S \right) \mathrm{d}t = 0.$$

Moreover, taking into account (4.4.24), we can write

$$\begin{aligned} &\int_{D_i} (\boldsymbol{w}(x,t_{k+1}) - \boldsymbol{w}(x,t_k))\,\mathrm{d}x \\ &\quad + \int_{t_k}^{t_{k+1}} \left(\sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} \int_{\Gamma_{ij}^{\alpha}} \sum_{s=1}^{N} \boldsymbol{f}_s(\boldsymbol{w}) n_s \,\mathrm{d}S \right) \mathrm{d}t = 0. \end{aligned} \tag{4.4.25}$$

Now we shall approximate the integral averages $\int_{D_i} \boldsymbol{w}(x,t_k)\mathrm{d}x/|D_i|$ of the quantity $\boldsymbol{w}$ over the finite volume D_i at time instant t_k by $\boldsymbol{w}_i^k$:

$$\boldsymbol{w}_i^k \approx \frac{1}{|D_i|} \int_{D_i} \boldsymbol{w}(x,t_k)\,\mathrm{d}x, \tag{4.4.26}$$

called the value of *the approximate solution* on D_i at time t_k. Further, we approximate the flux $\sum_{s=1}^{N} \boldsymbol{f}_s(\boldsymbol{w})(n_{ij}^{\alpha})_s$ of the quantity $\boldsymbol{w}$ through the face Γ_{ij}^{α} in the direction $\boldsymbol{n}_{ij}^{\alpha}$ with the aid of a *numerical flux* $\boldsymbol{H}(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, \boldsymbol{n}_{ij}^{\alpha})$. In this way we obtain the following explicit formula

$$\begin{aligned} \boldsymbol{w}_i^{k+1} &= \boldsymbol{w}_i^k - \frac{\tau_k}{|D_i|} \sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} \boldsymbol{H}(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, \boldsymbol{n}_{ij}^{\alpha}) |\Gamma_{ij}^{\alpha}|, \\ &D_i \in \mathcal{D}_h,\ t_k \in [0,T). \end{aligned} \tag{4.4.27}$$

for the computation of $\boldsymbol{w}_i^{k+1}$ from the known values $\boldsymbol{w}_i^k$, $i \in J$.

The FV method is equipped with *initial conditions* $\boldsymbol{w}_i^0$, $i \in J$, defined by

$$\boldsymbol{w}_i^0 = \frac{1}{|D_i|} \int_{D_i} \boldsymbol{w}^0(x)\,\mathrm{d}x, \tag{4.4.28}$$

under the assumption that the function $\boldsymbol{w}^0$ from (4.3.2) is locally integrable: $\boldsymbol{w}^0 \in L^1_{\mathrm{loc}}(\Omega)^m$.

4.4.3 *Properties of the numerical flux*

In what follows, we shall assume that the numerical flux $\boldsymbol{H}$ has the following properties:

1. $\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{n})$ is defined and continuous on $D \times D \times \mathcal{S}_1$, where D is the domain of definition of the fluxes $\boldsymbol{f}_s$ and $\mathcal{S}_1$ is the unit sphere in $\mathbb{R}^N$: $\mathcal{S}_1 = \{\boldsymbol{n} \in \mathbb{R}^N; |\boldsymbol{n}| = 1\}$.

2. $\boldsymbol{H}$ is *consistent*:

$$\boldsymbol{H}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{n}) = \boldsymbol{\mathcal{P}}(\boldsymbol{u},\boldsymbol{n}) = \sum_{s=1}^{N} \boldsymbol{f}_s(\boldsymbol{u})n_s, \quad \boldsymbol{u} \in D,\ \boldsymbol{n} \in \mathcal{S}_1. \tag{4.4.29}$$

3. $\boldsymbol{H}$ is *conservative*:

$$\boldsymbol{H}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{n}) = -\boldsymbol{H}(\boldsymbol{v},\boldsymbol{u},-\boldsymbol{n}), \quad \boldsymbol{u},\boldsymbol{v} \in D,\ \boldsymbol{n} \in \mathcal{S}_1. \tag{4.4.30}$$

If $\boldsymbol{H}$ satisfies conditions (4.4.29) and (4.4.30), the *method* is called *consistent* and *conservative*, respectively. (Note that the conservativity of the scheme means that the flux from the finite volume D_i into D_j through Γ_{ij}^{α} has the same magnitude, but opposite sign, as the flux from D_j into D_i.)

4.4.4 *Examples of numerical fluxes*

Taking into account that the matrix $\mathbb{P}$ is diagonalizable, i.e.

$$\mathbb{P}(\boldsymbol{w},\boldsymbol{n}) = \mathrm{T}\,\Lambda\,\mathrm{T}^{-1}, \quad \Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_m),$$

we define the matrices $\mathbb{P}^{\pm}$ and $|\mathbb{P}|$ by

$$\mathbb{P}^{\pm} = \mathrm{T}\,\Lambda^{\pm}\,\mathrm{T}^{-1}, \quad \Lambda^{\pm} = \mathrm{diag}(\lambda_1^{\pm},\dots,\lambda_4^{\pm}), \quad \lambda^{+} = \max(\lambda,0),\ \lambda^{-} = \min(\lambda,0),$$

$$|\mathbb{P}| = \mathrm{T}\,|\Lambda|\,\mathrm{T}^{-1}, \quad |\Lambda| = \mathrm{diag}(|\lambda_1|,\dots,|\lambda_4|).$$

a) The *Lax–Friedrichs numerical flux* is defined by

$$\boldsymbol{H}_{\mathrm{LF}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{n}) = \frac{1}{2}\left(\boldsymbol{\mathcal{P}}(\boldsymbol{u},\boldsymbol{n}) + \boldsymbol{\mathcal{P}}(\boldsymbol{v},\boldsymbol{n}) - \frac{1}{\lambda}(\boldsymbol{v}-\boldsymbol{u})\right), \quad \boldsymbol{u},\boldsymbol{v} \in D,\ \boldsymbol{n} \in \mathcal{S}_1. \tag{4.4.31}$$

Here $\lambda > 0$ is independent of $\boldsymbol{u},\boldsymbol{v}$, but depends, in general, on Γ_{ij}^{α} in the scheme.

b) *The Steger–Warming scheme* has the numerical flux

$$\boldsymbol{H}_{\mathrm{SW}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{n}) = \mathbb{P}^{+}(\boldsymbol{u},\boldsymbol{n})\boldsymbol{u} + \mathbb{P}^{-}(\boldsymbol{v},\boldsymbol{n})\boldsymbol{v}, \quad \boldsymbol{u},\boldsymbol{v} \in D,\ \boldsymbol{n} \in \mathcal{S}_1. \tag{4.4.32}$$

c) *The Vijayasundaram scheme*:

$$\boldsymbol{H}_{\mathrm{V}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{n}) = \mathbb{P}^{+}\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{2},\boldsymbol{n}\right)\boldsymbol{u} + \mathbb{P}^{-}\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{2},\boldsymbol{n}\right)\boldsymbol{v}. \tag{4.4.33}$$

d) *The Van Leer scheme*:

$$\boldsymbol{H}_{\mathrm{VL}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{n}) = \frac{1}{2}\left\{\boldsymbol{\mathcal{P}}(\boldsymbol{u},\boldsymbol{n}) + \boldsymbol{\mathcal{P}}(\boldsymbol{v},\boldsymbol{n}) - \left|\mathbb{P}\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{2},\boldsymbol{n}\right)\right|(\boldsymbol{v}-\boldsymbol{u})\right\}. \tag{4.4.34}$$

4.4.5 *Boundary conditions*

Let $D_i \in \mathcal{D}_h$ be a finite volume adjacent to the boundary $\partial\Omega_h$, i.e. ∂D_i is formed by faces $\Gamma = \Gamma^1_{ij} \subset \partial\Omega_h$ $(j \in \gamma(i))$ and let $\boldsymbol{n} = \boldsymbol{n}^1_{ij}$ be a unit outer normal to ∂D_i on Γ. (See Section 4.4.1.) In order to be able to compute the numerical flux $\boldsymbol{H}(\boldsymbol{w}^k_i, \boldsymbol{w}^k_j, \boldsymbol{n})$, it is necessary to specify the value $\boldsymbol{w}^k_j$. The choice of boundary conditions for nonlinear hyperbolic problems is a difficult question. One possibility is to use approach described in the monograph by *M. Feistauer, J. Felcman and I. Straškraba* - see Preface, based on the linearization and the use of the method of characteristics. In this way we get the following result:

On $\Gamma = \Gamma^\alpha_{ij} \subset \partial\Omega_h$ (i.e. $i \in J, j \in \gamma(i), \alpha = 1$) with normal $\boldsymbol{n} = \boldsymbol{n}^\alpha_{ij}$, we have to prescribe n_{pr} quantities characterizing the state vector $\boldsymbol{w}$, where n_{pr} is the number of negative eigenvalues of the matrix $\mathbb{P}(\boldsymbol{w}^k_i, \boldsymbol{n})$, whereas we extrapolate n_{ex} quantities to the boundary, where n_{ex} is the number of nonnegative eigenvalues of $\mathbb{P}(\boldsymbol{w}^k_i, \boldsymbol{n})$. The *extrapolation* of a quantity q to the boundary means in this case to set $q^k_j := q^k_i$. On the other hand, if we prescribe the boundary value of q, we set $q^k_j := q^k_{Bj}$ with a given value q^k_{Bj}, determined by the user on the basis of the physical character of the flow.

We present here one possibility, which is often used in practical computations. It is suitable to distinguish several cases given in Table 4.1 (for 2D flow, $N = 2,\ m = 4$).

4.4.6 *Stability of the finite volume schemes*

Let $\mathbf{w}^k = \{\boldsymbol{w}^k_i\}_{i\in J}$ be an approximate solution on the k-th time level obtained with the aid of the finite volume method. By $\|\mathbf{w}^k\|$ we denote a norm of the approximation $\mathbf{w}^k$. We call the *scheme stable*, if there exists a constant $c > 0$ independent of τ, h, k such that

$$\|\mathbf{w}^k\| \le c\|\mathbf{w}^0\|, \quad k = 0, 1, \ldots. \tag{4.4.35}$$

The stability conditions of schemes for the solution of the Euler equations is usually obtained by a heuristic extension of the stability conditions for the finite volume schemes applied to simple scalar equations. Let us consider Vijayasundaram and Steger–Warming schemes applied to the following scalar linear equation

$$\frac{\partial w}{\partial t} + \sum_{s=1}^{N} a_s \frac{\partial w}{\partial x_s} = 0, \tag{4.4.36}$$

where $a_s \in I\!R$. Let us denote $\boldsymbol{a} = (a_1, \ldots, a_N)^{\mathrm{T}}$. It is easy to see that the Vijayasundaram and Steger–Warming schemes applied to equation (4.4.36) become identical. The flux of the quantity w has the form

$$\mathcal{P}(w, \boldsymbol{n}) = w \sum_{s=1}^{N} a_s n_s = w(\boldsymbol{a} \cdot \boldsymbol{n}), \quad \boldsymbol{n} = (n_1, \ldots, n_N)^{\mathrm{T}} \in \mathcal{S}_1,\ w \in I\!R,$$

and the corresponding numerical flux becomes

Table 4.1 *Boundary conditions for 2D flow*

Type of boundary	Character of the flow	The sign of eigenvalues n_{pr} and n_{ex}	Quantities extrapolated	Quantities prescribed
INLET ($\boldsymbol{v}\cdot\boldsymbol{n} < 0$)	supersonic flow ($-\boldsymbol{v}\cdot\boldsymbol{n} > a$)	$\lambda_1 < 0$ $\lambda_2 = \lambda_3 < 0$ $\lambda_4 < 0$ $n_{pr} = 4,\ n_{ex} = 0$	—	ρ, v_1, v_2, p
	subsonic flow ($-\boldsymbol{v}\cdot\boldsymbol{n} \le a$)	$\lambda_1 < 0$ $\lambda_2 = \lambda_3 < 0$ $\lambda_4 \ge 0$ $n_{pr} = 3,\ n_{ex} = 1$	p	ρ, v_1, v_2
OUTLET ($\boldsymbol{v}\cdot\boldsymbol{n} > 0$)	supersonic flow ($\boldsymbol{v}\cdot\boldsymbol{n} \ge a$)	$\lambda_1 \ge 0$ $\lambda_2 = \lambda_3 > 0$ $\lambda_4 > 0$ $n_{pr} = 0,\ n_{ex} = 4$	ρ, v_1, v_2, p	—
	subsonic flow ($\boldsymbol{v}\cdot\boldsymbol{n} < a$)	$\lambda_1 < 0$ $\lambda_2 = \lambda_3 > 0$ $\lambda_4 > 0$ $n_{pr} = 1,\ n_{ex} = 3$	ρ, v_1, v_2	p
SOLID IMPER-MEABLE BOUNDARY	$\boldsymbol{v}\cdot\boldsymbol{n} = 0$	$\lambda_1 < 0$ $\lambda_2 = \lambda_3 = 0$ $\lambda_4 > 0$ $n_{pr} = 1,\ n_{ex} = 3$	p (ρ, v_t)	$\boldsymbol{v}\cdot\boldsymbol{n} = 0$

$$H(u,v,\boldsymbol{n}) = (\boldsymbol{a}\cdot\boldsymbol{n})^+u + (\boldsymbol{a}\cdot\boldsymbol{n})^-v, \quad u,v \in I\!R, \quad \boldsymbol{n} \in \mathcal{S}_1. \tag{4.4.37}$$

It is possible to show that the mentioned methods are L^∞-stable under the *stability condition*

$$\tau|\boldsymbol{a}||\partial D_i|/|D_i| \le 1, \quad i \in J. \tag{4.4.38}$$

4.4.7 *Extension of the stability conditions to the Euler equations*

In the above example, the vector $\boldsymbol{a}$ represents the characteristic speed of propagation of disturbances in the quantity w. For the Euler equations, we can consider 4 characteristic directions (in the 2D case) given by the eigenvectors of the matrix $\mathbb{P}(\boldsymbol{w},\boldsymbol{n})$ and the characteristic speeds are given by the corresponding eigenvalues $\lambda_s(\boldsymbol{w},\boldsymbol{n}),\ s = 1,\dots,4$. We generalize the stability condition (4.4.38) to the Euler equations in such a way that the speed $|\boldsymbol{a}|$ is replaced by the magnitudes of the eigenvalues $\lambda_s(\boldsymbol{w},\boldsymbol{n}), s = 1,\dots,4$. In this heuristic way we arrive at the CFL-stability condition of the form

$$\tau_k \max_{\substack{j\in S(i)\\ \alpha=1,\dots,\beta_{ij}}} \max_{r=1,\dots,4} \left|\lambda_r(\boldsymbol{w}_i^k, \boldsymbol{n}_{ij}^\alpha)\right| |\partial D_i|/|D_i| \le CFL, \quad i \in J, \tag{4.4.39}$$

where CFL is a positive constant. Usually we choose CFL < 1, e.g. CFL=0.85.

5

FINITE ELEMENT METHODS FOR COMPRESSIBLE FLOW

5.1 Combined finite volume–finite element method for viscous compressible flow

The finite volume method (FVM) represents an efficient and robust method for the solution of inviscid compressible flow. On the other hand, the finite element method (FEM) is suitable for the approximation of elliptic or parabolic problems. The use of advantages of both FE and FV techniques leads us to the *combined FV - FE method*. It is applied in such a way that the FVM is used for the discretization of inviscid Euler fluxes, whereas the FEM is applied to the approximation of viscous terms.

For simplicity we assume that volume force and heat sources are equal to zero. Then the complete system describing viscous compressible flow in a domain $\Omega \subset I\!R^N$ with Lipschitz-continuous boundary $\Gamma = \partial\Omega$ and in a time interval $(0, T)$ can be written in the form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{i=1}^{N} \frac{\partial \boldsymbol{f}_i(\boldsymbol{w})}{\partial x_i} = \sum_{i=1}^{N} \frac{\partial \boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_i} \quad \text{in } Q_T, \tag{5.1.1}$$

where $Q_T = \Omega \times (0, T)$ and

$$\begin{aligned}
\boldsymbol{w} &= (\rho, \rho v_1, \dots, \rho v_N, E)^{\mathrm{T}} \in I\!R^m, \qquad (5.1.2)\\
m &= N + 2,\ \boldsymbol{w} = \boldsymbol{w}(x, t),\ x \in \Omega,\ t \in (0, T),\\
\boldsymbol{f}_i(\boldsymbol{w}) &= (f_{i1}, \dots, f_{im})^{\mathrm{T}}\\
&= (\rho v_i, \rho v_1 v_i + \delta_{1i} p, \dots, \rho v_N v_i + \delta_{Ni} p, (E + p) v_i)^{\mathrm{T}}\\
\boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w}) &= (R_{i1}, \dots, R_{im})^{\mathrm{T}}\\
&= (0, \tau_{i1}, \dots, \tau_{iN}, \tau_{i1} v_1 + \dots + \tau_{iN} v_N + k \partial\theta/\partial x_i)^{\mathrm{T}},\\
\tau_{ij} &= \lambda \operatorname{div} \boldsymbol{v} \delta_{ij} + 2\mu d_{ij}(\boldsymbol{v}),\ d_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).
\end{aligned} \tag{5.1.3}$$

To system (5.1.1) we add the thermodynamical relations valid for a *perfect gas*:

$$p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2), \quad \theta = \left(\frac{E}{\rho} - \frac{1}{2}|\boldsymbol{v}|^2\right)/c_v. \tag{5.1.4}$$

As usual, we use the following *notation*: $\boldsymbol{v} = (v_1, \dots, v_N)^{\mathrm{T}}$ – velocity vector, ρ – density, p – pressure, θ – absolute temperature, E – total energy, γ – Poisson adiabatic constant, c_v – specific heat at constant volume, μ, λ – viscosity coefficients, k – heat conduction coefficient. We assume $\mu, k > 0$, $2\mu + 3\lambda \geq 0$. Usually we set $\lambda = -2\mu/3$. By τ_{ij} we denote here the of the viscous part of the stress tensor.

The system is equipped with *initial conditions* written in the form

$$\boldsymbol{w}(x, 0) = \boldsymbol{w}^0(x), \quad x \in \Omega, \tag{5.1.5}$$

where $\boldsymbol{w}^0(x)$ is a given vector-valued function defined in Ω.

5.1.0.1 *Boundary conditions* We write $\partial\Omega = \Gamma_I \cup \Gamma_O \cup \Gamma_W$, where Γ_I represents the inlet through which the gas enters the domain Ω, Γ_O is the outlet through which the gas should leave Ω and Γ_W represents impermeable fixed walls.

On Γ_I one can prescribe the conditions

$$\begin{aligned} &\text{a)}\ \ \rho\big|_{\Gamma_I \times (0,T)} = \rho_D, \quad \text{b)}\ \ \boldsymbol{v}\big|_{\Gamma_I \times (0,T)} = \boldsymbol{v}_D = (v_{D1}, \dots, v_{DN})^{\mathrm{T}}, \\ &\text{c)}\ \ \theta\big|_{\Gamma_I \times (0,T)} = \theta_D \end{aligned} \tag{5.1.6}$$

with given functions $\rho_D, \boldsymbol{v}_D, \theta_D$. The inlet Γ_I is characterized, of course, by the condition $\boldsymbol{v}_D \cdot \boldsymbol{n} < 0$ on Γ_I, where $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$.

On Γ_W we use the no-slip boundary conditions. Moreover, we use here the *condition of adiabatic wall* with zero heat flux. Hence,

$$\begin{aligned} &\text{a)}\ \ \boldsymbol{v}\big|_{\Gamma_W \times (0,T)} = 0, \\ &\text{b)}\ \ \frac{\partial \theta}{\partial n}\Big|_{\Gamma_W \times (0,T)} = 0. \end{aligned} \tag{5.1.7}$$

Finally on the outlet Γ_O we consider "natural boundary conditions"

$$\begin{aligned} &\sum_{i=1}^{N} \tau_{ij} n_i = 0, \quad j = 1, \dots, N, \\ &\frac{\partial \theta}{\partial n} = 0. \end{aligned} \tag{5.1.8}$$

The Dirichlet boundary conditions can be expressed in terms of the conservative variables in the form

$$\begin{aligned} w_1 = \rho_D,\ (w_2, \dots, w_{m-1})^{\mathrm{T}} = \rho_D \boldsymbol{v}_D, w_m = E_D \quad &\text{on } \Gamma_I \times (0, T), \\ w_2 = \ldots = w_{m-1} = 0 \quad &\text{on } \Gamma_W \times (0, T). \end{aligned} \tag{5.1.9}$$

This is reflected in the definition of the *space of test functions*

$$\boldsymbol{V} = \big\{\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)^{\mathrm{T}}; \varphi_i \in H^1(\Omega),\ i = 1, \dots, m, \tag{5.1.10}$$

$$\varphi_1, \varphi_2, \dots, \varphi_m = 0 \text{ on } \Gamma_I,\ \varphi_2, \dots, \varphi_{m-1} = 0 \text{ on } \Gamma_W \big\}.$$

We shall express the Dirichlet boundary conditions with the aid of a function $\boldsymbol{w}^*$ satisfying these condition. Then the fact that a solution $\boldsymbol{w}$ satisfies the Dirichlet conditions can be expressed as the condition

$$\boldsymbol{w}(t) - \boldsymbol{w}^*(t) \in \boldsymbol{V}, \quad t \in (0, T).$$

Now, assuming that $\boldsymbol{w}$ is a classical solution of problem (CFP), we multiply equation (5.1.1) by any $\boldsymbol{\varphi} \in \boldsymbol{V}$, integrate over Ω and apply Green's theorem to viscous terms. We obtain the identity

$$\begin{aligned} &\int_\Omega \frac{\partial \boldsymbol{w}}{\partial t} \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_\Omega \sum_{i=1}^N \frac{\partial \boldsymbol{f}_i(\boldsymbol{w})}{\partial x_i} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \\ &+ \int_\Omega \sum_{i=1}^N \boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \, \mathrm{d}x \\ &- \int_{\partial\Omega} \sum_{i=1}^N n_i \boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w}) \cdot \boldsymbol{\varphi} \, \mathrm{d}S = 0. \end{aligned} \tag{5.1.11}$$

From the representation of $\boldsymbol{R}_i$ in (5.1.2), boundary conditions and the definition of the space $\boldsymbol{V}$ we find that

$$\int_{\partial\Omega} \sum_{i=1}^N n_i \boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w}) \cdot \boldsymbol{\varphi} \, \mathrm{d}S = 0. \tag{5.1.12}$$

Let us introduce the notation

$$\begin{aligned} (\boldsymbol{w}, \boldsymbol{\varphi}) &= \int_\Omega \boldsymbol{w} \cdot \boldsymbol{\varphi} \, \mathrm{d}x, \\ a(\boldsymbol{w}, \boldsymbol{\varphi}) &= \int_\Omega \sum_{i=1}^N \boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \, \mathrm{d}x, \\ b(\boldsymbol{w}, \boldsymbol{\varphi}) &= \int_\Omega \sum_{i=1}^N \frac{\partial \boldsymbol{f}_i(\boldsymbol{w})}{\partial x_i} \cdot \boldsymbol{\varphi} \, \mathrm{d}x. \end{aligned} \tag{5.1.13}$$

Obviously, the forms given in (5.1.13). are linear with respect to $\boldsymbol{\varphi}$ and make sense for functions $\boldsymbol{w}$ with weaker regularity than that of the classical solution. We shall not specify it here. From the point of view of the FE solution, it is sufficient to write the *weak formulation* of problem (CFP) as the conditions

$$\begin{aligned} &\text{a)} \ \boldsymbol{w}(t) - \boldsymbol{w}^*(t) \in \boldsymbol{V}, \quad t \in (0, T), \\ &\text{b)} \ \left(\frac{\partial \boldsymbol{w}(t)}{\partial t}, \boldsymbol{\varphi} \right) + a(\boldsymbol{w}(t), \boldsymbol{\varphi}) + b(\boldsymbol{w}(t), \boldsymbol{\varphi}) = 0, \end{aligned} \tag{5.1.14}$$

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V},\ t \in (0,T),$$

$$\text{c)}\ \ \boldsymbol{w}(0) = \boldsymbol{w}^0.$$

A function $\boldsymbol{w}$ for which the individual terms in (5.1.14), b) make sense, satisfying conditions (5.1.14), a)-c) is called a *weak solution* of the compressible flow problem (CFP).

5.1.1 *Computational grids*

By Ω_h we denote a polygonal approximation of the domain Ω. In the combined FV–FE method we work with two meshes constructed in the domain Ω_h: a finite element mesh $\mathcal{T}_h = \{K_i\}_{i\in I}$ and a finite volume mesh $\mathcal{D}_h = \{D_j\}_{j\in J}$. Here, I and $J \subset Z^+$ are suitable index sets.

The FE mesh $\mathcal{T}_h$ satisfies the standard properties from the FEM. It is formed by a finite number of closed triangles covering the closure of Ω_h,

$$\overline{\Omega}_h = \bigcup_{K\in\mathcal{T}_h} K. \tag{5.1.15}$$

By σ_h we denote the set of all vertices of all elements $K \in \mathcal{T}_h$ and assume that $\sigma_h \cap \partial\Omega_h \subset \partial\Omega$. Moreover, let the common points of sets Γ_I, Γ_W and Γ_O belong to the set σ_h. The symbol $\mathcal{Q}_h$ will denote the set of all midpoints of sides of all elements $K \in \mathcal{T}$. By $|K|$ we denote the area of $K \in \mathcal{T}_h$ $h_K = \text{diam}(K)$ and $h = \max_{K\in\mathcal{T}_h} h_K$.

We shall also work with an FV mesh $\mathcal{D}_h$ in Ω_h, formed by a finite number of closed polygons such that

$$\overline{\Omega}_h = \bigcup_{D\in\mathcal{D}_h} D. \tag{5.1.16}$$

Various types of FV meshes were introduced in Section 4.3.

We use the same notation as in Section 4.4.1. The boundary ∂D_i of each finite volume $D_i \in \mathcal{D}_h$ can be expressed as

$$\partial D_i = \bigcup_{j\in S(i)} \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^{\alpha}, \tag{5.1.17}$$

where Γ_{ij}^{α} are straight segments, called faces of D_i, $\Gamma_{ij}^{\alpha} = \Gamma_{ji}^{\alpha}$, which either form the common boundary of neighbouring finite volumes D_i and D_j or are part of $\partial\Omega_h$. We denote by $|D_i|$ the area of D_i, $|\Gamma_{ij}^{\alpha}|$– the length of Γ_{ij}^{α}, $\boldsymbol{n}_{ij}^{\alpha}$ – the unit outer normal to ∂D_i on Γ_{ij}^{α}. Clearly, $\boldsymbol{n}_{ij}^{\alpha} = -\boldsymbol{n}_{ji}^{\alpha}$. $S(i)$ is a suitable index set written in the form

$$S(i) = s(i) \cup \gamma(i), \tag{5.1.18}$$

where $s(i)$ contains indexes of neighbours D_j of D_i and $\gamma(i)$ is formed by indexes j of $\Gamma_{ij}^1 \subset \partial\Omega_h$ (in this case we set $\beta_{ij} = 1$). For details see Section 4.4.1.

5.1.2 *FV and FE spaces*

The FE approximate solution will be sought in a finite dimensional space

$$\boldsymbol{X}_h = X_h^m, \tag{5.1.19}$$

called a *finite element space*. We shall consider two cases of the definition of X_h:

$$X_h = \left\{\varphi_h \in C(\overline{\Omega}_h); \varphi_h|_K \in P^1(K) \ \forall K \in \mathcal{T}_h\right\} \tag{5.1.20}$$

(conforming piecewise linear elements) and

$$X_h = \big\{\varphi_h \in L^2(\Omega); \varphi_h|_K \in P^1(K),\ \varphi_h \text{ are continuous} \\ \text{at midpoints } Q_j \in \mathcal{Q} \text{ of all faces of all } K \in \mathcal{T}_h\big\} \tag{5.1.21}$$

(nonconforming Crouzeix–Raviart piecewise linear elements – they were originally proposed for the approximation of the velocity of incompressible flow, see the previous chapter.

The finite volume approximation is an element of the finite volume space

$$\boldsymbol{Z}_h = Z_h^m, \tag{5.1.22}$$

where

$$Z_h = \left\{\varphi_h \in L^2(\Omega); \varphi_h|_D = \text{const } \forall D \in \mathcal{D}_h\right\}. \tag{5.1.23}$$

One of the most important concepts is a relation between the spaces $\boldsymbol{X}_h$ and $\boldsymbol{Z}_h$. We assume the existence of a mapping $L_h : \boldsymbol{X}_h \to \boldsymbol{Z}_h$, called a *lumping operator*.

In practical computations we use several combinations of the FV and FE spaces (see, for example, (Feistauer and Felcman, 1997), (Feistauer *et al.*, 1995), (Feistauer *et al.*, 1996), (Dolejší *et al.*, 2002)).

Let us mention, for example *conforming finite elements combined with dual finite volumes* In this case the FE space $\boldsymbol{X}_h$ is defined by (5.1.19) – (5.1.20). The mesh $\mathcal{D}_h$ is formed by dual FVs D_i constructed over the mesh $\mathcal{T}_h$, associated with vertices $P_i \in \sigma_h = \{P_i\}_{i \in J}$, defined in Sections 4.4.1.1, c). In this case, the lumping operator is defined as such a mapping $L_h : \boldsymbol{X}_h \to \boldsymbol{Z}_h$ that for each $\boldsymbol{\varphi}_h \in \boldsymbol{X}_h$

$$L_h \boldsymbol{\varphi}_h \in \boldsymbol{Z}_h, \quad L_h \boldsymbol{\varphi}_h|_{D_i} = \boldsymbol{\varphi}_h(P_i) \quad \forall i \in J. \tag{5.1.24}$$

Obviously, L_h is a one-to-one mapping of $\boldsymbol{X}_h$ onto $\boldsymbol{Z}_h$.

Another possibility is the combination of *nonconforming finite elements combined with barycentric finite volumes.*

5.1.3 *Space semidiscretization of the problem*

We use the following approximations: $\Omega \approx \Omega_h$, $\Gamma_I \approx \Gamma_{Ih} \subset \partial\Omega_h$, $\Gamma_W \approx \Gamma_{Wh} \subset \partial\Omega_h$, $\Gamma_O \approx \Gamma_{Oh} \subset \partial\Omega_h$, $\boldsymbol{w}(t) \approx \boldsymbol{w}_h(t) \in \boldsymbol{X}_h$, $\boldsymbol{\varphi} \approx \boldsymbol{\varphi}_h \in \boldsymbol{V}_h \approx \boldsymbol{V}$, where

$$\boldsymbol{V}_h = \Big\{\boldsymbol{\varphi}_h = (\varphi_{h1}, \dots, \varphi_{hm}) \in \boldsymbol{X}_h; \boldsymbol{\varphi}(P_i) = 0 \tag{5.1.25}$$

$$\text{at } P_i \in \Gamma_{Ih}, \varphi_{hn}(P_i) = 0 \text{ for } n = 2, \dots, m-1 \text{ at } P_i \in \Gamma_{Wh} \Big\}.$$

Here P_i denote nodes, i.e. vertices $P_i \in \sigma_h$ or midpoints of faces $P_i \in \mathcal{Q}_h$ in the case of conforming or nonconforming finite elements, respectively.

The form $a(\boldsymbol{w}, \boldsymbol{\varphi})$ defined in (5.1.13) is approximated by

$$a_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{N} \boldsymbol{R}_s(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} \, \mathrm{d}x, \quad \boldsymbol{w}_h, \boldsymbol{\varphi}_h \in \boldsymbol{X}_h. \quad (5.1.26)$$

In order to approximate the nonlinear convective terms containing inviscid fluxes $\boldsymbol{f}_s$, we start from the analogy with the form b from (5.1.13) written as $\int_\Omega \sum_{s=1}^N (\partial \boldsymbol{f}_s(\boldsymbol{w})/\partial x_s) \cdot \boldsymbol{\varphi} \, \mathrm{d}x$, where we use the approximation $\boldsymbol{\varphi} \approx L_h \boldsymbol{\varphi}_h$. Then Green's theorem is applied and the flux $\sum_{s=1}^N f_s(\boldsymbol{w}) n_s$ is approximated with the aid of a numerical flux $\boldsymbol{H}(\boldsymbol{w}, \boldsymbol{w}', \boldsymbol{n})$ from the FVM treated in Section 4.3:

$$\begin{aligned} \int_\Omega \sum_{s=1}^N \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} \cdot \boldsymbol{\varphi} \mathrm{d}x &\approx \sum_{i \in J} \int_{D_i} \sum_{s=1}^N \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} \cdot L_h \boldsymbol{\varphi}_h \, \mathrm{d}x \\ &= \sum_{i \in J} L_h \boldsymbol{\varphi}_h|_{D_i} \cdot \int_{D_i} \sum_{s=1}^N \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} \, \mathrm{d}x \\ &= \sum_{i \in J} L_h \boldsymbol{\varphi}_h|_{D_i} \cdot \int_{\partial D_i} \sum_{s=1}^N \boldsymbol{f}_s(\boldsymbol{w}) n_s \, \mathrm{d}S \\ &= \sum_{i \in J} L_h \boldsymbol{\varphi}_h|_{D_i} \cdot \sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} \int_{\Gamma_{ij}^\alpha} \sum_{s=1}^N \boldsymbol{f}_s(\boldsymbol{w}) n_s \, \mathrm{d}S \\ &\approx \sum_{i \in J} L_h \boldsymbol{\varphi}_h|_{D_i} \cdot \sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} \boldsymbol{H}(L_h \boldsymbol{w}_h|_{D_i}, L_h \boldsymbol{w}_h|D_j, \boldsymbol{n}_{ij}^\alpha) |\Gamma_{ij}^\alpha|. \end{aligned} \quad (5.1.27)$$

Hence, we set

$$b_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \sum_{i \in J} L_h \boldsymbol{\varphi}_h|_{D_i} \cdot \sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} \boldsymbol{H}(L_h \boldsymbol{w}_h|_{D_i}, L_h \boldsymbol{w}_h|_{D_j}, \boldsymbol{n}_{ij}^\alpha) |\Gamma_{ij}^\alpha|. \quad (5.1.28)$$

If $\Gamma_{ij}^\alpha \subset \partial \Omega_h$, it is necessary to give an interpretation of $L_h w_h|_{D_j}$ using inviscid boundary conditions – see Section 5.1.5.

Definition 5.1 *We define a finite volume–finite element approximate solution of the viscous compressible flow as a vector-valued function $\boldsymbol{w}_h = \boldsymbol{w}_h(x, t)$ defined for (a.a) $x \in \overline{\Omega}_h$ and all $t \in [0, T]$ satisfying the following conditions:*

$$a) \;\; \boldsymbol{w}_h \in C^1([0, T]; \boldsymbol{X}_h), \quad (5.1.29)$$

$$
\begin{aligned}
&b)\ \boldsymbol{w}_h(t) - \boldsymbol{w}_h^*(t) \in \boldsymbol{V}_h, \\
&c)\ \left(\frac{\partial \boldsymbol{w}_h(t)}{\partial t}, \boldsymbol{\varphi}_h\right) + b_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) \\
&\qquad + a_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) = 0 \\
&\qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_h,\ \forall t \in (0,T), \\
&d)\ \boldsymbol{w}_h(0) = \boldsymbol{w}_h^0.
\end{aligned}
$$

5.1.4 *Time discretization*

Problem (5.1.29) is equivalent to a large system of ordinary differential equations which is solved with the aid of a suitable time discretization. It is possible to use Runge–Kutta methods.

The simpliest possibility is the *Euler forward scheme*. Let $0 = t_0 < t_1 < t_2 \ldots$ be a partition of the time interval and let $\tau_k = t_{k+1} - t_k$. Then in (5.1.29), b), c) we use the approximations $\boldsymbol{w}_h^k \approx \boldsymbol{w}_h(t_k)$ and $(\partial \boldsymbol{w}_h/\partial t)(t_k) \approx (\boldsymbol{w}_h^{k+1} - \boldsymbol{w}_h^k)/\tau_k$ and obtain the scheme

$$
\begin{aligned}
&\text{a)}\ \boldsymbol{w}_h^{k+1} - \boldsymbol{w}_h^*(t_{k+1}) \in \boldsymbol{V}_h, \\
&\text{b)}\ (\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = (\boldsymbol{w}_h^k, \boldsymbol{\varphi}_h) - \tau_k a_h(\boldsymbol{w}_h^k, \boldsymbol{\varphi}_h) \\
&\qquad - \tau_k b_k(\boldsymbol{w}_h^k, \boldsymbol{\varphi}_h) \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_h,\ k = 0,1,\ldots.
\end{aligned}
\tag{5.1.30}
$$

5.1.5 *Realization of boundary conditions in the convective form b_h*

If $\Gamma_{ij}^\alpha \subset \partial\Omega_h$ (i.e. $j \in \gamma(i),\ \alpha = 1$), then there is no finite volume adjacent to Γ_{ij}^α from the opposite side to D_i and it is necessary to interpret the value $L_h \boldsymbol{w}_h|_{D_j}$ in the definition (5.1.28) of the form b_h. This means that we need to determine a boundary state $\tilde{\boldsymbol{w}}_j$ which will be substituted for $L_h \boldsymbol{w}_h|_{D_j}$ in (5.1.28).

We apply the approach used in the FVM and explained in Section 4.4.5.

5.1.5.1 *Stability of the combined FV–FE methods* Since scheme (5.1.30) is explicit, it is necessary to apply some stability condition. Unfortunately, there is no rigorous theory for the stability of schemes applied to the complete compressible Navier–Stokes system. We proceed heuristically. By virtue of the explicit FV discretization of inviscid terms, we apply the modification of the stability condition derived in Section 4.4.7 for the explicit FVM for the solution of the Euler equations:

$$
\frac{\tau_k}{|D_i|} \max_{\substack{j \in S(i) \\ \alpha=1,\ldots,\beta_{ij}}} \max_{\ell=1,\ldots,m} \{ |\partial D_i| |\lambda_\ell(\boldsymbol{w}_i^k, \boldsymbol{n}_{ij}^\alpha)| + \mu \} \le \mathrm{CFL} \approx 0.85, \quad i \in J, \tag{5.1.31}
$$

where $\lambda_\ell(\boldsymbol{w}_i^k, \boldsymbol{n}_{ij}^\alpha)$ are the eigenvalues of the matrix $\mathbb{P}(\boldsymbol{w}_i^k, \boldsymbol{n}_{ij}^\alpha)$ – see (4.3.6).

5.2 Discontinuous Galerkin finite element method

This section is concerned with the discontinuous Galerkin finite element method (DGFEM) for the numerical solution of compressible inviscid as well as viscous flow. The DGFEM is based on the use of piecewise polynomial approximations without any requirement on the continuity on interfaces between neighbouring elements. It uses advantages of the FVM and FEM.

5.2.1 *DGFEM for conservation laws*

In this section we shall discuss the discontinuous Galerkin finite element discretization of multidimensional initial-boundary value problems for conservation law equations and, in particular, for the Euler equations. Let $\Omega \subset I\!R^N$ be a bounded domain with a piecewise smooth Lipschitz-continuous boundary $\partial\Omega$ and let $T > 0$. In the space-time cylinder $Q_T = \Omega \times (0, T)$ we consider a system of m first order hyperbolic equations

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{N} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = 0. \tag{5.2.1}$$

This system is equipped with the initial condition

$$\boldsymbol{w}(x, 0) = \boldsymbol{w}^0(x), \quad x \in \Omega, \tag{5.2.2}$$

where $\boldsymbol{w}^0$ is a given function, and with boundary conditions

$$B(\boldsymbol{w}) = 0, \tag{5.2.3}$$

where B is a boundary operator. The choice of the boundary conditions is carried out similarly as in Section 4.4.5 in the framework of the discrete problem for the Euler equations describing gas flow.

5.2.1.1 *Discretization* Let Ω be a polygonal or polyhedral domain, if $N = 2$ or $N = 3$, respectively. Let $\mathcal{T}_h$ $(h > 0)$ denote a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed convex polygons (if $N = 2$) or polyhedra (if $N = 3$) K with mutually disjoint interiors. We call $\mathcal{T}_h$ a triangulation of Ω, but *do not* require the usual conforming properties from the FEM. In 2D problems we usually choose $K \in \mathcal{T}_h$ as triangles or quadrilaterals, in 3D, $K \in \mathcal{T}_h$ can be, for example, tetrahedra, pyramids or hexahedra, but we can allow even more general convex elements K.

We set $h_K = \mathrm{diam}(K)$, $h = \max_{K \in \mathcal{T}_h} h_K$. By $|K|$ we denote the N-dimensional Lebesgue measure of K. All elements of $\mathcal{T}_h$ will be numbered so that $\mathcal{T}_h = \{K_i\}_{i \in I}$, where $I \subset Z^+ = \{0, 1, 2, \ldots\}$ is a suitable index set. If two elements K_i, $K_j \in \mathcal{T}_h$ contain a nonempty open face which is a part of an $(N-1)$-dimensional hyperplane (i.e. straight line in 2D or plane in 3D), we call them *neighbouring elements* or *neighbours*. In this case we set $\Gamma_{ij} = \partial K_i \cap \partial K_j$ and assume that the whole set Γ_{ij} is a part of an $(N-1)$-dimensional hyperplane. For $i \in I$ we set $s(i) = \{j \in I; K_j \text{ is a neighbour of } K_i\}$. The boundary $\partial\Omega$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega$. We denote all these boundary faces by S_j, where $j \in I_b \subset Z^- = \{-1, -2, \ldots\}$, and set $\gamma(i) = \{j \in I_b; S_j \text{ is a face of } K_i\}, \Gamma_{ij} = S_j$ for $K_i \in \mathcal{T}_h$ such that $S_j \subset \partial K_i$, $j \in$

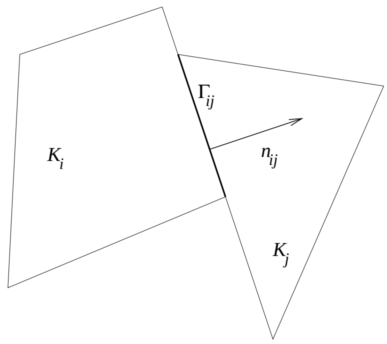


FIG. 5.1. Neighbouring elements $K_i,\ K_j$

I_b. For K_i not containing any boundary face S_j we set $\gamma(i) = \emptyset$. Obviously, $s(i) \cap \gamma(i) = \emptyset$ for all $i \in I$. Now, if we write $S(i) = s(i) \cup \gamma(i)$, we have

$$\partial K_i = \bigcup_{j \in S(i)} \Gamma_{ij}, \qquad \partial K_i \cap \partial\Omega = \bigcup_{j \in \gamma(i)} \Gamma_{ij}. \tag{5.2.4}$$

Furthermore, we use the following notation: $\boldsymbol{n}_{ij} = ((n_{ij})_1, \dots, (n_{ij})_N)$ is the unit outer normal to ∂K_i on the face Γ_{ij} ($\boldsymbol{n}_{ij}$ is a constant vector on Γ_{ij}), $d(\Gamma_{ij}) = \mathrm{diam}(\Gamma_{ij})$, and $|\Gamma_{ij}|$ is the $(N-1)$-dimensional Lebesgue measure of Γ_{ij}. See Fig. 5.1.

Over the triangulation $\mathcal{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K)\ \forall K \in \mathcal{T}_h\}. \tag{5.2.5}$$

For $v \in H^1(\Omega, \mathcal{T}_h)$ we introduce the following notation:

$$\begin{aligned} &v|_{\Gamma_{ij}} - \text{the trace of} \quad v|_{K_i} \quad \text{on} \quad \Gamma_{ij}, \\ &v|_{\Gamma_{ji}} - \text{the trace of} \quad v|_{K_j} \quad \text{on} \quad \Gamma_{ji} = \Gamma_{ij}. \end{aligned} \tag{5.2.6}$$

The approximate solution of problem (5.2.1)–(5.2.3) is sought in the space of discontinuous piecewise polynomial vector-valued functions $\boldsymbol{S}_h$ defined by

$$\begin{aligned} &\boldsymbol{S}_h = [S_h]^m, \\ &S_h = S^{r,-1}(\Omega, \mathcal{T}_h) = \{v; v|_K \in P^r(K)\ \forall K \in \mathcal{T}_h\}, \end{aligned} \tag{5.2.7}$$

where $r \in Z^+$ and $P^r(K)$ denotes the space of all polynomials on K of degree $\leq r$.

Let us assume that $\boldsymbol{w}$ is a classical C^1-solution of system (5.2.1). As usual, by $\boldsymbol{w}(t)$ we denote a function $\boldsymbol{w}(t) : \Omega \to I\!R^m$ such that $\boldsymbol{w}(t)(x) = \boldsymbol{w}(x,t)$ for $x \in \Omega$. In order to derive the discrete problem, we multiply (5.2.1) by a function $\boldsymbol{\varphi} \in H^1(\Omega, \mathcal{T}_h)^m$ and integrate over an element $K_i,\ i \in I$. With the use of Green's theorem, we obtain the integral identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{K_i} \boldsymbol{w}(t) \cdot \boldsymbol{\varphi}\, \mathrm{d}x - \int_{K_i} \sum_{s=1}^{N} \boldsymbol{f}_s(\boldsymbol{w}(t)) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s}\, \mathrm{d}x \tag{5.2.8}$$

$$+\sum_{j\in S(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{N}\boldsymbol{f}_s(\boldsymbol{w}(t))\cdot\boldsymbol{\varphi}\,n_s\,\mathrm{d}S=0.$$

Summing (5.2.8) over all $K_i\in\mathcal{T}_h$, we obtain the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in I}\int_{K_i}\boldsymbol{w}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x-\sum_{i\in I}\int_{K_i}\sum_{s=1}^{N}\boldsymbol{f}_s(\boldsymbol{w}(t))\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_s}\,\mathrm{d}x \qquad (5.2.9)$$

$$+\sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{N}\boldsymbol{f}_s(\boldsymbol{w}(t))\cdot\boldsymbol{\varphi}\,n_s\,\mathrm{d}S=0.$$

Under the notation

$$(\boldsymbol{w},\boldsymbol{\varphi})=\sum_{i\in I}\int_{K_i}\boldsymbol{w}\cdot\boldsymbol{\varphi}\,\mathrm{d}x=\int_{\Omega}\boldsymbol{w}\cdot\boldsymbol{\varphi}\mathrm{d}x \qquad (5.2.10)$$

($[L^2]^m$-scalar product) and

$$b(\boldsymbol{w},\boldsymbol{\varphi})=-\sum_{i\in I}\int_{K_i}\sum_{s=1}^{N}\boldsymbol{f}_s(\boldsymbol{w})\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_s}\,\mathrm{d}x \qquad (5.2.11)$$

$$+\sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{N}\boldsymbol{f}_s(\boldsymbol{w})\cdot\boldsymbol{\varphi}\,n_s\,\mathrm{d}S,$$

(5.2.8) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{w}(t),\boldsymbol{\varphi})+b(\boldsymbol{w}(t),\boldsymbol{\varphi})=0. \qquad (5.2.12)$$

This equality represents a *weak form* of system (5.2.1) in the sense of the broken Sobolev space $H^1(\Omega,\mathcal{T}_h)$.

5.2.1.2 *Numerical solution* Now we shall introduce the discrete problem approximating identity (5.2.12). For $t\in[0,T]$, the exact solution $\boldsymbol{w}(t)$ will be approximated by an element $\boldsymbol{w}_h(t)\in\boldsymbol{S}_h$. It is not possible to replace $\boldsymbol{w}$ formally in the definition (5.2.11) of the form b, because $\boldsymbol{w}_h$ is discontinuous on Γ_{ij} in general. Similarly as in the FVM we use here the concept of the *numerical flux* $\boldsymbol{H}=\boldsymbol{H}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{n})$ and write

$$\int_{\Gamma_{ij}}\sum_{s=1}^{N}\boldsymbol{f}_s(\boldsymbol{w}(t))\,n_s\cdot\boldsymbol{\varphi}\,\mathrm{d}S\approx\int_{\Gamma_{ij}}\boldsymbol{H}(\boldsymbol{w}_h|_{\Gamma_{ij}}(t),\boldsymbol{w}_h|_{\Gamma_{ji}}(t),\boldsymbol{n}_{ij})\cdot\boldsymbol{\varphi}|_{\Gamma ij}\,\mathrm{d}S. \qquad (5.2.13)$$

We assume that the numerical flux has the properties formulated in Section 4.4.3:

1) $\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{n})$ is defined and continuous on $D \times D \times \mathcal{S}_1$, where D is the domain of definition of the fluxes $\boldsymbol{f}_s$ and $\mathcal{S}_1$ is the unit sphere in $I\!\!R^N$: $\mathcal{S}_1 = \{\boldsymbol{n} \in I\!\!R^N; |\boldsymbol{n}| = 1\}$.
2) $\boldsymbol{H}$ is *consistent*:

$$\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{n}) = \mathcal{P}(\boldsymbol{u}, \boldsymbol{n}) = \sum_{s=1}^{N} \boldsymbol{f}_s(\boldsymbol{u})\, n_s, \quad \boldsymbol{u} \in D,\ \boldsymbol{n} \in \mathcal{S}_1. \tag{5.2.14}$$

3) $\boldsymbol{H}$ is *conservative*:

$$\boldsymbol{H}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{n}) = -\boldsymbol{H}(\boldsymbol{v}, \boldsymbol{u}, -\boldsymbol{n}), \quad \boldsymbol{u},\ \boldsymbol{v} \in D,\ \boldsymbol{n} \in \mathcal{S}_1. \tag{5.2.15}$$

The above considerations lead us to the definition of the approximation b_h of the convective form b:

$$\begin{aligned} b_h(\boldsymbol{w}, \boldsymbol{\varphi}) = &- \sum_{i \in I} \int_{K_i} \sum_{s=1}^{N} \boldsymbol{f}_s(\boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s}\, \mathrm{d}x \\ &+ \sum_{i \in I} \sum_{j \in S(i)} \int_{\Gamma_{ij}} \boldsymbol{H}(\boldsymbol{w}|_{\Gamma_{ij}}, \boldsymbol{w}|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}) \cdot \boldsymbol{\varphi}|_{\Gamma_{ij}} \mathrm{d}S, \quad \boldsymbol{w}, \boldsymbol{\varphi} \in H^1(\Omega, \mathcal{T}_h)^m. \end{aligned} \tag{5.2.16}$$

By $\boldsymbol{w}_h^0$ we denote an $\boldsymbol{S}_h$-approximation of $\boldsymbol{w}^0$, e.g. the $[L^2]^m$-projection on $\boldsymbol{S}_h$.

Now we come to the formulation of the *discrete problem*.

Definition 5.2 *We say that* $\boldsymbol{w}_h$ *is an approximate solution of* (5.2.12)*, if it satisfies the conditions*

$$\begin{aligned} &a)\ \ \boldsymbol{w}_h \in C^1([0,T]; \boldsymbol{S}_h), \\ &b)\ \ \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) + b_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) = 0 \quad \forall\, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h, \forall\, t \in (0,T), \\ &c)\ \ \boldsymbol{w}_h(0) = \boldsymbol{w}_h^0. \end{aligned} \tag{5.2.17}$$

The discrete problem (5.2.17) is equivalent to an initial value problem for a system of ordinary differential equations which can be solved by a suitable time stepping numerical method.

Remark 5.3 If we set $r = 0$, then the DGFEM for reduces to the FV method.

5.2.1.3 *Treatment of boundary conditions* If $\Gamma_{ij} \subset \partial\Omega_h$, then there is no neighbour K_j of K_i adjacent to Γ_{ij} and the values of $\boldsymbol{w}|_{\Gamma_{ij}}$ must be determined on the basis of *boundary conditions*. We use the same approach as in the FVM, explained in Section 4.4.5.

5.2.2 *Limiting of the order of accuracy*

In the DGFEM with $r \geq 1$, similarly as in other higher-order methods, we can observe the Gibbs phenomenon, manifested in approximate solutions by

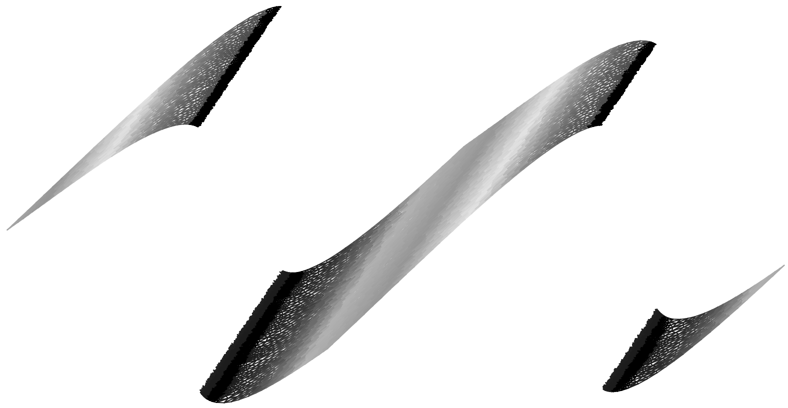


FIG. 5.2. Exact solution of the problem from Example 5.4 plotted at $t = 0.45$

nonphysical overshoots and undershoots near discontinuities. In order to avoid this effect, it is necessary to apply a suitable limiting (or stabilization) in the vicinity of discontinuities or steep gradients.

Example 5.4 In order to demonstrate the applicability of the described limiting procedure, let us consider the scalar 2D Burgers equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x_1} + u\,\frac{\partial u}{\partial x_2} = 0 \quad \text{in } \Omega \times (0, T), \tag{5.2.18}$$

where $\Omega = (-1, 1) \times (-1, 1)$, equipped with initial condition

$$u^0(x_1, x_1) = 0.25 + 0.5 \sin(\pi(x_1 + x_2)), \quad (x_1, x_2) \in \Omega, \tag{5.2.19}$$

and periodic boundary conditions. The exact entropy solution of this problem becomes discontinuous for $t \geq 0.3$ In Fig. 5.2, the graph of the exact solution at time $t = 0.45$ is plotted. If we apply scheme our method to this problem on the mesh from Fig. 5.3, with time step $\tau = 2.5 \cdot 10^{-4}$, we obtain the numerical solution shown in Fig. 5.4. It can be seen here that the numerical solution contains spurious overshoots and undershoots near discontinuities. The application of the described limiting procedure avoids them, as shown in Fig. 5.5.

5.2.3 *Approximation of the boundary*

In the FVM applied to conservation laws or in the FEM using piecewise linear approximations applied to elliptic or parabolic problems, it is sufficient to use a polygonal or polyhedral approximation Ω_h of the 2D or 3D domain Ω, respectively. However, numerical experiments show that in some cases the DGFEM does not give a good resolution in the neighbourhood of curved parts of the boundary $\partial\Omega$, if the mentioned approximations of Ω are used. The quality of approximate solutions near curved boundaries approximated in a piecewise linear way may lead to a bad quality of the numerical solution. This effect can be

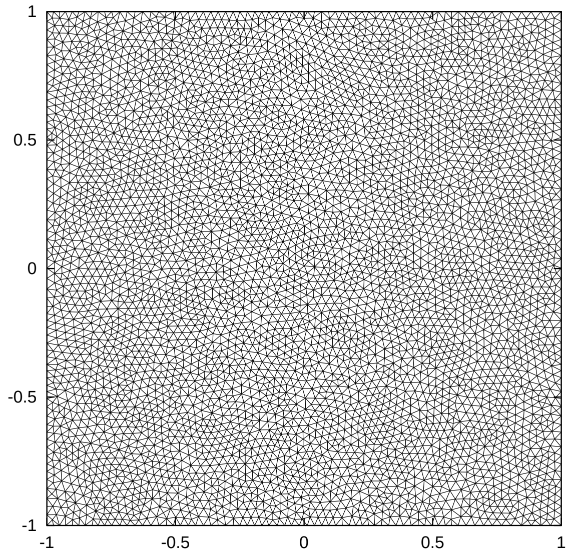


FIG. 5.3. Triangulation used for the numerical solution

avoided by the use of the so-called *isoparametric elements*. For example, if we use piecewise linear elements, then one must use a bilinear transformation of a reference element on a curved boundary element, see Fig. 5.6 and Fig. 5.7.

Then triangles $K_i,\ i \in I_c$, are replaced by the curved triangles and integrals are evaluated on the reference triangle with the aid of a substitution theorem.

5.2.4 *DGFEM for convection–diffusion problems and viscous flow*

5.2.4.1 *Example of a scalar problem* First let us consider a simple scalar nonstationary nonlinear convection-diffusion problem to find $u : Q_T = \Omega \times (0,T) \to I\!R$ such that

$$\text{a)}\quad \frac{\partial u}{\partial t} + \sum_{s=1}^{N} \frac{\partial f_s(u)}{\partial x_s} = \nu \Delta u + g \quad \text{in } Q_T, \tag{5.2.20}$$

$$\text{b)}\quad u\big|_{\Gamma_D \times (0,T)} = u_D, \quad \text{c)}\quad \nu\,\frac{\partial u}{\partial n}\Big|_{\Gamma_N \times (0,T)} = g_N,$$

$$\text{d)}\quad u(x,0) = u^0(x), \quad x \in \Omega.$$

We assume that $\Omega \subset I\!R^N$ is a bounded polygonal domain, if $N = 2$, or polyhedral domain, if $N = 3$, with a Lipschitz boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and $T > 0$. The diffusion coefficient $\nu > 0$ is a given constant, $g : Q_T \to I\!R$, $u_D : \Gamma_D \times (0,T) \to I\!R$, $g_N : \Gamma_N \times (0,T) \to I\!R$ and $u^0 : \Omega \to I\!R$ are given functions, $f_s \in C^1(I\!R)$, $s = 1, \ldots, N$, are given inviscid fluxes.

We define a *classical solution* of problem (5.2.20) as a sufficiently regular function in $\overline{Q}_T$ satisfying (5.2.20), a)–d) pointwise.

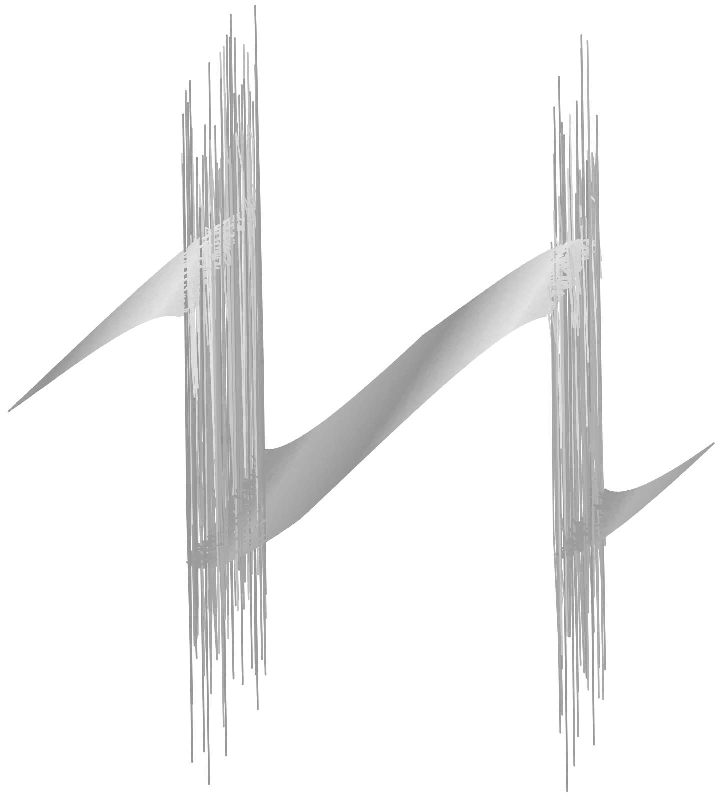


FIG. 5.4. Numerical solution of the problem from Example 5.4 computed by DGFEM, plotted at $t = 0.45$

We leave to the reader the definition of a weak solution to problem (5.2.20) as an exercise.

5.2.4.2 *Discretization* The discretization of convective terms is carried out in the same way as in Section 5.2.1.1. There are several approaches to the discretization of the diffusion term. We shall apply here the technique, used, for example, in (Oden *et al.*, 1998), (Babuška *et al.*, 1999).

We use the notation from Section 5.2.1.1. Moreover, for $i \in I$, by $\gamma_D(i)$ and $\gamma_N(i)$ we denote the subsets of $\gamma(i)$ formed by such indexes j that the faces Γ_{ij} approximate the parts Γ_D and Γ_N, respectively, of $\partial\Omega$. Thus, we suppose that

$$\gamma(i) = \gamma_D(i) \cup \gamma_N(i), \quad \gamma_D(i) \cap \gamma_N(i) = \emptyset. \tag{5.2.21}$$

For $v \in H^1(\Omega, \mathcal{T}_h)$ we set

$$\langle v \rangle_{\Gamma_{ij}} = \frac{1}{2} \left(v\big|_{\Gamma_{ij}} + v\big|_{\Gamma_{oj}} \right), \tag{5.2.22}$$

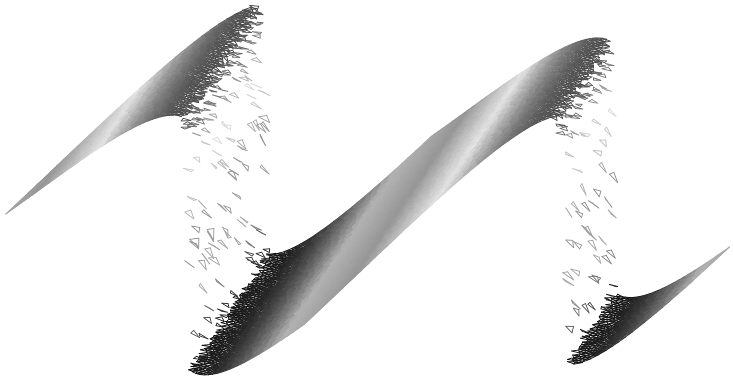


FIG. 5.5. Numerical solution of the problem from Example 5.4 computed by DGFEM with limiting, plotted at $t = 0.45$

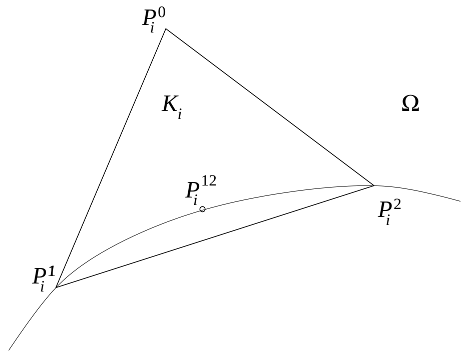


FIG. 5.6. Triangle K_i lying on a curved part of $\partial\Omega$

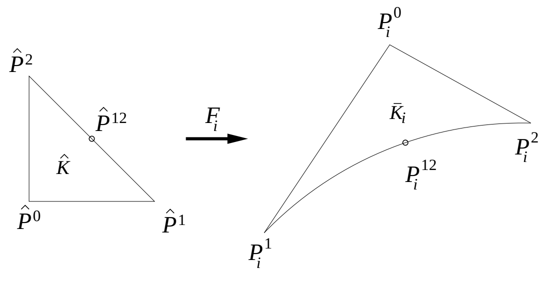


FIG. 5.7. Bilinear mapping $F_i : \hat{K} \to K_i$

$$[v]_{\Gamma_{ij}} = v\big|_{\Gamma_{ij}} - v\big|_{\Gamma_{ji}},$$

denoting the *average* and *jump of the traces* of v on $\Gamma_{ij} = \Gamma_{ji}$ defined in (5.2.6). The approximate solution as well as test functions are supposed to be elements of the space $S_h = S^{p,-1}(\Omega, \mathcal{T}_h)$ introduced in (5.2.7). Obviously, $\langle v\rangle_{\Gamma_{ij}} = \langle v\rangle_{\Gamma_{ji}}$, $[v]_{\Gamma_{ij}} = -[v]_{\Gamma_{ji}}$ and $[v]_{\Gamma_{ij}}\boldsymbol{n}_{ij} = [v]_{\Gamma_{ji}}\boldsymbol{n}_{ji}$.

In order to derive the discrete problem, we assume that u is a classical solution of problem (5.2.20). The regularity of u implies that $u(\cdot,t) \in H^2(\Omega) \subset H^2(\Omega, \mathcal{T}_h)$ and

$$\begin{aligned} \langle u(\cdot,t)\rangle_{\Gamma_{ij}} &= u(\cdot,t)|_{\Gamma_{ij}},\ [u(\cdot,t)]_{\Gamma_{ij}} = 0, \\ \langle \nabla u(\cdot,t)\rangle_{\Gamma_{ij}} &= \nabla u(\cdot,t)|_{\Gamma_{ij}} = \nabla u(\cdot,t)|_{\Gamma_{ji}}, \\ &\qquad \text{for each } t \in (0,T). \end{aligned} \tag{5.2.23}$$

We multiply equation (5.2.20), a) by any $\varphi \in H^2(\Omega, \mathcal{T}_h)$, integrate over $K_i \in \mathcal{T}_h$, apply Green's theorem and sum over all $K_i \in \mathcal{T}_h$. After some manipulation we obtain the identity

$$\begin{aligned} &\int_\Omega \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}x + \sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}}\sum_{s=1}^N f_s(u)\,(n_{ij})_s\,\varphi|_{\Gamma_{ij}}\,\mathrm{d}S \\ &-\sum_{i\in I}\int_{K_i}\sum_{s=1}^N f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x + \sum_{i\in I}\int_{K_i}\nu\,\nabla u\cdot\nabla\varphi\,\mathrm{d}x \\ &-\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\nu\langle\nabla u\rangle\cdot\boldsymbol{n}_{ij}[\varphi]\,\mathrm{d}S \\ &-\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\nu\,\nabla u\cdot\boldsymbol{n}_{ij}\,\varphi\,\mathrm{d}S \\ =&\int_\Omega g\,\varphi\,\mathrm{d}x + \sum_{i\in I}\sum_{j\in\gamma_N(i)}\int_{\Gamma_{ij}}\nu\,\nabla\,u\cdot\boldsymbol{n}_{ij}\,\varphi\,\mathrm{d}S. \end{aligned} \tag{5.2.24}$$

To the left-hand side of (5.2.24) we now add the terms

$$\pm\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\nu\langle\nabla\varphi\rangle\cdot\boldsymbol{n}_{ij}[u]\,\mathrm{d}S = 0, \tag{5.2.25}$$

as follows from (5.2.23). Further, to the left-hand side and the right-hand side we add the terms

$$\pm\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\nu\,\nabla\varphi\cdot\boldsymbol{n}_{ij}\,u\,\mathrm{d}S$$

and

$$\pm \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \nu \, \nabla\varphi \cdot \boldsymbol{n}_{ij} \, u_D \, \mathrm{d}S,$$

respectively, which are identical by the Dirichlet condition (5.2.20), b). We can add these terms equipped with the + sign (the so-called *nonsymmetric DG discretization* of diffusion terms) or with the − sign (*symmetric DG discretization* of diffusion terms). Both possibilities have their advantages and disadvantages. Here we shall use the nonsymmetric discretization.

In view of the Neumann condition (5.2.20), c), we replace the second term on the right-hand side of (5.2.24) by

$$\sum_{i\in I} \sum_{j\in\gamma_N(i)} \int_{\Gamma_{ij}} g_N \, \varphi \, \mathrm{d}S. \tag{5.2.26}$$

Because of the stabilization of the scheme we introduce the *interior penalty*

$$\sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \sigma[u] \, [\varphi] \, \mathrm{d}S \tag{5.2.27}$$

and the *boundary penalty*

$$\sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma \, u \, \varphi \, \mathrm{d}S = \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma u_D \varphi \mathrm{d}S \tag{5.2.28}$$

where σ is a *weight* defined by

$$\sigma|_{\Gamma_{ij}} = C_W \nu / \mathrm{d}(\Gamma_{ij}), \tag{5.2.29}$$

where $C_W > 0$ is a suitable constant. In the considered nonsymmetric formulation we can set $C_W = 1$. (In the case of the symmetric formulation, the choice of C_W follows from a detailed theoretical analysis.)

On the basis of the above considerations we introduce the following forms defined for u, $\varphi \in H^2(\Omega, \mathcal{T}_h)$:

$$\begin{aligned} a_h(u,\varphi) = & \sum_{i\in I} \int_{K_i} \nu \, \nabla u \cdot \nabla\varphi \, \mathrm{d}x \\ & - \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \nu \langle \nabla u \rangle \cdot \boldsymbol{n}_{ij} [\varphi] \, \mathrm{d}S \\ & + \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \nu \langle \nabla \varphi \rangle \cdot \boldsymbol{n}_{ij} [u] \, \mathrm{d}S \\ & - \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \nu \, \nabla \, u \cdot \boldsymbol{n}_{ij} \, \varphi \, \mathrm{d}S \end{aligned} \tag{5.2.30}$$

$$+\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}} \nu\,\nabla\,\varphi\cdot\boldsymbol{n}_{ij}\,u\,\mathrm{d}S$$

(nonsymmetric variant of the diffusion form - it is obvious what form would have the symmetric variant),

$$J_h(u,\varphi)=\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\sigma[u]\,[\varphi]\,\mathrm{d}S \qquad (5.2.31)$$
$$+\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\sigma\,u\,\varphi\,\mathrm{d}S$$

(interior and boundary penalty jump terms),

$$\ell_h(\varphi)(t)=\int_\Omega g(t)\,\varphi\,\mathrm{d}x+\sum_{i\in I}\sum_{j\in\gamma_N(i)}\int_\Gamma g_N(t)\,\varphi\,\mathrm{d}S \qquad (5.2.32)$$
$$+\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\nu\,\nabla\,\varphi\cdot\boldsymbol{n}_{ij}\,u_D(t)\,\mathrm{d}S+\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\sigma\,u_D(t)\,\varphi\,\mathrm{d}S$$

(right-hand side form).

Finally, the convective terms are approximated with the aid of a numerical flux $H=H(u,v,\boldsymbol{n})$ by the form $b_h(u,\varphi)$ defined analogously as in Section 5.2.1.2:

$$b_h(u,\varphi)=-\sum_{i\in I}\int_K\sum_{s=1}^N f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \qquad (5.2.33)$$
$$+\sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}} H\left(u|_{\Gamma_{ij}},u|_{\Gamma_{ji}},\boldsymbol{n}_{ij}\right)\varphi|_{\Gamma_{ij}}\,\mathrm{d}S,\quad u,\,\varphi\in H^2(\Omega,\mathcal{T}_h).$$

We assume that the numerical flux H is (locally) Lipschitz-continuous, consistent and conservative – see Section 5.2.1.2.

Now we can introduce the *discrete problem.*

Definition 5.5 *We say that u_h is a DGFE solution of the convection-diffusion problem* (5.2.20)*, if*

a) $u_h\in C^1([0,T];S_h)$, (5.2.34)

b) $\frac{\mathrm{d}}{\mathrm{d}t}(u_h(t),\varphi_h)+b_h(u_h(t),\varphi_h)+a_h(u_h(t),\varphi_h)+J_h(u_h(t),\varphi_h)=\ell_h(\varphi_h)\,(t)$
$\quad\forall\,\varphi_h\in S_h,\ \forall\,t\in(0,T)$,

c) $u_h(0)=u_h^0$.

By u_h^0 we denote an S_h-approximation of the initial condition u^0.

This discrete problem has been obtained with the aid of the method of lines, i.e. the space semidiscretization. In practical computations suitable time discretization is applied (Euler forward or backward scheme, Runge–Kutta methods or discontinuous Galerkin time discretization) and integrals are evaluated with the aid of numerical integration. Let us note that we do not require here that the approximate solution satisfies the essential Dirichlet boundary condition pointwise, e.g. at boundary nodes. In the DGFEM, this condition is represented in the framework of the 'integral identity' (5.2.34), b).

The above DGFE discrete problem was investigated theoretically in (Dolejší *et al.*, 2002) and (Dolejší *et al.*, 2005), where error estimates were analysed.

5.2.4.3 *DGFE discretization of the Navier–Stokes equations* Similarly as above, one can proceed in the case of the compressible Navier-Stokes equations, but the situation is much more complicated, because the diffusion, i.e. viscous terms, are nonlinear. Therefore, we shall treat the discretization of the viscous terms in a special way, as described in (Dolejší, 2004) or (Feistauer *et al.*, 2005). To this end, we shall linearize partially the viscous terms $\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})$ in a suitable way. From (5.1.2) we obtain

$$\boldsymbol{R}_1(\boldsymbol{w}, \nabla \boldsymbol{w}) \tag{5.2.35}$$
$$= \begin{pmatrix} 0 \\ \frac{2}{3}\frac{\mu}{w_1}\left[2\left(\frac{\partial w_2}{\partial x_1} - \frac{w_2}{w_1}\frac{\partial w_1}{\partial x_1}\right) - \left(\frac{\partial w_3}{\partial x_2} - \frac{w_3}{w_1}\frac{\partial w_1}{\partial x_2}\right)\right] \\ \frac{\mu}{w_1}\left[\left(\frac{\partial w_3}{\partial x_1} - \frac{w_3}{w_1}\frac{\partial w_1}{\partial x_1}\right) + \left(\frac{\partial w_2}{\partial x_2} - \frac{w_2}{w_1}\frac{\partial w_1}{\partial x_2}\right)\right] \\ \frac{w_2}{w_1}\boldsymbol{R}_1^{(2)} + \frac{w_3}{w_1}\boldsymbol{R}_1^{(3)} + \frac{k}{c_v w_1}\left[\frac{\partial w_4}{\partial x_1} - \frac{w_4}{w_1}\frac{\partial w_1}{\partial x_1}\right. \\ \left. -\frac{1}{w_1}\left(w_2\frac{\partial w_2}{\partial x_1} + w_3\frac{\partial w_3}{\partial x_1}\right) + \frac{1}{w_1^2}\left(w_2^2 + w_3^2\right)\frac{\partial w_1}{\partial x_1}\right] \end{pmatrix},$$

$$\boldsymbol{R}_2(\boldsymbol{w}, \nabla \boldsymbol{w})$$
$$= \begin{pmatrix} 0 \\ \frac{\mu}{w_1}\left[\left(\frac{\partial w_3}{\partial x_1} - \frac{w_3}{w_1}\frac{\partial w_1}{\partial x_1}\right) + \left(\frac{\partial w_2}{\partial x_2} - \frac{w_2}{w_1}\frac{\partial w_1}{\partial x_2}\right)\right] \\ \frac{2}{3}\frac{\mu}{w_1}\left[2\left(\frac{\partial w_3}{\partial x_2} - \frac{w_3}{w_1}\frac{\partial w_1}{\partial x_2}\right) - \left(\frac{\partial w_2}{\partial x_1} - \frac{w_2}{w_1}\frac{\partial w_1}{\partial x_1}\right)\right] \\ \frac{w_2}{w_1}\boldsymbol{R}_2^{(2)} + \frac{w_3}{w_1}\boldsymbol{R}_2^{(3)} + \frac{k}{c_v w_1}\left[\frac{\partial w_4}{\partial x_1} - \frac{w_4}{w_1}\frac{\partial w_1}{\partial x_2}\right. \\ \left. -\frac{1}{w_1}\left(w_2\frac{\partial w_2}{\partial x_2} + w_3\frac{\partial w_3}{\partial x_2}\right) + \frac{1}{w_1^2}\left(w_2^2 + w_3^2\right)\frac{\partial w_1}{\partial x_2}\right] \end{pmatrix},$$

where $\boldsymbol{R}_s^{(r)} = \boldsymbol{R}_s^{(r)}(\boldsymbol{w}, \nabla \boldsymbol{w})$ denotes the r-th component of $\boldsymbol{R}_s$ ($s = 1, 2,\ r = 2, 3$). Now for $\boldsymbol{w} = (w_1, \dots, w_4)^{\mathrm{T}}$ and $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_4)^{\mathrm{T}}$ we define the vector-valued functions

$$\boldsymbol{D}_1(\boldsymbol{w}, \nabla \boldsymbol{w}, \boldsymbol{\varphi}, \nabla \boldsymbol{\varphi}) \tag{5.2.36}$$

$$= \begin{pmatrix} 0 \\ \frac{2}{3}\frac{\mu}{w_1}\left[2\left(\frac{\partial\varphi_2}{\partial x_1} - \frac{\varphi_2}{w_1}\frac{\partial w_1}{\partial x_1}\right) - \left(\frac{\partial\varphi_3}{\partial x_2} - \frac{\varphi_3}{w_1}\frac{\partial w_1}{\partial x_2}\right)\right] \\ \frac{\mu}{w_1}\left[\left(\frac{\partial\varphi_3}{\partial x_1} - \frac{\varphi_3}{w_1}\frac{\partial w_1}{\partial x_1}\right) + \left(\frac{\partial\varphi_2}{\partial x_2} - \frac{\varphi_2}{w_1}\frac{\partial w_1}{\partial x_2}\right)\right] \\ \frac{w_2}{w_1}\boldsymbol{D}_1^{(2)} + \frac{w_3}{w_1}\boldsymbol{D}_1^{(3)} + \frac{k}{c_v w_1}\left[\frac{\partial\varphi_4}{\partial x_1} - \frac{\varphi_4}{w_1}\frac{\partial w_1}{\partial x_1}\right. \\ \qquad - \frac{1}{w_1}\left(w_2\frac{\partial\varphi_2}{\partial x_1} + w_3\frac{\partial\varphi_3}{\partial x_1}\right) \\ \qquad \left. + \frac{1}{w_1^2}(w_2\varphi_2 + w_3\varphi_3)\frac{\partial w_1}{\partial x_1}\right] \end{pmatrix},$$

$$\boldsymbol{D}_2(\boldsymbol{w}, \nabla\boldsymbol{w}, \boldsymbol{\varphi}, \nabla\boldsymbol{\varphi})$$

$$= \begin{pmatrix} 0 \\ \frac{\mu}{w_1}\left[\left(\frac{\partial\varphi_3}{\partial x_1} - \frac{\varphi_3}{w_1}\frac{\partial w_1}{\partial x_1}\right) + \left(\frac{\partial\varphi_2}{\partial x_2} - \frac{\varphi_2}{w_1}\frac{\partial w_1}{\partial x_2}\right)\right] \\ \frac{2}{3}\frac{\mu}{w_1}\left[2\left(\frac{\partial\varphi_3}{\partial x_2} - \frac{\varphi_2}{w_1}\frac{\partial w_1}{\partial x_2}\right) - \left(\frac{\partial\varphi_2}{\partial x_1} - \frac{\varphi_2}{w_1}\frac{\partial w_1}{\partial x_1}\right)\right] \\ \frac{w_2}{w_1}\boldsymbol{D}_2^{(2)} + \frac{w_3}{w_1}\boldsymbol{D}_2^{(3)} + \frac{k}{c_v w_1}\left[\frac{\partial\varphi_4}{\partial x_2} - \frac{\varphi_4}{w_1}\frac{\partial w_1}{\partial x_2}\right. \\ \qquad - \frac{1}{w_1}\left(w_2\frac{\partial\varphi_2}{\partial x_2} + w_2\frac{\partial\varphi_3}{\partial x_2}\right) \\ \qquad \left. + \frac{1}{w_1^2}(w_2\varphi_2 + w_3\varphi_3)\frac{\partial w_1}{\partial x_2}\right] \end{pmatrix},$$

where $\boldsymbol{D}_s^{(r)}$ denotes the r-th component of $\boldsymbol{D}_s$ ($s = 1, 2,\ r = 2, 3$). Obviously, $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are linear with respect to $\boldsymbol{\varphi}$ and $\nabla\boldsymbol{\varphi}$ and

$$\boldsymbol{D}_s(\boldsymbol{w}, \nabla\boldsymbol{w}, \boldsymbol{w}, \nabla\boldsymbol{w}) = \boldsymbol{R}_s(\boldsymbol{w}, \nabla\boldsymbol{w}), \quad s = 1, 2. \tag{5.2.37}$$

Now we introduce the following forms defined for functions $\boldsymbol{w}_h, \boldsymbol{\varphi}_h \in \mathcal{S}_h$:

$$(\boldsymbol{w}_h, \boldsymbol{\varphi}_h)_h = \int_{\Omega_h} \boldsymbol{w}_h \cdot \boldsymbol{\varphi}_h \,\mathrm{d}x \tag{5.2.38}$$

($L^2(\Omega_h)$-scalar product),

$$a_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \sum_{i\in I}\int_{K_i}\sum_{s=1}^{2}\boldsymbol{R}_s(\boldsymbol{w}_h, \nabla\boldsymbol{w}_h)\cdot\frac{\partial\boldsymbol{\varphi}_h}{\partial x_s}\,\mathrm{d}x \tag{5.2.39}$$

$$-\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\sum_{s=1}^{2}\langle\boldsymbol{R}_s(\boldsymbol{w}_h, \nabla\boldsymbol{w}_h)\rangle\,(n_{ij})_s\cdot[\boldsymbol{\varphi}_h]\,\mathrm{d}S$$

$$+\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}$$

$$\int_{\Gamma_{ij}} \sum_{s=1}^{2} \langle \boldsymbol{D}_s(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \boldsymbol{\varphi}_h, \nabla \boldsymbol{\varphi}_h)\rangle \, (n_{ij})_s \cdot [\boldsymbol{w}_h] \, \mathrm{d}S$$

$$-\sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} \boldsymbol{R}_s(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h) \, (n_{ij})_s \cdot \boldsymbol{\varphi}_h \, \mathrm{d}S$$

$$+\sum_{i\in I} \sum_{j\in\gamma_D(i)}$$

$$\int_{\Gamma_{ij}} \sum_{s=1}^{2} \boldsymbol{D}_s(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \boldsymbol{\varphi}_h, \nabla \boldsymbol{\varphi}_h) \, (n_{ij})_s \cdot \boldsymbol{w}_h \, \mathrm{d}S$$

(nonsymmetric version of the diffusion form). The use of the above special approach does not yield some additional terms in the discrete analogy to the continuity equation. This appears important for a good quality of the approximate solution.

Further, we introduce the following forms:

$$J_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \sigma [\boldsymbol{w}_h] \cdot [\boldsymbol{\varphi}_h] \, \mathrm{d}S + \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma \, \boldsymbol{w}_h \cdot \boldsymbol{\varphi}_h \, \mathrm{d}S \tag{5.2.40}$$

with

$$\sigma|_{\Gamma_{ij}} = \mu / d(\Gamma_{ij}) \tag{5.2.41}$$

(interior and boundary penalty terms),

$$\beta_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \Big(\sigma \, \boldsymbol{w}_B \cdot \boldsymbol{\varphi}_h + \sum_{s=1}^{2} \boldsymbol{D}_s(\boldsymbol{w}_h, \nabla \boldsymbol{w}_h, \boldsymbol{\varphi}_h, \nabla \boldsymbol{\varphi}_h) \, (n_{ij})_s \cdot \boldsymbol{w}_B \Big) \, \mathrm{d}S \tag{5.2.42}$$

(right-hand side form). The boundary state $\boldsymbol{w}_B$ will be defined later. Finally, we define the form approximating viscous terms:

$$A_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) = a_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) + J_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h) - \beta_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h). \tag{5.2.43}$$

The convective terms are represented by the form b_h defined by (5.2.16).

Now the *discrete problem* reads: Find a vector-valued function $\boldsymbol{w}_h$ such that

$$\text{a)}\ \ \boldsymbol{w}_h \in C^1([0,T]; \boldsymbol{S}_h), \tag{5.2.44}$$

b) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h\right)_h + b_h\left(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h\right) + A_h\left(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h\right)$

$$= 0 \quad \forall\, \boldsymbol{\varphi}_h \in \boldsymbol{S}_h,\ t \in (0, T),$$

c) $\boldsymbol{w}_h(0) = \boldsymbol{w}_h^0$,

where $\boldsymbol{w}_h^0$ is an $\boldsymbol{S}_h$-approximation of $\boldsymbol{w}^0$.

5.2.4.4 *Boundary conditions* If $\Gamma_{ij} \subset \partial\Omega_h$, i.e. $j \in \gamma(i)$, it is necessary to specify boundary conditions.

The boundary state $\boldsymbol{w}_B = (w_{B1}, \ldots, w_{B4})^{\mathrm{T}}$ is determined with the aid of the prescribed Dirichlet conditions and extrapolation:

$$\boldsymbol{w}_B = (\rho_{ij}, 0, 0, c_v \rho_{ij} \theta_{ij}) \quad \text{on } \Gamma_{ij} \text{ approximating } \Gamma_W, \tag{5.2.45}$$

$$\boldsymbol{w}_B = \left(\rho_{Dh}, \rho_{Dh} v_{Dh1}, \rho_D v_{Dh2}, c_v \rho_{Dh} \theta_{Dh} + \frac{1}{2}\rho_{Dh} |\boldsymbol{v}_{Dh}|^2\right)$$
$$\text{on } \Gamma_{ij} \text{ approximating } \Gamma_I,$$

where ρ_{Dh} and $\boldsymbol{v}_{Dh} = (v_{Dh1}, v_{Dh2})$ are approximations of the given density and velocity from the boundary conditions and ρ_{ij}, θ_{ij} are the values of the density and absolute temperature extrapolated from K_i onto Γ_{ij}.

The boundary state $\boldsymbol{w}|_{\Gamma_{ji}}$ appearing in the form b_h is defined in the same way as in Section 5.2.1.3 above.

5.2.5 *Numerical examples*

In the DGFE solution of problems presented here, the forward Euler time discretization was used.

5.2.5.1 *Application of the DGFEM to the solution of inviscid compressible flow* The first numerical example deals with inviscid transonic flow through the GAMM channel with inlet Mach number = 0.67. In order to obtain a steady-state solution, the time stabilization method for $t \to \infty$ is applied.

We demonstrate the influence of the use of superparametric elements at the curved part of $\partial\Omega$, explained in Section 5.2.3. The computations were performed on a coarse grid shown in Fig. 5.8 having 784 triangles. Figures 5.9 and 5.10 show the density distribution along the lower wall obtained by the DGFEM without and with the use of a bilinear mapping, respectively. One can see a difference in the quality of the approximate solutions.

Figure 5.11 shows the computational grid constructed with the aid of the anisotropic mesh adaptation (AMA) technique ((Dolejší, 1998), (Dolejší, 2001)). Figure 5.12 shows the density distribution along the lower wall obtained with the aid of the bilinear mapping on a refined mesh. As we can see, a very sharp shock wave and the so-called Zierep (small local maximum behind the shock wave) singularity were obtained.

In the above examples, the forward Euler time stepping and limiting of the order of accuracy from Section 5.2.2 were used. This method requires to satisfy

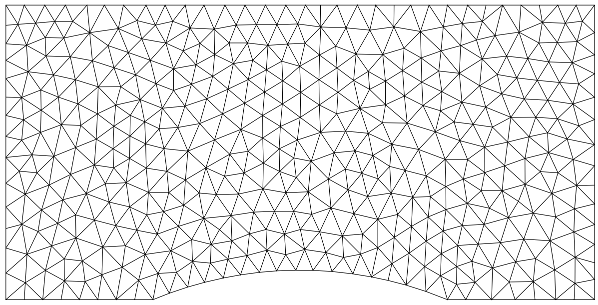


FIG. 5.8. Coarse triangular mesh (784 triangles) in the GAMM channel

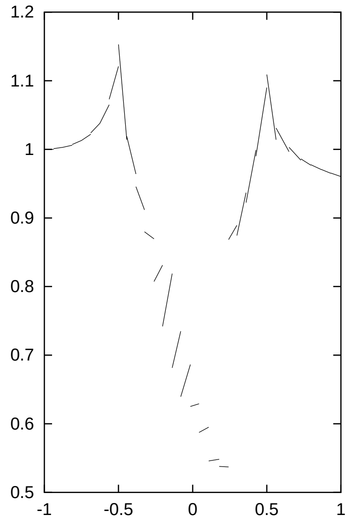


FIG. 5.9. Density distribution along the lower wall in the GAMM channel without the use of a bilinear mapping at $\partial\Omega$

the CFL stability condition representing a severe restriction of the time step. In (Dolejší and Feistauer, 2003), an efficient semi-implicit time stepping scheme was developed for the numerical solution of the Euler equations, allowing to use a long time step in the DGFEM.

5.2.5.2 *Application of the DGFEM to the solution of viscous compressible flow* We present here results from (Dolejší, 2004) on the numerical solution of the viscous flow past the airfoil NACA 0012 by the DGFEM. The computation was performed for the solution of viscous compressible supersonic flow past the profile NACA0012 with far field Mach number $M_\infty = 2$ and Reynolds number $Re =$ 1000. In Fig. 5.13 we see the mesh obtained by the with the aid of the anisotropic

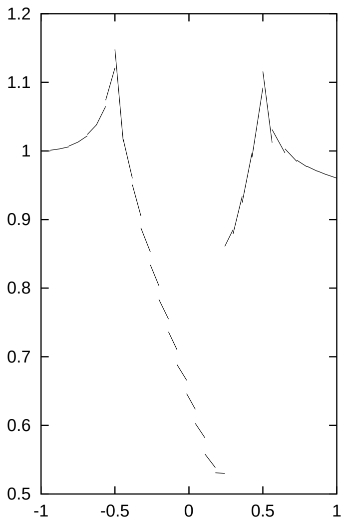


FIG. 5.10. Density distribution along the lower wall in the GAMM channel with the use of a bilinear mapping at $\partial\Omega$

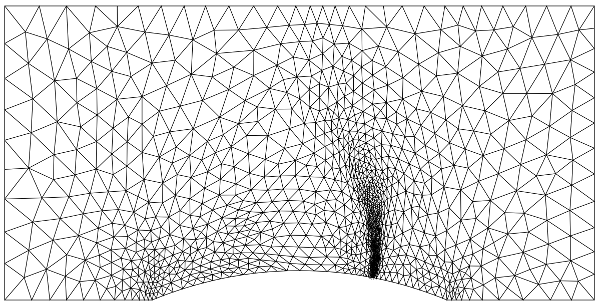


FIG. 5.11. Adapted triangular mesh (2131 triangles) in the GAMM channel

mesh adaptation (AMA) technique. Fig. 5.14 shows the Mach number isolines. Here we see a shock wave in front of the profile, wake and a shock wave leaving the profile.

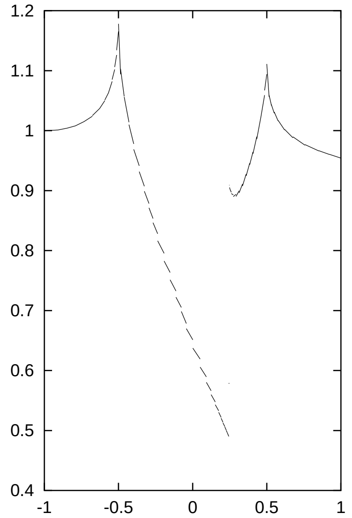


FIG. 5.12. Density distribution along the lower wall in the GAMM channel with the use of a bilinear mapping on an adapted mesh

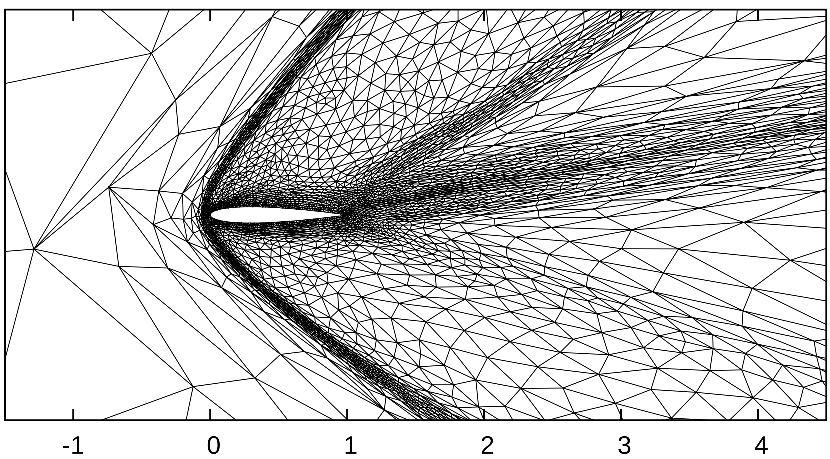


FIG. 5.13. Viscous supersonic flow past the profile NACA 0012: triangulation

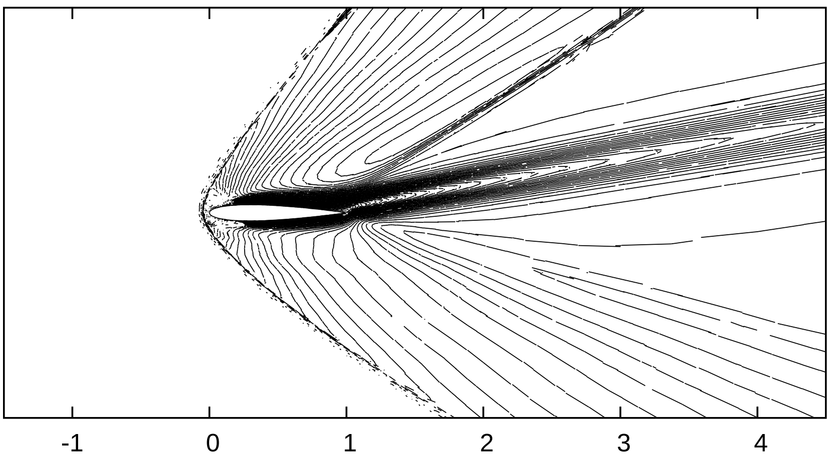


FIG. 5.14. Viscous supersonic flow past the profile NACA 0012: Mach number isolines

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