Three views on potential theory

A course at Charles University (Prague)

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Contents

Chapter 1

Balayage spaces

1.1 Definition of balayage spaces

In the following let X be a locally compact space with countable base. Let $\mathcal{B}(X)$ ($\mathcal{C}(X)$, respectively) denote the set of all Borel measurable numerical (continuous real, respectively) functions on X. We shall write $A \in \mathcal{B}(X)$, if $A \subset X$ and the characteristic function 1_A is Borel measurable. Let

$$\begin{aligned} \mathcal{C}_0(X) &:= \{f \in \mathcal{C}(X) \colon f \text{ vanishes at infinity}\}, \\ \mathcal{K}(X) &:= \{f \in \mathcal{C}(X) \colon f \text{ has compact support} \operatorname{supp}(f)\}. \end{aligned}$$

For every numerical function f on X, let $\hat{f}$ denote the largest lower semicontinuous minorant of f, that is,

$$\hat{f}(x) = \liminf_{y \to x} f(y) \qquad (x \in X).$$

Finally, let $\mathcal{M}(X)$ be the set of all (positive) Radon measures on X.

Let $\mathcal{W}$ be a convex cone of positive lower semicontinuous numerical functions on X. The coarsest topology on X which is at least as fine as the initial topology and for which all functions of $\mathcal{W}$ are continuous will be called the ($\mathcal{W}$)-*fine topology.* Topological notions with respect to the fine topology are distinguished by the term "fine(ly)" or the affix "f" from those pertaining to the initial topology on X.

LEMMA 1.1.1. *Let $x \in X$ and let V be a fine neighborhood of x. Then there exists a compact neighborhood K of x, a function $u \in \mathcal{W}$, and $\alpha \in (0, \infty)$ such that $u(x) < \alpha$ and $K \cap \{u \le \alpha\} \subset V$.*

In particular, x has a fundamental system of fine neighborhoods which are compact in the initial topology.

Proof. By definition of the fine topology, there exist an open set U in X, functions $u_i \in \mathcal{W}$ and $\alpha_i \in \mathbb{R}$, $1 \le i \le n$, such that

$$x \in U \cap \bigcap\nolimits_{i=1}^{n} \{u_i < \alpha_i\} \subset V.$$

Let

$$\varepsilon := \frac{1}{n+1} \min\{\alpha_i - u_i(x) \colon 1 \le i \le n\}, \quad u := \sum_{i=1}^{n} u_i, \quad \alpha := u(x) + \varepsilon.$$

Of course, $u \in \mathcal{W}$ and $u(x) < \alpha < \infty$. Since the functions in $\mathcal{W}$ are lower semicontinuous, there exists a compact neighborhood K of x in U such that $u_i(x) - \varepsilon < u_i(z)$ for all $z \in K$ and $1 \le i \le n$. Then $K \cap \{u \le \alpha\}$ is a fine neighborhood of x. If $z \in K \cap \{u \le \alpha\}$, then $z \in V$, since $K \subset U$ and, for every $1 \le i \le n$,

$$u_i(z) = u(z) - \sum_{j \neq i} u_j(z) \le \alpha - \sum_{j \neq i} (u_j(x) - \varepsilon) = u_i(x) + n\varepsilon < \alpha_i.$$

□

PROPOSITION 1.1.2. *X endowed with the fine topology is a Baire space. More generally, every fine G_δ-set Y in X is a Baire space with respect to the fine topology (that is, with respect to topology on Y which is induced by the fine topology on X).*

Proof. Easy modification of the corresponding proof for locally compact metric spaces (see [1, II.4.2]). □

The following definition of a balayage space $(X, \mathcal{W})$ will lead to an intrinsic characterization of cones of excessive functions for sub-Markov semigroups, where all excessive functions are lower semicontinuous and there exist sufficiently many continuous excessive functions. Moreover, we shall see that $(X, \mathcal{W})$ is a balayage space if and only if $\mathcal{W}$ is the set of all positive hyperharmonic functions for a family of harmonic kernels on X. These equivalences alone could justify the introduction of the concept of a balayage space. However, as the name already indicates, its main importance lies in the fact that it offers the tools which are necessary for a rich theory of balayage of functions and measures.

DEFINITION 1.1.3. $(X, \mathcal{W})$ *is a* balayage space*, if the following holds:*

(B_1) $\mathcal{W}$ *is* σ-stable*, that is,* $\sup v_n \in \mathcal{W}$ *for every increasing sequence* (v_n) *in* $\mathcal{W}$.

(B_2) $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$ *for every subset* $\mathcal{V}$ *of* $\mathcal{W}$.

(B_3) *If* $u, v', v'' \in \mathcal{W}$ *such that* $u \le v' + v''$*, there exist* $u', u'' \in \mathcal{W}$ *such that* $u = u' + u''$*,* $u' \le v'$*, and* $u'' \le v''$.

(B_4) $\mathcal{W}$ *is linearly separating*[1]*, there exist strictly positive* $u_0, v_0 \in \mathcal{W} \cap \mathcal{C}(X)$ *such that* $u_0/v_0 \in \mathcal{C}_0(X)$*, and*

$$w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) \colon v \le w\} \qquad \textit{for every } w \in \mathcal{W}. \tag{1.1}$$

EXAMPLES 1.1.4. 1. If X is a discrete space (at most countable) and $\mathcal{W}$ is the set of all positive numerical functions on X, then $(X, \mathcal{W})$ is a balayage space.

2. If X is the compact subspace $\mathbb{N} \cup \{\infty\}$ of $[0, \infty]$ (with the induced topology), then the convex cone $\mathcal{W}$ of all lower semicontinuous functions $u \colon X \to [0, \infty]$ satisfies (B_1), (B_3), (B_4), but not (B_2), and this in spite of the fact that, of course, $\widehat{\inf \mathcal{V}} \in \mathcal{W}$ for every subset $\mathcal{V}$ of $\mathcal{W}$ (see Section 6.5 for details).

[1] That is, for all $x, y \in X$, $x \neq y$, and $\lambda \in [0, \infty)$, there exists $v \in \mathcal{W}$ such that $v(x) \neq \lambda v(y)$.

In the following let $(X, \mathcal{W})$ be a balayage space.

The functions in $\mathcal{W}$ may be called positive *hyperharmonic functions* on X (see Definition 4.2.1 and Proposition 4.2.3). By (B_2), $\mathcal{W}$ is $\wedge$-stable. Another important consequence is the following.

PROPOSITION 1.1.5. *For all* $\mathcal{V} \subset \mathcal{W}$*, the set* $\{\widehat{\inf \mathcal{V}}^f < \inf \mathcal{V}\}$ *is finely meager, and* $\widehat{\inf \mathcal{V}}^f = \widehat{\inf \mathcal{V}}$.

Proof. Let $\mathcal{V} \subset \mathcal{W}$, $u := \inf \mathcal{V}$. By definition of the fine topology, we immediately obtain that $\hat{u} \le \hat{u}^f$. On the other hand, $\hat{u}^f \le u$ and $\hat{u}^f$ is lower semicontinuous, since it is contained in $\mathcal{W}$ by (B_2). Hence $\hat{u}^f \le \hat{u}$ ($\hat{u}$ being the largest lower semicontinuous minorant of u). Thus $\hat{u}^f = \hat{u}$.

Given $\varepsilon > 0$, let

$$E_\varepsilon := \{\hat{u} \le (u - 2\varepsilon) \wedge \varepsilon^{-1}\}.$$

Obviously, $\{\hat{u} < u\}$ is the union of the sets $E_{1/n}$, $n \in \mathbb{N}$, and these sets are finely closed. So it suffices to show that the sets E_ε have no fine interior points.

To that end let us fix $\varepsilon > 0$ and $x \in E := E_\varepsilon$. Then $V := \{\hat{u} > \hat{u}(x) - \varepsilon\}$ is a neighborhood of x and

$$E \cap V \subset \{u > \hat{u}(x) + \varepsilon\}.$$

Assuming that E is a fine neighborhood of x, the set $E \cap V$ is a fine neighborhood of x and therefore

$$\hat{u}(x) = \hat{u}^f(x) \ge \inf u(E \cap V) \ge \hat{u}(x) + \varepsilon$$

which is impossible. Thus E has no fine interior points. □

For every numerical function f on X, we define

$$R_f := \inf\{w \in \mathcal{W} \colon w \ge f\}. \tag{1.2}$$

Of course, $R_f = R_{f^+}$. By Proposition 1.1.5, $\hat{R}_f \in \mathcal{W}$, and $R_f = \hat{R}_f \in \mathcal{W}$ if f is finely lower semicontinuous.

LEMMA 1.1.6. *Let f be the limit of an increasing sequence (f_n) of finely lower semicontinuous numerical functions on X. Then $R_f = \lim_{n\to\infty} R_{f_n}$.*

Proof. The sequence (R_{f_n}) is increasing to a function $u \in \mathcal{W}$. Obviously, $u \ge f$ and hence $u \ge R_f$. The reverse inequality is trivial. □

DEFINITION 1.1.7. *A* potential cone *is a convex cone S in $\mathcal{B}^+(X)$ such that, for all $u, v \in S$,*

$$R_{u,v} := {}^S R_{u,v} := \inf\{w' \in S \colon w' \ge u - v \text{ on } \{v < \infty\}\} \in S, \tag{1.3}$$

and $u = w + R_{u,v}$ for some $w \in S$.

LEMMA 1.1.8. *$\mathcal{W}$ is a potential cone.*

Proof. Let $u, v \in \mathcal{W}$ and $f := 1_{\{v<\infty\}}(u-v)^+$. Since f is finely lower semicontinuous, we know that $R_f \in \mathcal{W}$. Of course, $u \le v + R_f$. By (B_3), there exist $w, w' \in \mathcal{W}$ such that

$$u = w + w', \quad w \le v, \quad \text{and} \quad w' \le R_f.$$

Then $u \le v + w'$ and hence $R_f \le w'$. Thus $R = w'$ and $u = w + w' = w + R_f$. □

For later use let us note the following converse.

LEMMA 1.1.9. *Let S be a $\wedge$-stable potential cone. Then S satisfies* (B_3).

Proof. Let $u, v, v' \in S$ such that $u \le v + v'$. Clearly, $w' := {}^S R_{u,v} \le v'$, $u \le v + w'$. Let $w \in S$ such that $u = w + w'$. Then

$$u = (v + w') \wedge (w + w') = v \wedge w + w',$$

where $v \wedge w, w' \in S$. Thus S satisfies (B_3). □

1.2 Continuous potentials

Let $(X, \mathcal{W})$ be a balayage space. The functions in

$$\mathcal{P} := \Big\{p \in \mathcal{W} \colon \frac{p}{v} \in \mathcal{C}_0(X) \text{ for some } v \in \mathcal{W} \cap \mathcal{C}(X),\, v > 0\Big\}$$

(which, obviously, are continuous) will be called *continuous potentials* (on X).

PROPOSITION 1.2.1. *$\mathcal{W}$ is the set of all limits of increasing sequences in $\mathcal{P}$, and $p \wedge w \in \mathcal{P}$ for all $p \in \mathcal{P}$, $w \in \mathcal{W} \cap \mathcal{C}(X)$. If $(p_n) \subset \mathcal{P}$ such that $p := \sum_{n=1}^\infty p_n \in \mathcal{C}(X)$, then $p \in \mathcal{P}$.*

Moreover, $\mathcal{P}$ is a function cone, *that is, $\mathcal{P}$ is a convex cone in $\mathcal{C}(X)$, $\mathcal{P}$ is linearly separating and, for every $p \in \mathcal{P}$, there exists a strictly positive $q \in \mathcal{P}$ such that $p/q \in \mathcal{C}_0(X)$.*

Proof. It is immediately seen that $\mathcal{P}$ is a convex cone and $p \wedge w \in \mathcal{P}$ for all $p \in \mathcal{P}$, $w \in \mathcal{W} \cap \mathcal{C}(X)$. By ($B_4$), there exist strictly positive functions $u_0, v_0 \in \mathcal{W} \cap \mathcal{C}(X)$ such that $u_0/v_0 \in \mathcal{C}_0(X)$. Of course, $u_0 \in \mathcal{P}$.

By (B_1), every limit of an increasing sequence in $\mathcal{P}$ is contained in $\mathcal{W}$. To prove the converse we first note that, for every $v \in \mathcal{W} \cap \mathcal{C}(X)$, the functions $p_n := (nu_0) \wedge v$, $n \in \mathbb{N}$, are contained in $\mathcal{P}$ and $p_n \uparrow v$. So property (1.1) in (B_4) implies that, for every $w \in \mathcal{W}$,

$$w = \sup\{q \in \mathcal{P} \colon q \le w\}. \tag{2.1}$$

Let us now fix $w \in \mathcal{W}$ and let $\mathcal{F}$ denote the set of all $q \in \mathcal{P}$ such that $q \le w$. Then the set $\mathcal{F}$ is increasingly filtered. Indeed, let $q_1, q_2 \in \mathcal{F}$ and define $q := R_{q_1 \vee q_2}$. Clearly, $q_1 \vee q_2 \le q \le w$. Moreover,

$$q = R_{q_1 + q_2 - q_1 \wedge q_2},$$

where the functions $q_1 + q_2$ and $q_1 \wedge q_2$ are contained in $\mathcal{P}$. By Lemma 1.1.8, $q \in \mathcal{W}$ and there exists a function $w' \in \mathcal{W}$ such that $q + w' = q_1 + q_2$. This shows that q is a continuous minorant of q_1+q_2, and hence $q \in \mathcal{P}$. So, by Lemma 6.1.1, there exists an increasing sequence (p_n) in $\mathcal{P}$ such that

$$w = \sup \mathcal{F} = \sup p_n.$$

Since $\mathcal{W}$ is linearly separating, we see, in particular, that $\mathcal{P}$ is linearly separating.

Next let $p = \sum_{n=1}^\infty p_n \in \mathcal{C}(X)$, $p_n \in \mathcal{P}$. We want to show that $p \in \mathcal{P}$. Let U_n be relatively compact open sets in X such that $U_n \uparrow X$. For every $n \in \mathbb{N}$, there exist $v_n \in \mathcal{W} \cap \mathcal{C}(X)$, $v_n > 0$, and $m_n \in \mathbb{N}$ such that

$$p_n/v_n \in \mathcal{C}_0(X), \quad v_n \le 2^{-n} \text{ on } U_n, \quad w_n := \sum_{m=m_n+1}^\infty p_m \le 2^{-n} \text{ on } U_n.$$

Of course, we may assume that $m_n \le m_{n+1}$, $n \in \mathbb{N}$, and hence the sequence (w_n) is decreasing. Then

$$v := \sum_{n=1}^\infty (v_n + w_n) \in \mathcal{W} \cap \mathcal{C}(X), \quad v > 0.$$

Moreover, $p/v \in \mathcal{C}_0(X)$. Indeed, given $k \in \mathbb{N}$, there exists a compact set K in X such that $kp_n \le v_n$ on K^c, $1 \le n \le m_k$, and therefore

$$kp = \sum_{n=1}^{m_k} kp_n + kw_k \le \sum_{n=1}^{m_k} v_n + \sum_{j=1}^{k} w_j \le v \quad \text{ on } K^c.$$

Finally, let p be any function in $\mathcal{P}$. To show that $\mathcal{P}$ is a function cone it remains to find a function $q \in \mathcal{P}$, $q > 0$, such that $p/q \in \mathcal{C}_0(X)$. Let $v \in \mathcal{W} \cap \mathcal{C}(X)$, $v > 0$, such that $p/v \in \mathcal{C}_0(X)$. Then $q := \sum_{n=1}^\infty (2^{-n}v) \wedge p \in \mathcal{C}(X)$ and hence $q \in \mathcal{P}$ by the preceding considerations (obviously, this follows as well observing that $q/v = \sum_{n=1}^\infty 2^{-n} \wedge (p/v) \in \mathcal{C}_0(X)$). Moreover, $p/q \in \mathcal{C}_0(X)$, since, for every $n \in \mathbb{N}$, there is a compact set K in X such that $p \le 2^{-n}v$ on K^c, and hence $np \le q$ on K^c. □

A real function f on X is called $\mathcal{P}$-*bounded*, if there exists $p \in \mathcal{P}$ such that $|f| \le p$. Let $\mathcal{C}_{\mathcal{P}}(X)$ denote the set of all $\mathcal{P}$-bounded functions on X.

COROLLARY 1.2.2. 1. *For every $\mathcal{P}$-bounded upper semicontinuous function f on X, R_f is upper semicontinuous, $R_f = \inf\{p \in \mathcal{P} : p \ge f\}$.*

2. *For every $f \in \mathcal{C}_{\mathcal{P}}(X)$, $R_f \in \mathcal{P}$.*

3. *If (f_n) is a decreasing sequence of $\mathcal{P}$-bounded upper semicontinuous functions on X and $f := \lim_{n\to\infty} f_n$, then $R_f = \lim_{n\to\infty} R_{f_n}$.*

Proof. Let (f_n) be a decreasing sequence of positive $\mathcal{P}$-bounded upper semicontinuous functions on X and $f := \lim_{n\to\infty} f_n$. Of course, the sequence (R_{f_n}) is decreasing and $\lim_{n\to\infty} R_{f_n} \ge R_f$.

Let us fix $p_0, q_0 \in \mathcal{P}$ such that $f_1 \le p_0$, $q_0 > 0$, and $p_0/q_0 \in \mathcal{C}_0(X)$. Now let $w \in \mathcal{W}$ such that $w \ge f$, and $\varepsilon > 0$. There is a compact set K in X such that

$p_0 \le \varepsilon q_0$ on K^c and hence $f_1 \le \varepsilon q_0$ on K^c. By Proposition 1.2.1, there are $p_n \in \mathcal{P}$, $n \in \mathbb{N}$, such that $p_n \uparrow w$. Since the functions $p_n - f_n$ are lower semicontinuous and increasing to $w - f$, there exists $n \in \mathbb{N}$ such that $p_n + \varepsilon q_0 > f_n$ on K, and therefore

$$w + \varepsilon q_0 \ge p_n + \varepsilon q_0 \ge f_n \ge f \quad \text{ on } X.$$

This implies that $\lim_{n\to\infty} R_{f_n} \le R_f$ and $R_f \ge \inf\{p \in \mathcal{P} \colon p \ge f\}$. Since the reverse inequalities hold trivially, we obtain (3) and (1).

If f is even continuous, then we know, in addition, that $R_f \in \mathcal{W}$, that is, R_f is lower semicontinuous as well, hence $R_f \in \mathcal{P}$. To finish the proof, it suffices to recall that, for every function g on X, $R_g = R_{g^+}$. □

PROPOSITION 1.2.3. *Let (μ_n) be a sequence of measures on X such that, for every $p \in \mathcal{P}$, the sequence $(\mu_n(p))$ is bounded.*

Then (μ_n) admits a weakly convergent subsequence. Moreover, for any subsequence (ν_n) of (μ_n) which converges weakly to a measure ν,

$$\lim_{n\to\infty} \nu_n(p) = \nu(p) \qquad \textit{for every } p \in \mathcal{P}.$$

Proof. Clearly, the assumption implies that, for every compact K in X, the sequence $(\mu_n(K))$ is bounded. Hence (μ_n) admits a weakly convergent subsequence.

Let (ν_n) be any such subsequence, $\nu_\infty := \lim_{n\to\infty} \nu_n$, and $p \in \mathcal{P}$. Let $q \in \mathcal{P}$, $q > 0$, such that $p/q \in \mathcal{C}_0(X)$. Then $\gamma := \sup \nu_n(q) \ge \nu_\infty(q)$. Let $\varepsilon > 0$ and $\varphi := (p - \varepsilon q)^+$. Then, for every $n \in \mathbb{N} \cup \{\infty\}$,

$$\nu_n(\varphi) \le \nu_n(p) \le \nu_n(\varphi) + \varepsilon\gamma.$$

Hence $|\nu_n(p) - \nu_\infty(p)| < |\nu_n(\varphi) - \nu_\infty(\varphi)| + \varepsilon\gamma$, where $\lim_{n\to\infty} \nu_n(\varphi) = \nu_\infty(\varphi)$. Thus $\lim_{n\to\infty} \nu_n(p) = \nu_\infty(p)$. □

Chapter 2

Semigroups and resolvents

2.1 Supermedian functions

To prepare the study of supermedian functions for sub-Markov semigroups $\mathbb{P} = (P_t)_{t>0}$ and sub-Markov resolvents $(V_\lambda)_{\lambda>0}$, let us first look at functions which are supermedian with respect to a *single* kernel P on X.

DEFINITION 2.1.1. *A function $v \in \mathcal{B}^+(X)$ is called* P-supermedian *if*

$$Pv \le v.$$

The set of all P-supermedian functions is denoted by S_P.

It is immediately seen that the following holds.

LEMMA 2.1.2. 1. *S_P is a convex cone.*

2. *If $v \in S_P$ and $w \in \mathcal{B}^+(X)$ such that $Pv \le w \le v$, then $w \in S_P$.*
3. *If (v_n) is an increasing sequence in S_P, then $\sup v_n \in S_P$.*
4. *If (v_n) is a sequence in S_P, then $\inf v_n \in S_P$.*

If f is any numerical function on X, then obviously

$$R_f := \inf\{v \in S_P : v \ge f\} \in S_P$$

provided $R_f \in \mathcal{B}(X)$. The next result shows that this is always true, if $f \in \mathcal{B}(X)$.

LEMMA 2.1.3. *Let $f \in \mathcal{B}(X)$, $g_1 := f^+$ and $g_{n+1} := g_n \vee Pg_n$, $n \in \mathbb{N}$. Then $R_f = \sup g_n \in S_P$.*

Proof. The sequence (g_n) is increasing to a function $g \in \mathcal{B}^+(X)$ such that

$$Pg = \sup Pg_n \le \sup g_{n+1} = g,$$

and hence $g \in S_P$. Of course, $R_f \le g$, since $f^+ = g_1 \le g$. If $v \in S_P$ such that $v \ge f$, we obtain by induction that $v \ge g_n$ for every $n \in \mathbb{N}$ (if $v \ge g_n$, then $v \ge Pv \ge Pg_n$ and hence $v \ge g_{n+1}$). Thus $R_f \ge g$. □

EXERCISE 2.1.4. For every $f \in \mathcal{B}(X)$, $R_f = f \vee PR_f$.

PROPOSITION 2.1.5. *S_P is a potential cone.*

Proof. Let $u, v \in S_P$ and $f := 1_{\{v<\infty\}}(u-v)^+$. We recursively define a sequence (g_n) as in Lemma 2.1.3, that is,

$$g_1 := f \quad \text{and} \quad g_{n+1} = g_n \vee Pg_n.$$

Next we fix $g \in \mathcal{B}^+(X)$ such that $u = Pu + g$, and define a sequence (w_n) by

$$w_1 := v, \qquad w_{n+1} := w_n \wedge (Pw_n + g).$$

Then $w_n \in S_P$ for every $n \in \mathbb{N}$. Indeed, $w_1 = v \in S_P$ and, if $n \in \mathbb{N}$ such that $w_n \in S_P$, then $Pw_n \le w_{n+1} \le w_n$ and hence $w_{n+1} \in S_P$. Moreover,

$$(1.1) \qquad u = w_n + g_n \qquad (n \in \mathbb{N}).$$

Indeed, $u = v + f = w_1 + g_1$ and, if $n \in \mathbb{N}$ such that $u = w_n + g_n$, then $u = Pu + g = Pw_n + g + Pg_n$ and therefore

$$u = w_n \wedge (Pw_n + g) + g_n \vee Pg_n = w_{n+1} + g_{n+1}.$$

Clearly, $w := \lim_{n\to\infty} w_n = \inf w_n \in S_P$. By Lemma 2.1.3, $\lim_{n\to\infty} g_n = R_f \in S_P$, and (1.1) implies that $u = w + R_f$. $\square$

2.2 Semigroups, resolvents, excessive functions

DEFINITION 2.2.1. *A family* $\mathbb{P} = (P_t)_{t>0}$ *of kernels on* X *is a* semigroup*, if*

$$P_{s+t} = P_s P_t \qquad (s, t > 0).$$

It is sub-Markov (Markov*, respectively*)*, if, for every* $t > 0$*,*

$$P_t 1 \le 1 \qquad (P_t 1 = 1,\ \textit{respectively.}).$$

Moreover,

$$S_{\mathbb{P}} := \{u \in \mathcal{B}^+(X) \colon P_t u \le u \text{ for every } t > 0\} = \bigcap_{t>0} S_{P_t}.$$

is the set of all $\mathbb{P}$-supermedian *functions and*

$$E_{\mathbb{P}} := \{u \in \mathcal{B}^+(X) \colon \sup_{t>0} P_t u = u\} \subset S_{\mathbb{P}}.$$

is set of all $\mathbb{P}$-excessive *functions.*

EXAMPLES 2.2.2. 1. *Brownian semigroup.* $X = \mathbb{R}^d$, $d \ge 1$, and

$$P_t f(x) := (2\pi t)^{-d/2} \int \exp\bigl(-\frac{|x-y|^2}{2t}\bigr) f(y)\, dy.$$

2. *Semigroup* $\mathbb{T} = (T_t)_{t>0}$ *of uniform translation* (to the left). $X = \mathbb{R}$ and

$$T_t(x, \cdot) := \varepsilon_{x-t} \qquad (t > 0, x \in X).$$

$S_{\mathbb{T}}$ is the set of all increasing function $v \colon \mathbb{R} \to [0, \infty]$ and $E_{\mathbb{T}}$ the set of all left continuous functions in $S_{\mathbb{T}}$.

3. *Pseudo-Poisson semigroup.* X discrete, (at most) countable, P a sub-Markov kernel on X and, for every $t > 0$,

$$P_t := e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} P^k.$$

Then the semigroup $\mathbb{P} = (P_t)_{t>0}$ is sub-Markov (Markov, if P is Markov). Moreover,

$$E_{\mathbb{P}} = S_{\mathbb{P}} = S_P.$$

Indeed, obviously $S_P \subset S_{\mathbb{P}}$. Moreover, for all $t > 0$ and $f \in \mathcal{B}^+(X)$, $e^{-t} f \le P_t f$. Therefore $E_{\mathbb{P}} = S_{\mathbb{P}}$. Finally, let $u \in S_{\mathbb{P}}$. Then, for every $t > 0$,

$$e^{-t}(u + tPu) \le P_t u \le u,$$

and hence

$$e^{-t} Pu \le \frac{1 - e^{-t}}{t}\, u.$$

Letting t tend to zero we see that $u \in S_P$.

If $X = \mathbb{Z}$ and $P(x, \cdot) = \varepsilon_{x+1}$, then $\mathbb{P}$ is the *Poisson semigroup* and $E_{\mathbb{P}}$ is the set of all decreasing positive numerical functions on $\mathbb{Z}$.

In the following let $\mathbb{P}$ be a sub-Markov semigroup on X.

LEMMA 2.2.3. *For every $u \in S_{\mathbb{P}}$, the function $t \mapsto P_t u$ is decreasing. For every $u \in E_{\mathbb{P}}$, the function $t \mapsto P_t u$ is decreasing and right continuous.*

Proof. Let $u \in S_{\mathbb{P}}$. Then, for all $s, t > 0$,

$$P_{t+s}u = P_t P_s u \leq P_t u.$$

If $u \in E_{\mathbb{P}}$, then $P_s u$ is increasing to u as s decreases to zero and hence, for every $t > 0$,

$$\lim_{s \to 0} P_{t+s}u = \lim_{s \to 0} P_t P_s u = P_t u.$$

□

PROPOSITION 2.2.4. *$S_{\mathbb{P}}$ and $E_{\mathbb{P}}$ are σ-stable convex cones. Moreover, for every $t > 0$,*

$$P_t(S_{\mathbb{P}}) \subset S_{\mathbb{P}} \quad \text{and} \quad P_t(E_{\mathbb{P}}) \subset E_{\mathbb{P}}.$$

Proof. By Lemma 2.2.3,

$$E_{\mathbb{P}} = \{u \in S_{\mathbb{P}} \colon \lim_{t \to 0} P_t u = u\},$$

hence both $S_{\mathbb{P}}$ and $E_{\mathbb{P}}$ are convex cones. The remaining statements are easily verified (see [1, II.3.3]). □

Let us now suppose that $\mathbb{P}$ is *measurable*, that is, for every $f \in \mathcal{B}^+(X)$, the function $(x,t) \mapsto P_t f(x)$ on $X \times (0,\infty)$ is measurable. Then, for all $\lambda > 0$ and $f \in \mathcal{B}^+(X)$, we may define a function $V_\lambda f \in \mathcal{B}^+(X)$, by

$$V_\lambda f(x) := \int_0^\infty e^{-\lambda t} P_t f(x)\, dt.$$

PROPOSITION 2.2.5. *The family $\mathbb{V} = (V_\lambda)_{\lambda > 0}$ has the following properties.*

1. *For every $\lambda > 0$, λV_λ is a sub-Markov kernel on X.*
2. *For all $\lambda, \mu \in (0, \infty)$,*

$$V_\lambda = V_\mu + (\mu - \lambda) V_\lambda V_\mu \qquad (\textit{resolvent equation}). \tag{2.1}$$

Proof. (1) is immediately verified. To prove (2) let $f \in \mathcal{B}^+(X)$ and $\lambda, \mu \in (0,\infty)$, $\lambda \neq \mu$. Then

$$\begin{aligned}
V_\lambda V_\mu f &= \int_0^\infty e^{-\lambda s} P_s V_\mu f\, ds \\
&= \int_0^\infty e^{-\lambda s} \Big(\int_0^\infty e^{-\mu t} P_s P_t f\, dt \Big)\, ds = \int_0^\infty e^{-(\lambda-\mu)s} \Big(\int_0^\infty e^{-\mu(s+t)} P_{s+t} f\, dt \Big)\, ds \\
&= \int_0^\infty e^{-(\lambda-\mu)s} \Big(\int_s^\infty e^{-\mu t} P_t f\, dt \Big)\, ds = \int_0^\infty \Big(\int_0^t e^{-(\lambda-\mu)s}\, ds \Big) e^{-\mu t} P_t f\, dt \\
&= \frac{1}{\lambda-\mu} \int_0^\infty (1 - e^{-(\lambda-\mu)t}) e^{-\mu t} P_t f\, dt = \frac{1}{\lambda - \mu}(V_\mu f - V_\lambda f).
\end{aligned}$$

□

The family $\mathbb{V}$ will be useful when studying $E_{\mathbb{P}}$. Moreover, semigroups will be constructed by first constructing their resolvents. Hence the following definition.

DEFINITION 2.2.6. *A sub-Markov resolvent* $\mathbb{V} = (V_\lambda)_{\lambda>0}$ *on* X *is a family of kernels* V_λ *on* X *such that, for every* $\lambda > 0$, *the kernel* λV_λ (!) *is sub-Markov and the resolvent equation* (2.1) *holds.*

For a while, let us assume that $\mathbb{V} = (V_\lambda)_{\lambda>0}$ is an arbitrary sub-Markov resolvent on X (not necessarily obtained from a sub-Markov semigroup). The resolvent equation implies that, for all $\lambda, \mu \in (0, \infty)$,

$$V_\lambda V_\mu = V_\mu V_\lambda \tag{2.2}$$

and that, for all $f \in \mathcal{B}^+(X)$,

$$\lambda \mapsto V_\lambda f \quad \text{is decreasing.} \tag{2.3}$$

Hence we may define a kernel V_0 on X by

$$V_0 f := \sup_{\lambda>0} V_\lambda f = \lim_{\lambda\to 0} V_\lambda f \qquad (f \in \mathcal{B}^+(X)). \tag{2.4}$$

V_0 is called the *potential kernel of* $\mathbb{V}$. By the resolvent equation, for every $\lambda > 0$,

$$V_0 = V_\lambda + \lambda V_\lambda V_0 \quad \text{and} \quad V_0 V_\lambda = V_\lambda V_0. \tag{2.5}$$

Let us observe that, for every $\alpha > 0$,

$$\mathbb{V}^\alpha := (V_{\alpha+\lambda})_{\lambda>0} \tag{2.6}$$

is a sub-Markov resolvent on X. In some situations, it is useful to pass from $\mathbb{V}$ to $\mathbb{V}^\alpha$, $\alpha > 0$, since the potential kernel of $\mathbb{V}^\alpha$ is the *bounded* kernel V_α.

Let us now assume again that $\mathbb{V}$ is the resolvent of a sub-Markov semigroup $\mathbb{P}$. Then, for all $f \in \mathcal{B}^+$ and $x \in X$,

$$V_0 f(x) = \int_0^\infty P_t f(x)\, dt. \tag{2.7}$$

Therefore V_0 is also called the *potential kernel of* $\mathbb{P}$. For every $\alpha > 0$, $\mathbb{P}^\alpha := (e^{-\alpha t} P_t)_{t>0}$ is a measurable sub-Markov semigroup, $\mathbb{V}^\alpha$ is the resolvent of $\mathbb{P}^\alpha$, and its potential kernel is the bounded kernel V_α.

REMARK 2.2.7. The potential kernels for the semigroups in the Examples 2.2.2 are easily determined.

1. Brownian semigroup on $\mathbb{R}^d$, $d \geq 3$ (cf. Corollary 3.1.6 and the subsequent discussion):

$$V_0 f(x) = c_d \int \frac{f(y)}{|x-y|^{d-2}}\, dy, \qquad \text{where } c_d := \frac{\Gamma((d/2)-1)}{2\pi^{d/2}}.$$

2. Translation semigroup on $\mathbb{R}$:

$$V_0 f(x) = \int_{-\infty}^{x} f(t)\,dt.$$

3. Pseudo-Poisson semigroup:

$$V_0 = \sum_{k=0}^{\infty} P^k.$$

If $u \in S_{\mathbb{P}}$, then, for every $\lambda > 0$,

$$\lambda V_\lambda u = \lambda \int_0^\infty e^{-\lambda t} P_t u\,dt \le \lambda \int_0^\infty e^{-\lambda t} u\,dt = u, \tag{2.8}$$

and $\sup_{\lambda>0} \lambda V_\lambda u = u$ provided $u \in E_{\mathbb{P}}$.

Returning again to a general sub-Markov resolvent we hence define the following.

DEFINITION 2.2.8. *A function $u \in \mathcal{B}^+(X)$ is called* $\mathbb{V}$-supermedian, *if*

$$\lambda V_\lambda u \le u \qquad \text{for every } \lambda > 0.$$

It is called $\mathbb{V}$-excessive, *if*

$$\sup_{\lambda>0} \lambda V_\lambda u = u.$$

Let $S_{\mathbb{V}}$ and $E_{\mathbb{V}}$ denote the set of all $\mathbb{V}$-supermedian functions and $\mathbb{V}$-excessive functions, respectively.

Since $S_{\mathbb{V}} = \bigcap_{\lambda>0} S_{\lambda V_\lambda}$, we shall be able to prove that $S_{\mathbb{V}}$ is a potential cone. We start with a simple observation:

LEMMA 2.2.9. *Let $f \in \mathcal{B}^+(X)$ and $\mu > 0$ such that $\mu V_\mu f \le f$. Then*

$$\lambda V_\lambda f \le \mu V_\mu f \qquad \text{for all } \lambda \in (0,\mu).$$

In particular, for every $u \in S_{\mathbb{V}}$, the function $\lambda \mapsto \lambda V_\lambda u$ is increasing, and

$$E_{\mathbb{V}} = \{u \in S_{\mathbb{V}} \colon \lim_{\lambda\to\infty} \lambda V_\lambda u = u\}.$$

Proof. Let $0 < \lambda < \mu$ and let us assume first that f is bounded. By the resolvent equation,

$$V_\lambda f = V_\mu f + (\mu-\lambda) V_\lambda V_\mu f \le \frac{f}{\mu} + (\mu-\lambda) V_\lambda(\frac{f}{\mu}).$$

Hence $\lambda V_\lambda \le f$ and even, using the identity $V_\lambda V_\mu = V_\mu V_\lambda$,

$$\lambda V_\lambda f = \lambda V_\mu f + \lambda(\mu-\lambda) V_\lambda V_\mu f \le \lambda V_\mu f + (\mu-\lambda) V_\mu f = \mu V_\mu f.$$

The general statement now follows applying the preceding considerations to the functions $f \wedge n$, $n \in \mathbb{N}$, and then letting n tend to infinity. □

PROPOSITION 2.2.10. *$S_{\mathbb{V}}$ is a σ-stable potential cone such that* $\inf u_n \in S_{\mathbb{V}}$ *for every sequence (u_n) in $S_{\mathbb{V}}$.*

Proof. It is immediately verified that $S_{\mathbb{V}}$ is a σ-stable convex cone and $\inf u_n \in S_{\mathbb{V}}$ for every sequence (u_n) in $S_{\mathbb{V}}$. It remains to show that $S_{\mathbb{V}}$ is a potential cone. To that end let $u, v \in S_{\mathbb{V}}$ and $f := 1_{\{v<\infty\}}(u-v)^+$. Replacing v by $u \wedge v$ we may assume that $v \le u$. We define

$$w_n := \inf\{w' \in S_{nV_n} \colon w' \ge f\} \qquad (n \in \mathbb{N}).$$

By Proposition 2.1.5, $w_n \in S_{nV_n}$ and there exist $v_n \in S_{nV_n}$ such that

$$u = v_n + w_n \qquad (n \in \mathbb{N}). \tag{2.9}$$

By Lemma 2.2.9, the sequence (S_{nV_n}) of convex cones is decreasing to $S_{\mathbb{V}}$ as $n \uparrow \infty$. Therefore the sequence (w_n) is increasing and $w_\infty := \sup w_n \in S_{\mathbb{V}}$, since, for every $n \in \mathbb{N}$, $w_\infty = \sup_{m \ge n} w_m \in S_{nV_n}$. Of course, $w_\infty \ge w_1 \ge f$ and hence

$$w_\infty \ge \inf\{w' \in S_{\mathbb{V}} \colon w' \ge f\} =: R_f.$$

On the other hand, if $w' \in S_{\mathbb{V}}$ such that $w' \ge f$, then $w' \in S_{nV_n}$ and hence $w' \ge w_n$, $n \in \mathbb{N}$. Therefore $R_f \ge \sup w_n = w_\infty$. Thus $R_f = w_\infty \in S_{\mathbb{V}}$.

Since the sequence (w_n) is increasing, (2.9) implies that the sequence $(v_n + (1/n)u)$ is decreasing. Its infimum v_∞ is contained in $S_{\mathbb{V}}$, since $v_n + (1/n)u \in S_{nV_n}$, $n \in \mathbb{N}$. By (2.9), for every $n \in \mathbb{N}$, $u + (1/n)u = v_n + (1/n)u + w_n$. Letting n tend to infinity, we see that $u = v_\infty + R_f$. □

PROPOSITION 2.2.11. 1. $E_{\mathbb{V}} = \{u \in S_{\mathbb{V}} \colon \lim_{\lambda\to\infty} \lambda V_\lambda u = u\}$.

2. *$E_{\mathbb{V}}$ is a σ-stable potential cone containing $V_0(\mathcal{B}^+(X))$.*

3. *For every $\lambda > 0$, $V_\lambda(S_{\mathbb{V}}) \subset E_{\mathbb{V}}$.*

4. *For every $u \in S_{\mathbb{V}}$, $\tilde{u} := \lim_{\lambda\to\infty} \lambda V_\lambda u$ is the largest $\mathbb{V}$- excessive minorant of u.*

Proof. (1) follows immediately from Lemma 2.2.9. In particular, $E_{\mathbb{V}}$ is a σ-stable convex cone. For every bounded $f \in \mathcal{B}^+(X)$,

$$V_0 f = V_\lambda f + \lambda V_\lambda V_0 f, \qquad \lambda > 0,$$

where $V_\lambda f \le \|f\| V_\lambda 1 \le \|f\|/\lambda$, and so $\sup_{\lambda>0} \lambda V_\lambda V_0 f = V_0 f$, that is, $V_0 f \in E_{\mathbb{V}}$. Given any $f \in \mathcal{B}^+(X)$, we now obtain that $V_0 f = \sup_n V_0(f \wedge n) \in E_{\mathbb{V}}$.

If $\lambda > 0$ and $u \in S_{\mathbb{V}}$ is bounded, then $V_\lambda u \in E_{\mathbb{V}}$, since, by the resolvent equation,

$$V_\lambda u - \mu V_\mu V_\lambda u = V_\mu u - V_\mu(\lambda V_\lambda u),$$

where $0 \le V_\mu u - V_\mu(\lambda V_\lambda u) \le V_\mu u \le u/\mu$. Given any $u \in S_{\mathbb{V}}$, we obtain that $V_\lambda u = \sup_m V_\lambda(u \wedge m) \in E_{\mathbb{V}}$. In particular, $\tilde{u} = \sup_n n V_n u \in E_{\mathbb{V}}$. Of course, $\tilde{u} \le u$ and, if $w \in E_{\mathbb{V}}$ such that $w \le u$, then $w = \tilde{w} \le \tilde{u}$.

To prove that $E_{\mathbb{V}}$ is a potential cone, let us finally consider $u, v \in E_{\mathbb{V}}$ and let $f := 1_{\{v<\infty\}}(u-v)^+$. By Proposition 2.2.10,

$$w_f := \inf\{s \in S_{\mathbb{V}} \colon s \ge f\} \in S_{\mathbb{V}}$$

and there exists $w \in S_{\mathbb{V}}$ such that $u = w + w_f$. Clearly,

$$u = \tilde{u} = \tilde{w} + \tilde{w}_f.$$

Since $u \le v + f \le v + w_f$, we see that $u \le \tilde{v} + \tilde{w}_f = v + \tilde{w}_f$ and hence $\tilde{w}_f \ge f$. So

$$\tilde{w}_f \ge \inf\{s \in E_{\mathbb{V}} \colon s \ge f\} =: R_f.$$

Further, since $E_{\mathbb{V}} \subset S_{\mathbb{V}}$,

$$R_f \ge \inf\{s \in S_{\mathbb{V}} \colon s \ge f\} = w_f \ge \tilde{w}_f.$$

Thus $R_f = \tilde{w}_f \in E_{\mathbb{V}}$, and the proof is finished. □

In many cases $E_{\mathbb{V}}$ can be completely characterized by the potential kernel V.

THEOREM 2.2.12. *Let $\mathbb{V}$ be a sub-Markov resolvent on X such that its potential kernel V_0 is* proper*, that is, there exists $g \in \mathcal{B}(X)$ such that $g > 0$ and $V_0 g < \infty$. Then $E_{\mathbb{V}}$ is the set of all increasing limits of functions in $V_0(\mathcal{B}^+(X))$.*

Proof. By Proposition 2.2.11, every limit of an increasing sequence in $V_0(\mathcal{B}^+(X))$ is contained in $E_{\mathbb{V}}$. To prove the converse, let us fix $u \in E_{\mathbb{V}}$ and $g \in \mathcal{B}(X)$ such that $g > 0$, $V_0 g < \infty$. We define

$$u_n := \min\{u, n, nV_0 g\}, \qquad n \in \mathbb{N}.$$

Of course, (u_n) is an increasing sequence in $\mathcal{B}_b^+(X)$. For every $\lambda > 0$,

$$V_\lambda u \le V_0 u = \sup_m V_0(u \wedge (mg_0)) \le \sup_m mV_0 g = 0 \quad \text{ on } \{V_0 g = 0\},$$

and hence $u = \sup_{\lambda>0} \lambda V_\lambda u = 0$ on $\{V_0 g = 0\}$. Therefore $\sup u_n = u$.

Further, $u, nV_0 g \in E_{\mathbb{V}} \subset S_{\mathbb{V}}$ and $1 \in S_{\mathbb{V}}$. So (u_n) is a sequence in $S_{\mathbb{V}}$ and hence, by Lemma 2.2.9, each sequence $(mV_m u_n)_{m\in\mathbb{N}}$, $n \in \mathbb{N}$, is increasing. Thus the sequence $(nV_n u_n)$ is increasing and

$$\sup nV_n u_n = \sup_m \sup_n mV_m u_n = \sup_m mV_m u = u.$$

For every $n \in \mathbb{N}$, let

$$f_n := n(u_n - nV_n u_n).$$

Obviously, the proof is finished once we have shown that,

$$V_0 f_n = nV_n u_n \qquad (n \in \mathbb{N}). \tag{2.10}$$

Clearly, (2.10) follows from (2.5), if V_0 is bounded. Knowing only that $V_0 g < \infty$, we have to go back to the resolvent equation itself. Let $n \in \mathbb{N}$. For every $\lambda > 0$,

$$V_\lambda f_n = n(V_\lambda u_n - (n-\lambda)V_\lambda V_n u_n) - n\lambda V_\lambda V_n u_n = nV_n u_n - n\lambda V_\lambda V_n u_n,$$

where

$$n\lambda V_\lambda V_n u_n \le \lambda V_\lambda u_n \le n\lambda V_\lambda V_0 g = n(V_0 g - V_\lambda g).$$

Since $\lim_{\lambda\to 0} V_\lambda g = V_0 g$, we conclude that $V_0 f_n = \lim_{\lambda\to 0} V_\lambda f_n = nV_n u_n$. Hence (2.10) holds, and the proof is finished. □

LEMMA 2.2.13. $E_{\mathbb{V}} = \bigcap_{\alpha>0} E_{\mathbb{V}^\alpha}$.

Proof. Let $u \in E_{\mathbb{V}}$ and $\alpha > 0$. Then, for every $\lambda > 0$,

$$\lambda V_{\alpha+\lambda} u \le (\lambda+\alpha) V_{\alpha+\lambda} u \le u,$$

and $\lim_{\lambda\to\infty} \lambda V_{\alpha+\lambda} u = \lim_{\lambda\to\infty} (\lambda+\alpha) V_{\alpha+\lambda} u = u$, since $\lim_{\lambda\to\infty} \lambda/(\lambda+\alpha) = 1$. Hence $u \in E_{\mathbb{V}^\alpha}$.

Conversely, let $u \in E_{\mathbb{V}^\alpha}$ for every $\alpha > 0$. Then, for all $0 < \alpha < \lambda$,

$$(\lambda-\alpha) V_\lambda u = (\lambda-\alpha) V_{\alpha+(\lambda-\alpha)} u \le u,$$

hence $\lambda V_\lambda u \le u$ and $\lim_{\lambda\to\infty} \lambda V_\lambda u = \lim_{\lambda\to\infty} (\lambda-\alpha) V_{\alpha+(\lambda-\alpha)} u = u$. Thus $u \in E_{\mathbb{V}}$. □

COROLLARY 2.2.14. *If $\mathbb{V}$ is the resolvent of a measurable semigroup $\mathbb{P}$, then $S_{\mathbb{P}} \subset S_{\mathbb{V}}$ and $E_{\mathbb{P}} = E_{\mathbb{V}}$.*

Proof. We already observed that $S_{\mathbb{P}} \subset S_{\mathbb{V}}$ and $E_{\mathbb{P}} \subset E_{\mathbb{V}}$ (see (2.8)).

If $f \in \mathcal{B}^+(X)$, then, for every $s > 0$,

$$P_s V_0 f = \int_0^\infty P_s P_t f \, dt = \int_0^\infty P_{s+t} f \, dt = \int_s^\infty P_t f \, dt, \tag{2.11}$$

and hence $\sup_{s>0} P_s V_0 f = V_0 f$, that is, $V_0 f \in E_{\mathbb{P}}$. If V_0 is a proper kernel, the inclusion $E_{\mathbb{V}} \subset E_{\mathbb{P}}$ then follows by Proposition 2.2.12.

In the general case, we therefore obtain that $E_{\mathbb{V}^\alpha} \subset E_{\mathbb{P}^\alpha}$ for every $\alpha > 0$. Thus, by Lemma 2.2.13,

$$E_{\mathbb{V}} = \bigcap_{\alpha>0} E_{\mathbb{V}^\alpha} \subset \bigcap_{\alpha>0} E_{\mathbb{P}^\alpha} = E_{\mathbb{P}}.$$

□

REMARK 2.2.15. $S_{\mathbb{P}}$ may be a proper subset of $S_{\mathbb{V}}$. Indeed, if $\mathbb{V}$ is the resolvent of translation on $\mathbb{R}$, then, of course, $u := 1_{\{0\}} \notin S_{\mathbb{P}}$, but $u \in S_{\mathbb{V}}$, since $V_0 u = 0$.

2.3 Excessive functions and balayage spaces

DEFINITION 2.3.1. *A sub-Markov resolvent* $\mathbb{V} = (V_\lambda)_{\lambda>0}$ (a *sub-Markov semigroup* $\mathbb{P} = (P_t)_{t>0}$*, respectively) is called* right continuous*, if, for every* $\varphi \in \mathcal{K}(X)$,

$$\lim_{\lambda\to\infty} \lambda V_\lambda \varphi = \varphi \qquad (\lim_{t\to 0} P_t\varphi = \varphi, \text{ respectively}).$$

It is easily verified that every right continuous semigroup is measurable. Moreover, let us observe that the following two results have obvious analogs for sub-Markov semigroups.

LEMMA 2.3.2. *A sub-Markov resolvent* $\mathbb{V} = (V_\lambda)_{\lambda>0}$ *is right continuous if and only if*

$$\lim_{\lambda\to\infty} \lambda V_\lambda(x, W) = 1 \qquad \text{for all open } W \text{ in } X \text{ and } x \in W. \tag{3.1}$$

If $\mathbb{V} = (V_\lambda)_{\lambda>0}$ *is right continuous, then* $1 \in E_{\mathbb{V}}$*, every lower semicontinuous* $\mathbb{V}$*-supermedian function is* $\mathbb{V}$*-excessive, and* $\lim_{\lambda\to\infty} \lambda V_\lambda f = f$ *for every* $f \in \mathcal{C}_b(X)$.

Proof. Let us suppose first that $\mathbb{V}$ is right continuous. Given $x \in X$ and an open neighborhood W of x, we choose $\varphi \in \mathcal{K}(X)$ such that $0 \le \varphi \le 1_W$ and $\varphi(x) = 1$. Then $0 \le \lambda V_\lambda \varphi(x) \le \lambda V_\lambda(x, W) \le 1$ for every $\lambda > 0$. So (3.1) follows, since $\lim_{\lambda\to\infty} \lambda V_\lambda \varphi(x) = \varphi(x) = 1$.

Now let $u \in S_{\mathbb{V}}$ be lower semicontinuous, $x \in X$, and $a < u(x)$. Then x is contained in the open set $W := \{u > a\}$ and hence, $\lambda V_\lambda u(x) \ge a\lambda V_\lambda(x, W)$ for every $\lambda > 0$. Therefore $\lim_{\lambda\to\infty} \lambda V_\lambda u(x) \ge a$ proving that $u \in E_{\mathbb{V}}$. In particular, $1 \in E_{\mathbb{V}}$.

Finally, let us suppose that (3.1) holds. Let $f \in \mathcal{C}_b(X)$, $|f| \le 1$, $x \in X$, and $\varepsilon > 0$. There exists an open neighborhood W of x such that $|f - f(x)| < \varepsilon$ on W. Let $\lambda > 0$ with $\lambda V_\lambda(x, W) > 1 - \varepsilon$. Then $1 - \varepsilon \le \lambda V_\lambda(x, X) \le 1$ and $\lambda V_\lambda(x, X \setminus W) \le 1 - \lambda V_\lambda(x, W) < \varepsilon$. Hence

$$|\lambda V_\lambda f(x) - f(x)| \le \lambda V_\lambda |f - f(x)|(x) + \varepsilon \le \lambda V_\lambda(1_W |f - f(x)|)(x) + 3\varepsilon \le 4\varepsilon.$$

So $\lim_{\lambda\to\infty} \lambda V_\lambda f = f$. In particular, $\mathbb{V}$ is right continuous. □

LEMMA 2.3.3. *If* $\mathbb{V} = (V_\lambda)_{\lambda>0}$ *is a sub-Markov resolvent on* X *such that* $(X, E_{\mathbb{V}})$ *is a balayage space, then, for every* $\varphi \in \mathcal{K}(X)$,

$$\lim_{\lambda\to\infty} \lambda V_\lambda \varphi = \varphi \qquad \text{locally uniformly.} \tag{3.2}$$

Proof. For every $p \in \mathcal{P}$, the functions $\lambda V_\lambda p$, $\lambda > 0$, are $\mathbb{V}$-excessive, hence lower semicontinuous, and they are increasing to p as $\lambda \to \infty$. By Dini's theorem, this convergence is locally uniform. Now (3.2) follows by the approximation in Proposition 6.2.2. □

Here is our first main result.

THEOREM 2.3.4. *For every sub-Markov resolvent* $\mathbb{V}$ *on* X*, the following statements are equivalent.*

1. *$(X, E_{\mathbb{V}})$ is a balayage space.*

2. *The resolvent $\mathbb{V}$ is right continuous, and $E_{\mathbb{V}}$ satisfies (B_4).*

Moreover, if $(X, E_{\mathbb{V}})$ is a balayage space, then $\lim_{\lambda\to\infty} \lambda V_\lambda(x, U) = 1$ for all $x \in X$ and fine neighborhoods $U \in \mathcal{B}(X)$ of x. In particular, $E_{\mathbb{V}}$ is the set of all finely lower semicontinuous functions in $S_{\mathbb{V}}$.

Proof. (1) $\Rightarrow$ (2): Lemma 2.3.3.

(2) $\Rightarrow$ (1): Since $E_{\mathbb{V}}$ satisfies (B_4), it is a convex cone of lower semicontinuous functions. Let $U \in \mathcal{B}(X)$ be a finely open set (with respect to $E_{\mathbb{V}}$) and $x \in U$. We want to prove that

$$\lim_{\lambda\to\infty} \lambda V_\lambda(x, U) = 1 \tag{3.3}$$

By Lemma 1.1.1, there exists a compact neighborhood K of x, a function $u \in E_{\mathbb{V}}$ and $\alpha \in \mathbb{R}$ such that $u(x) < \alpha$ and $K \cap \{u \le \alpha\} \subset U$. If $u(x) = 0$, then $\lambda V_\lambda(x, \{u > 0\}) = 0$, and hence $\lambda V_\lambda(K \cap \{u \le \alpha\}) = \lambda V_\lambda(x, K) \to 1$ as $\lambda \to \infty$. So let $u(x) > 0$ and $\beta \in (0, u(x))$. Then there exists a compact neighborhood L of x such that $L \subset K$ and $u > \beta$ on L. For every $\lambda > 0$,

$$\begin{aligned} u(x) &\ge \lambda V_\lambda u(x) \ge \alpha\lambda V_\lambda(x, K \cap \{u > \alpha\}) + \beta\lambda V_\lambda(x, L \cap \{u \le \alpha\}) \\ &= (\alpha - \beta)\lambda V_\lambda(x, K \cap \{u > \alpha\}) \\ &\qquad + \beta\lambda V_\lambda(x, (K \cap \{u > \alpha\}) \cup (L \cap \{u \le \alpha\})) \\ &\ge (\alpha - u(x))\lambda V_\lambda(x, K \cap \{u > \alpha\}) + \beta\lambda V_\lambda(x, L). \end{aligned}$$

Since $\lim_{\lambda\to\infty} \lambda V_\lambda(x, L) = 1$ and $\beta \in (0, u(x))$ was arbitrary, we conclude that $\lim_{\lambda\to\infty} \lambda V_\lambda(x, (K \cap \{u > \alpha\}) = 0$, and hence

$$1 = \lim_{\lambda\to\infty} \lambda V_\lambda(x, K) = \lim_{\lambda\to\infty} \lambda V_\lambda(x, (K \cap \{u \le \alpha\}) \le \liminf_{\lambda\to\infty} \lambda V_\lambda(x, U).$$

Thus (3.3) follows (since λV_λ is sub-Markov). Consequently (cf. Lemma 2.3.2),

$$\lim_{\lambda\to\infty} \lambda V_\lambda \psi \ge \psi \qquad \text{for every finely lower semicontinuous } \psi \in \mathcal{B}^+(X). \tag{3.4}$$

This shows that every finely lower semicontinuous function in $S_{\mathbb{V}}$ is excessive, and hence $E_{\mathbb{V}}$ is the set of all finely lower semicontinuous functions in $S_{\mathbb{V}}$.

By Proposition 2.2.11, $E_{\mathbb{V}}$ is a σ-stable potential cone. In particular, (B_3) holds by Lemma 1.1.9.

It remains to verify (B_2). So let $\mathcal{V} \subset E_{\mathbb{V}}$, $\mathcal{V} \ne \emptyset$, $v := \inf \mathcal{V}$. By Lemma 6.1.2, there exists a countable subset $\mathcal{V}_0$ of $\mathcal{V}$ such that its infimum v_0 satisfies $\hat{v}_0 = \hat{v}$. Then $v_0 \in S_{\mathbb{V}}$ and, by Proposition 2.2.11, $\tilde{v}_0 := \lim_{\lambda\to\infty} \lambda V_\lambda v_0 \in E_{\mathbb{V}}$, $\tilde{v}_0 \le v_0$. Since the functions in $E_{\mathbb{V}}$ are lower semicontinuous, we obtain that

$$\tilde{v}_0 \le \widehat{v}_0 = \widehat{v} \le \widehat{v}^f \le \widehat{v}_0^f.$$

On the other hand, by (3.4), $\widehat{v}_0^f \le \lim_{\lambda\to\infty} \lambda V_\lambda \widehat{v}_0^f \le \lim_{\lambda\to\infty} \lambda V_\lambda v_0 = \tilde{v}_0$. Thus $\hat{v}^f = \tilde{v}_0 \in E_{\mathbb{V}}$ proving (B_2). $\square$

LEMMA 2.3.5. *Let $\mathbb{V} = (V_\lambda)_{\lambda>0}$ be a right continuous sub-Markov resolvent on X such that V_0 is proper. Then $E_\mathbb{V}$ is linearly separating.*

Proof. Let us fix $g_0 \in \mathcal{B}^+(X)$, $g_0 > 0$, such that $V_0 g_0 < \infty$. Let $x, y \in X$, $x \neq y$, and $a \in [0, \infty)$. We choose $\varphi \in \mathcal{K}(X)$ such that $\varphi(x) \neq a\varphi(y)$. Since $\lim_{\lambda\to\infty} \lambda V_\lambda \varphi = \varphi$, there exists $\lambda > 0$ such that $V_\lambda\varphi(x) \neq aV_\lambda\varphi(y)$. Since $g_n := \varphi \wedge (ng_0) \uparrow \varphi$ as $n \to \infty$, there exists $n \in \mathbb{N}$ such that $V_\lambda g_n(x) \neq aV_\lambda g_n(y)$. By Proposition 2.2.11, the functions $u := V_0 g_n$ and $v := V_0(\lambda V_\lambda g_n)$ are $\mathbb{V}$-excessive. Moreover, by the resolvent equation, $u = V_\lambda g_n + v$, where $u \leq nV_0 g_0 < \infty$. Thus $u(x) \neq au(y)$ or $v(x) \neq av(y)$. □

DEFINITION 2.3.6. *A* strong Feller kernel *on X is a bounded kernel V on X such that $V(\mathcal{B}_b(X)) \subset \mathcal{C}_b(X)$. A family $(V_i)_{i\in I}$ of kernels on X is* strong Feller, *if all kernels V_i, $i \in I$, are strong Feller.*

COROLLARY 2.3.7. *Let $\mathbb{V} = (V_\lambda)_{\lambda>0}$ be a right continuous strong Feller sub-Markov resolvent.*

Then $(X, E_\mathbb{V})$ is a balayage space provided the potential kernel V_0 of $\mathbb{V}$ is proper and there exists $u \in E_\mathbb{V}$, $u > 0$, such that u vanishes at infinity or, more generally, such that u/v vanishes at infinity for some strictly positive $v \in E_\mathbb{V} \cap \mathcal{C}(X)$.

Proof. For every $u \in E_\mathbb{V}$, u is the limit of the increasing sequence $(nV_n(u \wedge n))$ which, by Proposition 2.2.11 and our assumption on V_n, is contained in $E_\mathbb{V} \cap \mathcal{C}(X)$. Let $u, v \in E_\mathbb{V}$ be strictly positive such that $v \in \mathcal{C}(X)$ and u/v vanishes at infinity (if u vanishes at infinity, we take $v := 1$). There exists a sequence (u_n) in $E_\mathbb{V} \cap \mathcal{C}(X)$ which is increasing to u. Let us choose a compact sets K_m, $m \in \mathbb{N}$, covering X. For every $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that $u_{n_m} > 0$ on K_m. Then

$$u_0 := \sum_{m=1}^{\infty} 2^{-m} u_{n_m} \wedge 1 \in E_\mathbb{V} \cap \mathcal{C}(X).$$

Obviously, $0 < u_0 \leq u$. In particular, u_0/v vanishes at infinity. An application of Lemma 2.3.5 and Theorem 2.3.4 finishes the proof. □

COROLLARY 2.3.8. *For every sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X the following holds.*

1. *$(X, E_\mathbb{P})$ is a balayage space if and only if $\mathbb{P}$ is right continuous and $E_\mathbb{P}$ satisfies* (B_4).

2. *If $\mathbb{P}$ is right continuous, the potential kernel of $\mathbb{P}$ is proper, and the resolvent $\mathbb{V}$ of $\mathbb{P}$ (or even $\mathbb{P}$ itself) is strong Feller, then $E_\mathbb{P}$ satisfies* (B_4) *provided there exists $u \in E_\mathbb{V}$, $u > 0$, such that u vanishes at infinity or, more generally, such that u/v vanishes at infinity for some strictly positive $v \in E_\mathbb{V} \cap \mathcal{C}(X)$.*

3. *If $(X, E_\mathbb{P})$ is a balayage space, then $\lim_{t\to 0} P_t(x, U) = 1$ for all $x \in X$ and fine neighborhoods $U \in \mathcal{B}(X)$ of x. In particular, $E_\mathbb{P}$ is the set of all finely lower semicontinuous functions in $S_\mathbb{P}$.*

Proof. Let us suppose first that $(X, E_{\mathbb{P}})$ is a balayage space. By Proposition 6.2.2, $\lim_{t\to 0} P_t\varphi = \varphi$ for every $\varphi \in \mathcal{K}(X)$. Proceeding as in the proof of Theorem 2.3.4 (replacing λV_λ by P_t and letting t tend to 0) we obtain (3).

The remaining statements follow from Theorem 2.3.4 and the fact that $E_{\mathbb{P}} = E_{\mathbb{V}}$ by Corollary 2.2.14. □

REMARK 2.3.9. It can be shown that, for every balayage space $(X, \mathcal{W})$ with $1 \in \mathcal{W}$, there exist (many) strong Feller sub-Markov resolvents $\mathbb{V} = (V_\lambda)_{\lambda>0}$ and sub-Markov semigroups $\mathbb{P}$ such that $\mathcal{W} = E_{\mathbb{V}} = E_{\mathbb{P}}$.

Let us finish this section looking at the examples we have until now (see Examples 2.2.2).

EXAMPLES 2.3.10. 1. *Classical potential theory.* Let $\mathbb{P}$ be the Brownian semigroup on $\mathbb{R}^d$, $d \geq 3$. Then $(\mathbb{R}^d, E_{\mathbb{P}})$ is a balayage space having the following properties:

(i) $E_{\mathbb{P}}$ is the set of all increasing limits of sequences of Newtonian potentials G^μ (where $\mu \in \mathcal{M}(\mathbb{R}^d)$, $G^\mu(x) := \int |x-y|^{2-d}\, d\mu(y)$, and $G^\mu \not\equiv \infty$).

(ii) $\lambda^d(U) > 0$ for every finely open Borel set $U \neq \emptyset$ in $\mathbb{R}^d$.

(iii) Every countable subset of $\mathbb{R}^d$ is finely closed. In particular, only finite subsets of $\mathbb{R}^d$ are finely compact.

Indeed, if $\mu \in \mathcal{M}(\mathbb{R}^d)$, then G^μ is lower semicontinuous (by Fatou's lemma) and $\mathbb{P}$-supermedian (by Fubini's theorem). Hence $G^\mu \in E_{\mathbb{P}}$ (see Lemma 2.3.2 and the preceding remark). In particular, $G(\cdot, 0) \wedge 1 \in E_{\mathbb{P}} \cap \mathcal{C}_0(X)$. Thus, by Corollary 2.3.8, $(\mathbb{R}^d, E_{\mathbb{P}})$ is a balayage space.

To finish the proof of (i), let us consider an arbitrary $v \in E_{\mathbb{P}}$. By Proposition 2.2.12 and Corollary 2.2.14, there exists a sequence (f_n) in $\mathcal{B}_b^+(X)$ such that $V_0 f_n \uparrow v$, where $V_0 f_n = G^{f_n \lambda^d}$, $n \in \mathbb{N}$.

Statement (ii) follows immediately from the fact that, for every finely open Borel set U and for every $x \in U$, $\lim_{t\to 0} P_t(x, U) = 1$ (see Corollary 2.3.8).

Finally, let $A := \{x_m : m \in \mathbb{N}\}$ be a countable set in $\mathbb{R}^d$, $x_n \neq x_m$ if $n \neq m$, and $x \in \mathbb{R}^d \setminus A$. Let us define $u \in E_{\mathbb{V}}$ by

$$u := \sum_{m=1}^{\infty} \big(2^m G(x, x_m)\big)^{-1} G(\cdot, x_m).$$

Then $u(x) = 1$ and $u(x_m) = \infty$ for every $m \in \mathbb{N}$. Hence $V := \{u < \infty\}$ is a fine neighborhood of x such that $V \cap A = \emptyset$. So A is closed. If $A \subset B \subset \mathbb{R}^d$, then the sets $(\mathbb{R}^d \setminus A) \cup \{x_n\} = \mathbb{R}^d \setminus (A \setminus \{x_n\})$, $n \in \mathbb{N}$, form an f-open covering of B (its union is $\mathbb{R}^d$) which obviously has no finite subcovering. Thus (iii) holds.

2. *Translation to the left.* $(\mathbb{R}, E_{\mathbb{T}})$ is a balayage space. Its fine topology is generated by the half-open intervals $(a, b]$, $-\infty < a < b < \infty$.

Indeed, let us first recall that $E_{\mathbb{T}}$ is the set of all increasing functions $u \colon \mathbb{R} \to [0, \infty]$ which are lower semicontinuous (left continuous). Of course, the functions 1 and e^x linearly separate the points in $\mathbb{R}$. Defining $u_0(x) := 1 \wedge e^x$ and $v_0(x) := 1 \vee e^x$

we have $u_0, v_0 \in E_{\mathbb{T}}$ and $u_0/v_0 \in \mathcal{C}_0(\mathbb{R})$. Given any $u \in E_{\mathbb{T}}$, we may choose $\varphi_n \in \mathcal{K}^+(X)$, $n \in \mathbb{N}$, such that $\varphi_n \uparrow u$ and then the functions u_n, defined by $u_n(x) := \sup_{y\le x} \varphi_n(y)$, are contained in $E_{\mathbb{T}}$ and satisfy $u_n \uparrow u$.

3. *Pseudo-Poisson semigroup.* Let P be a sub-Markov kernel on X, where X is a countable set with discrete topology. Then (X, S_P) is a balayage space if S_P separates the points of X (or the kernel $\sum_{k=0}^{\infty} P^k$ is proper). (For the existence of strictly positive $u_0, v_0 \in S_P$ such that $u_0/v_0 \in C_0(X)$ see Example 5.1.3,3 and Proposition 5.3.10.)

Chapter 3

Examples given by semigroups

3.1 Subordination, Riesz potentials

DEFINITION 3.1.1. *A family* $(\mu_t)_{t>0}$ *of measures on* $(0,\infty)$ *is called a* (*weakly continuous*) convolution semigroup *on* $(0,\infty)$*, provided* $\mu_t(1) \le 1$*,* $\mu_{s+t} = \mu_s * \mu_t$ *for all* $s,t \in (0,\infty)$*, and* $\lim_{t\to 0} \mu_t = \varepsilon_0$ (*that is,* $\lim_{t\to 0} \mu_t(f) = f(0)$ *for all functions* $f \in \mathcal{K}([0,\infty))$)*.*

EXAMPLES 3.1.2. 1. For every $\alpha > 0$, $(e^{-\alpha t}\varepsilon_t)_{t>0}$ is a convolution semigroup on $(0,\infty)$.

2. For every $\alpha \in (0,2]$, there exists a unique convolution semigroup $(\eta_t^\alpha)_{t>0}$ of probability measures on $(0,\infty)$ such that, for every $t > 0$, the Laplace transform of η_t^α is given by

$$\mathcal{L}\eta_t^\alpha(s) = \exp(-ts^{\alpha/2}) \qquad (s > 0).$$

$(\eta_t^\alpha)_{t>0}$ is called *one-sided stable semigroup.*

Obviously, $\eta_t^2 = \varepsilon_t$ for every $t > 0$. To prove the existence of $(\eta_t^\alpha)_{t>0}$ for arbitrary $\alpha \in (0,2]$, let us fix $t > 0$ and define $f\colon (0,\infty) \to (0,\infty)$ by

$$f(s) := \exp(-ts^{\alpha/2}) \qquad (s > 0).$$

Since $\alpha/2 \le 1$, it is easily verified that, for every $m \in \mathbb{N}$, there are $a_{1m}, \dots, a_{mm} \ge 0$ such that

$$f^{(m)} = (-1)^m \sum_{l=1}^{m} a_{lm} s^{l\frac{\alpha}{2}-m} f,$$

and hence $(-1)^m f^{(m)} \ge 0$. By the theorem of Bernstein, there exists a unique measure η_t^α on $[0,\infty)$ such that $\mathcal{L}\eta_t^\alpha = f$. Moreover, $\eta_t^\alpha([0,\infty)) = \lim_{s\to 0} f(s) = 1$, $\eta_t^\alpha(\{0\}) = \lim_{s\to\infty} f(s) = 0$. So η_t^α is a probability measure on $(0,1)$.

Since obviously $\mathcal{L}\eta_{t_1}^\alpha \cdot \mathcal{L}\eta_{t_2}^\alpha = \mathcal{L}\eta_{t_1+t_2}^\alpha$, we obtain that $\eta_{t_1}^\alpha * \eta_{t_2}^\alpha = \eta_{t_1+t_2}^\alpha$ for all $t_1, t_2 > 0$. Finally, $\lim_{t\to 0} \mathcal{L}\eta_t^\alpha = 1$ and hence $\lim_{t\to 0} \eta_t^\alpha = \varepsilon_0$. Thus $(\eta_t^\alpha)_{t>0}$ is a convolution semigroup.

DEFINITION 3.1.3. *Given a convolution semigroup* $(\mu_t)_{t>0}$ *on* $(0,\infty)$ *and a measurable sub-Markov semigroup* $\mathbb{P} = (P_t)_{t>0}$ *on* X*, let* $\mathbb{P}^\mu = (P_t^\mu)_{t>0}$*, where*

$$P_t^\mu f(x) := \int P_s f(x)\, d\mu_t(s) \qquad (f \in \mathcal{B}^+(X)). \tag{1.1}$$

$\mathbb{P}^\mu$ *is called the* (*sub-Markov*) semigroup subordinated to $\mathbb{P}$ by means of $(\mu_t)_{t>0}$.

This definition is justified since, of course, all kernels P_t^μ are sub-Markov, and for all $t_1, t_2 > 0$ and $f \in \mathcal{B}^+(X)$,

$$
\begin{aligned}
P_{t_1}^\mu P_{t_2}^\mu f &= \int_0^\infty P_{s_1}(P_{t_2}^\mu f)\, d\mu_{t_1}(s_1) \\
&= \int_0^\infty \Big(\int_0^\infty P_{s_1} P_{s_2} f\, d\mu_{t_2}(s_2)\Big)\, d\mu_{t_1}(s_1) = \int_0^\infty \Big(\int_0^\infty P_{s_1+s_2} f\, d\mu_{t_2}(s_2)\Big)\, d\mu_{t_1}(s_1) \\
&= \int_0^\infty P_s f\, d(\mu_{t_1} * \mu_{t_2})(s) = \int_0^\infty P_s f\, d(\mu_{t_1+t_2})(s) = P_{t_1+t_2}^\mu f,
\end{aligned}
$$

proving that $\mathbb{P}^\mu$ is a semigroup.

Moreover,

$$E_{\mathbb{P}} \subset E_{\mathbb{P}^\mu}. \tag{1.2}$$

Indeed, let $u \in E_{\mathbb{V}}$. Obviously, (1.1) implies that $P_t^\mu u \le u$ for every $t > 0$. Let $x \in X$, $a < u(x)$, and $b \in (0,1)$. Then there exist $s, t \in (0,\infty)$ such that $P_s u(x) > a$ and $\mu_t((0,s)) > b$. Since $r \mapsto P_r u$ is decreasing, we obtain that

$$P_t^\mu u(x) \ge P_s u(x) \mu_t((0,s)) \ge ab.$$

This implies that $\sup_{t>0} P_t^\mu u \ge u$. Thus $u \in E_{\mathbb{P}^\mu}$.

Finally, it is immediately verified (using Lebesgue's theorem) that $\mathbb{P}^\mu$ is right continuous (strong Feller, respectively), if $\mathbb{P}$ is right continuous (strong Feller, respectively).

The following general result is an immediate consequence of Corollary 2.3.8.

PROPOSITION 3.1.4. *Let $\mathbb{P}$ be a strong Feller semigroup on X such that $(X, E_{\mathbb{P}})$ is a balayage space, and let $(\mu_t)_{t>0}$ be a convolution semigroup on $(0,\infty)$. Then $\mathbb{P}^\mu$ is a strong Feller semigroup, $(X, E_{\mathbb{P}^\mu})$ is a balayage space, and $E_{\mathbb{P}} \subset E_{\mathbb{P}^\mu}$.*

DEFINITION 3.1.5. *If $\mathbb{P}$ is the Brownian semigroup on $\mathbb{R}^d$, $0 < \alpha < 2 \wedge d$, the sub-Markov semigroup subordinated to $\mathbb{P}$ by means of $(\eta_t^\alpha)_{t>0}$ is the* symmetric stable semigroup $\mathbb{P}^{\eta^\alpha} = (P_t^{\eta^\alpha})_{t>0}$ of index α on $\mathbb{R}^d$.

To avoid double superscripts we may write η, α instead of η^α.

Knowing that the Brownian semigroup on $\mathbb{R}^d$, $d \ge 3$, leads to a balayage space, Proposition 3.1.4 immediately yields the following result (if $d \le 2$, we have to go back to Corollary 2.3.4, see Theorem 3.1.7).

COROLLARY 3.1.6. *If $d \ge 3$, then $(\mathbb{R}^d, E_{\mathbb{P}^{\eta,\alpha}})$ is a balayage space.*

To cover also the case $d \le 2$ and to obtain further information on $\mathbb{P}^{\eta,\alpha}$-excessive functions, we intend to prove that the potential kernel $V^{\eta,\alpha}$ of $\mathbb{P}^{\eta,\alpha}$ is given by

$$V^{\eta,\alpha} f = c_d^\alpha k_\alpha * f, \qquad \text{where } k_\alpha(x) := |x|^{\alpha-d}$$

(and c_d^α is a constant). To that end we define a measure κ_α on $(0,\infty)$ by

$$\kappa_\alpha := \int_0^\infty \eta_t^\alpha\, dt. \tag{1.3}$$

Then

$$\mathcal{L}\kappa_\alpha(s) = \int_0^\infty \mathcal{L}\eta_t^\alpha(s)\,dt = \int_0^\infty \exp(-t\,s^{\alpha/2})\,dt = s^{-\alpha/2}.$$

On the other hand, for every $s > 0$,

$$\int_0^\infty e^{-ts} t^{(\alpha/2)-1}\,dt = s^{-\alpha/2}\int_0^\infty e^{-t} t^{(\alpha/2)-1}\,dt = \Gamma(\alpha/2) s^{-\alpha/2}.$$

Therefore

$$\kappa_\alpha = \Gamma(\alpha/2)^{-1} t^{(\alpha/2)-1}\lambda_{(0,\infty)} \tag{1.4}$$

(where $\lambda_{(0,\infty)}$ denotes Lebesgue measure on $(0,\infty)$). Hence, for every $f \in \mathcal{B}^+(\mathbb{R}^d)$,

$$V^{\eta,\alpha} f = \int_0^\infty P_t^{\eta,\alpha} f\,dt = \int_0^\infty \Big(\int_0^\infty P_s f\,d\eta_t^\alpha(s)\Big)\,dt = \int_0^\infty g_s * f\,d\kappa_\alpha(s) = k_\alpha * f,$$

where

$$\begin{aligned} k_\alpha(x) &= \int_0^\infty g_s(x)\,d\kappa_a(s) = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty (2\pi s)^{-d/2}\exp\Big(-\frac{|x|^2}{2s}\Big)\,s^{\frac{\alpha}{2}-1}\,ds \\ &= \frac{|x|^{\alpha-d}}{\Gamma(\frac{\alpha}{2})2^{\alpha/2}\pi^{d/2}}\int_0^\infty s^{\frac{d-\alpha}{2}-1}\,e^{-s}\,ds = c_n^\alpha |x|^{\alpha-d} \end{aligned}$$

with

$$c_n^\alpha := \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^{\alpha/2}\pi^{d/2}}.$$

Moreover, we see that $V^{\eta,\alpha}$ is a proper kernel, since, for every $R > 0$,

$$\int_{B(0,R)} |y|^{\alpha-d}\,dy = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\int_0^R d r^{d-1} r^{\alpha-d}\,dr = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\cdot\frac{d}{\alpha}R^\alpha.$$

For every $\mu \in \mathcal{M}(\mathbb{R}^d)$, let

$$G^\alpha\mu(x) := \int |x-y|^{\alpha-d}\,d\mu(y).$$

If the function $G^\alpha\mu$ is not identically infinite, it is called the *Riesz potential of* μ.

We obtain the following result.

THEOREM 3.1.7. *For every $0 < \alpha < 2 \wedge d$, $(X, E_{\mathbb{P}^{\eta,\alpha}})$ is a balayage space. For every $x \in \mathbb{R}^d$, the functions $k_\alpha^x \colon y \mapsto |x-y|^{\alpha-d}$ are $\mathbb{P}^\alpha$-excessive.*

In particular, $E_{\mathbb{V}^{\eta,\alpha}}$ is the set of all increasing limits of Riesz potentials.

Proof. For a moment, let us fix $x \in X$. Then $k_\alpha^x \in E_{\mathbb{P}^{\eta,\alpha}}$. Indeed, if we choose a sequence (φ_n) in $\mathcal{K}^+(\mathbb{R}^d)$ such that $\lambda^d(\varphi_n) = 1$ and $S(\varphi_n) \subset B(0,1/n)$, then

$$\lim_{n\to\infty} k_a * \varphi_n = k_\alpha^x.$$

Since $k_\alpha * \varphi_n = V^{\eta,\alpha}\varphi_n \in E_{\mathbb{P}^\alpha}$, we obtain that, for every $t > 0$,

$$P_t^\alpha k_\alpha^x \le \liminf_{n\to\infty} P_t^\alpha(k_\alpha * \varphi_n) \le \liminf_{n\to\infty} k_\alpha * \varphi_n = k_\alpha^x.$$

So $k_\alpha^x \in S_{\mathbb{P}^{\eta,\alpha}}$. Since $k_\alpha \colon \mathbb{R}^d \to [0,\infty]$ is continuous and the semigroup $\mathbb{P}^{\eta,\alpha}$ is right continuous and strong Feller, we obtain that $k_\alpha^x \in E_{\mathbb{P}^{\eta,\alpha}}$. In particular, the function $k_a \wedge 1$ is contained in $E_{\mathbb{P}^{\eta,\alpha}} \cap C_0(X)$.

Thus, by Corollary 2.3.8, $(\mathbb{R}^d, E_{\mathbb{P}^{\eta,\alpha}})$ is a balayage space.

If $\mu \in \mathcal{M}(\mathbb{R}^d)$, then $G^\alpha\mu$ is lower semicontinuous (by Fatou's lemma) and $\mathbb{P}^\alpha$-supermedian (by Fubini's theorem). Hence $G^\alpha\mu \in E_{\mathbb{P}^\alpha}$ by Lemma 2.3.2.

To finish the proof, let us consider an arbitrary $v \in E_{\mathbb{P}^{\eta,\alpha}}$. By Proposition 2.2.12, there exists a sequence (f_n) in $\mathcal{B}_b^+(X)$ such that $V^{\eta,\alpha} f_n \uparrow v$. Since $V^{\eta,\alpha} f_n = G^\alpha(f_n\lambda^d)$, $n \in \mathbb{N}$, we see that v is the limit of an increasing sequence of Riesz potentials. □

Proposition 3.1.4 states that subordination by *any* convolution semigroup leads to a balayage space, if we start with a balayage space $(X, E_\mathbb{P})$ given by a *strong Feller semigroup*. Let us now ask which convolution semigroups will admit a subordination on *any* balayage space, where the positive constants are hyperharmonic. To that end we recall from Remark 2.3.9 that, for every balayage space $(X, \mathcal{W})$ with $1 \in \mathcal{W}$, there exist sub-Markov semigroups $\mathbb{P}$ such that $E_\mathbb{P} = \mathcal{W}$ and $\mathbb{P}$ has a *strong Feller resolvent*. A first step to an answer is the following result.

PROPOSITION 3.1.8. *Let $(\mu_t)_{t>0}$ be a convolution semigroup on $(0,\infty)$ and let $\kappa := \int_0^\infty \mu_t\,dt$. Then the following statements are equivalent:*

1. *For all spaces X and sub-Markov semigroups $\mathbb{P}$ on X with strong Feller resolvent, the semigroup $\mathbb{P}^\mu$ has a strong Feller resolvent.*

2. *The semigroup $\mathbb{T}^\mu$ (on $\mathbb{R}$) has a strong Feller resolvent.*

3. *For every $A \in \mathcal{B}(X)$, the function $1_A * \kappa$ is lower semicontinuous.*

4. *The measure κ is absolutely continuous with respect to $\lambda_{(0,\infty)}$.*

Proof. (1) ⇒ (2): Trivial, since $\mathbb{T}$ has a strong Feller resolvent.

(2) ⇒ (3): For all $A \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\begin{aligned} 1_A * \kappa(x) &= \int_0^\infty 1_A(x-s)\,d\kappa(s) = \int_0^\infty \int_0^\infty T_s 1_A(x)\,d\mu_t(s)dt \\ &= \int_0^\infty T_t^\mu 1_A(x)\,dt = V^\mu 1_A(x) = \sup_{\alpha>0} V_\alpha^\mu 1_A(x). \end{aligned}$$

(3) ⇒ (4): Let $A \in \mathcal{B}((0,\infty))$ be a Lebesgue null set. Then $\kappa(A) = 1_{-A} * \kappa(0)$ and

$$\int_\mathbb{R} 1_{-A} * \kappa(x)\,dx = \int_0^\infty \Big(\int_\mathbb{R} 1_{-A}(x-s)\,dx\Big) d\kappa(s) = 0.$$

Since the function $1_{-A} * \kappa$ is positive and lower semicontinuous, we see that $1_{-A} * \kappa$ is identically zero.

(4) ⇒ (1): Let $\mathbb{P} = (P_t)_{t>0}$ be a sub-Markov semigroup on a space X such that the resolvent of $\mathbb{P}$ is strong Feller. Let us fix $\alpha > 0$. Since the finite measure

$\nu := \int_0^\infty e^{-\alpha t}\mu_t\,dt$ is majorized by κ, we obtain by the theorem of Radon-Nikodym that ν has an integrable density g with respect to $\lambda_{(0,\infty)}$. Let $f \in \mathcal{B}_b(X)$. Then

$$\begin{aligned} V_\alpha^\mu f &= \int_0^\infty e^{-\alpha t} P_t^\mu\,dt = \int_0^\infty e^{-\alpha t}\Big(\int_0^\infty P_s f\,d\mu_t(s)\Big)dt \\ &= \int_0^\infty P_s f\,d\nu(x) = \int_0^\infty (P_s f) g\,ds. \end{aligned}$$

So the continuity of $V_\alpha^\mu f$ follows from Lemma 6.4.4. □

Now the following result answers the question we raised.

THEOREM 3.1.9. *Let $(\mu_t)_{t>0}$ be a convolution semigroup on $(0,\infty)$, $\kappa := \int_0^\infty \mu_t\,dt$. Then the following statements are equivalent:*

1. *For all spaces X and sub-Markov semigroups $\mathbb{P}$ on X such that $(X, E_{\mathbb{P}})$ is a balayage space and $\mathbb{P}$ has a strong Feller resolvent, $(X, E_{\mathbb{P}^\mu})$ is a balayage space.*

2. *$(\mathbb{R}, E_{\mathbb{T}^\mu})$ is a balayage space.*

3. *κ is absolutely continuous with respect to $\lambda_{(0,\infty)}$.*

Proof. (1) ⇒ (2): Trivial.

(2) ⇒ (3): If $A \in \mathcal{B}(\mathbb{R})$, then $1_A * \kappa = V^\mu 1_A \in E_{\mathbb{T}^\mu}$, and hence $1_A * \kappa$ is lower semicontinuous. Thus (3) follows from Proposition 3.1.8.

(3) ⇒ (1): Let $(X, E_{\mathbb{P}})$ be a balayage space such that $\mathbb{P}$ has a strong Feller resolvent. Then, by Proposition 3.1.8, $\mathbb{P}^\mu$ has a strong Feller resolvent. Using Corollary 2.3.8 and the fact that $E_{\mathbb{P}} \subset E_{\mathbb{P}^\mu}$, we finally obtain that $(X, E_{\mathbb{P}^\mu})$ is a balayage space. □

Finally, by (1.4), we may note the following.

PROPOSITION 3.1.10. *For every $\alpha \in (0,2)$, $(\mathbb{R}, E_{\mathbb{T}^{\eta,\alpha}})$ is a balayage space with $T_t^{\eta,\alpha} = f * \eta_t^\alpha$.*

We have seen that the balayage space $(\mathbb{R}, E_{\mathbb{T}})$ may serve as a test for subordination. It will turn out that $(\mathbb{R}, E_{\mathbb{T}})$ plays the same crucial role for products of balayage spaces (see Theorem 3.2.10).

3.2 Products of semigroups, heat semigroup

Throughout this section let $\mathbb{P}$ be a sub-Markov semigroup on X and $\tilde{\mathbb{P}}$ a sub-Markov semigroup on $\tilde{X}$ (where $\tilde{X}$ is any locally compact space with countable base). Defining

$$(P_t \otimes \tilde{P}_t) f(x,\tilde{x}) := \iint f(y,\tilde{y})\,P_t(x,dy)\,\tilde{P}_t(\tilde{x},d\tilde{y})$$

we obtain a sub-Markov semigroup $\mathbb{P} \otimes \tilde{\mathbb{P}} = (P_t \otimes \tilde{P}_t)_{t>0}$ on $X \times \tilde{X}$.

Will $(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ be a balayage space provided that $(X, E_{\mathbb{P}})$ and $(\tilde{X}, E_{\tilde{\mathbb{P}}})$ are balayage spaces? A first answer is the following.

PROPOSITION 3.2.1. *Let $(X, \mathbb{P})$ and $(\tilde{X}, \tilde{\mathbb{P}})$ be balayage spaces. Then the product space $(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ is a balayage space if and only if every function in $E_{\mathbb{P}\otimes\tilde{\mathbb{P}}}$ is the supremum of functions in $\mathcal{C}(X \times \tilde{X}) \cap E_{\mathbb{P}\otimes\tilde{\mathbb{P}}}$. In particular, $(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ is a balayage space, if $\mathbb{P} \otimes \tilde{\mathbb{P}}$ has a strong Feller resolvent.*

Before proving this result we note the following.

LEMMA 3.2.2. *The product semigroup $\mathbb{P} \otimes \tilde{\mathbb{P}}$ has the following properties:*

1. *$E_{\mathbb{P}} \otimes E_{\tilde{\mathbb{P}}} \subset E_{\mathbb{P}\otimes\tilde{\mathbb{P}}}$, that is, if $u \in E_{\mathbb{P}}$ and $\tilde{u} \in E_{\tilde{\mathbb{P}}}$, then $u \otimes \tilde{u} \colon (x, \tilde{x}) \mapsto u(x)\tilde{u}(\tilde{x})$ is excessive with respect to $\mathbb{P} \otimes \tilde{\mathbb{P}}$.*

2. *If $\lim_{t\to 0} P_t\varphi = \varphi$ (locally uniformly) for every $\varphi \in \mathcal{K}(X)$ and $\lim_{t\to 0} \tilde{P}_t\tilde{\varphi} = \tilde{\varphi}$ (locally uniformly) for every $\tilde{\varphi} \in \mathcal{K}(\tilde{X})$, then $\lim_{t\to 0}(P_t \otimes \tilde{P}_t)\psi = \psi$ (locally uniformly) for every $\psi \in \mathcal{K}(X \times \tilde{X})$.*

3. *If $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are measurable and if the potential kernel V of $\mathbb{P}$ (or the potential kernel $\tilde{V}$ of $\tilde{\mathbb{P}}$) is proper, then the potential kernel W of $\mathbb{P} \otimes \tilde{\mathbb{P}}$ is proper.*

4. *$\tilde{P} \otimes \tilde{\mathbb{P}}$ is a strong Feller semigroup if (and only if) $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are strong Feller semigroups.*

Proof. Since $(P_t \otimes \tilde{P}_t)(u \otimes \tilde{u}) = (P_t u) \otimes (\tilde{P}_t\tilde{u})$, (1) follows immediately. In order to prove (2) it suffices to consider functions $\psi \in \mathcal{K}(X \times \tilde{X})$ which are of the form $\psi = \varphi \otimes \tilde{\varphi}$, $\varphi \in \mathcal{K}(X)$, $\tilde{\varphi} \in \mathcal{K}(\tilde{X})$. Then, for every $t > 0$,

$$\begin{aligned} & |(P_t \otimes \tilde{P}_t)\psi - \psi| = |(P_t\varphi) \otimes (\tilde{P}_t\tilde{\varphi}) - \varphi \otimes \tilde{\varphi}| \\ \leq\ & |(P_t\varphi - \varphi) \otimes (\tilde{P}_t\tilde{\varphi})| + |\varphi \otimes (\tilde{P}_t\tilde{\varphi} - \tilde{\varphi})| \\ \leq\ & \|\tilde{\varphi}\|_\infty |P_t\varphi - \varphi| + \|\varphi\|_\infty |\tilde{P}_t\tilde{\varphi} - \tilde{\varphi}|. \end{aligned}$$

Now (2) follows easily.

Let us suppose next that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are measurable and that V is proper. There exists $f \in \mathcal{B}^+(X)$, $f > 0$, such that $Vf < \infty$. Then, for all $(x, \tilde{x}) \in X \times \tilde{X}$,

$$W(f \otimes 1)(x, \tilde{x}) = \int_0^\infty P_t f(x)\tilde{P}_t 1(\tilde{x})\, dt \leq \int_0^\infty P_t f(x)\, dt = Vf(x) < \infty.$$

So W is a proper kernel, that is, (3) holds.

Property (4) is an immediate consequence of Corollary 6.4.2. □

Proof of Proposition 3.2.1. Since $E_{\mathbb{P}}$ and $E_{\tilde{\mathbb{P}}}$ are linearly separating, the points of $X \times \tilde{X}$ are linearly separated by $\{u \otimes 1 \colon u \in E_{\mathbb{P}}\} \cup \{1 \otimes \tilde{u} \colon \tilde{u} \in E_{\tilde{\mathbb{P}}}\}$. Therefore $E_{\mathbb{P}\otimes\tilde{\mathbb{P}}}$ is linearly separating.

Moreover, there exist strictly positive $u, v \in E_{\mathbb{P}} \cap \mathcal{C}(X)$, $\tilde{u}, \tilde{v} \in E_{\tilde{\mathbb{P}}} \cap \mathcal{C}(\tilde{X})$ such that $u/v \in \mathcal{C}_0(X)$, $\tilde{u}/\tilde{v} \in \mathcal{C}_0(\tilde{X})$. Then $u \otimes \tilde{u}, v \otimes \tilde{v}$ are strictly positive functions in $E_{\mathbb{P}\otimes\tilde{\mathbb{P}}} \cap \mathcal{C}(X \times \tilde{X})$ such that $(u \otimes \tilde{u})/(v \otimes \tilde{v}) = (u/v) \otimes (\tilde{u}/\tilde{v}) \in \mathcal{C}_0(X \times \tilde{X})$.

By Lemma 3.2.2, $\lim_{t\to 0}(P_t \otimes \tilde{P}_t)\psi = \psi$ for every $\psi \in \mathcal{K}(X \times \tilde{X})$.

An application of Corollary 2.3.8 finishes the proof. □

COROLLARY 3.2.3. *Let $(X, E_{\mathbb{P}})$ and $(\tilde{X}, E_{\tilde{\mathbb{P}}})$ be balayage spaces and suppose that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are strong Feller semigroups. Then $(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ is a balayage space.*

Proof. Lemma 3.2.2 and Proposition 3.2.1. □

Next shall investigate the case, where $\tilde{X} = \mathbb{R}$ and $\tilde{\mathbb{P}}$ is the translation semigroup $\mathbb{T} = (T_t)_{t>0}$, which is not strong Feller, but has a strong Feller resolvent. Clearly, for all $f \in \mathcal{B}_b(X \times \mathbb{R})$, $t > 0$, and $(x, s) \in X \times \mathbb{R}$,

$$(P_t \otimes T_t)f(x, s) = P_t f(\cdot, s - t)(x).$$

To show that $\mathbb{P} \otimes \mathbb{T}$ has a strong Feller resolvent, if $\mathbb{P}$ is strong Feller and $(X, E_{\mathbb{P}})$ is a balayage space, we shall need the following two results.

LEMMA 3.2.4. *Let $P_0 := I$ and $t \geq 0$ such that $P_t(\mathcal{C}_b(X)) \subset \mathcal{C}_b(X)$ and, for every $\varphi \in \mathcal{K}(X)$, $\lim_{s\downarrow t} P_s\varphi = P_t\varphi$ locally uniformly. Then, for every $f \in \mathcal{C}_b(X)$, $\lim_{s\downarrow t} P_s f = P_t f$.*

Proof. Let $f \in \mathcal{C}_b(X)$, $0 \leq f \leq 1$, K a compact subset of X, and $\varepsilon > 0$. Let (φ_n) be a sequence in $\mathcal{K}(X)$, $\varphi_n \uparrow 1$. Then $(P_t(1 - \varphi_n))$ is a decreasing sequence in $\mathcal{C}_b(X)$ and $\inf_n P_t(1 - \varphi_n) = 0$. Hence there exists $n \in \mathbb{N}$ such that $\varphi := \varphi_n$ satisfies

$$P_t(1 - \varphi) < \varepsilon \qquad \text{on } K.$$

By assumption, there exists $\delta > 0$ such that, for every $s \in [t, t + \delta]$,

$$P_s(\varphi f) \geq P_t(\varphi f) - \varepsilon \quad \text{on } K, \qquad P_s(\varphi(1 - f)) \geq P_t(\varphi(1 - f)) - \varepsilon \quad \text{on } K.$$

Let $s \in [t, t + \delta]$. Since $P_s f \geq P_s(\varphi f)$ and $P_t(\varphi f) \geq P_t f - P_t(1 - \varphi)$, we obtain the lower estimate $P_s f \geq P_t f - 2\varepsilon$ on K. Similarly, $P_s(1 - f) \geq P_t(1 - f) - 2\varepsilon$ leading to the the upper upper estimate $P_s f \leq P_s 1 - P_t 1 + P_t f + 2\varepsilon \leq P_t f + 2\varepsilon$ on K, since $P_s 1 \leq P_t 1$. □

REMARK 3.2.5. If $\mathbb{P}$ is strong Feller and $\lim_{t\to 0} P_t\varphi = \varphi$ locally uniformly for every $\varphi \in \mathcal{K}(X)$, then $\lim_{s\downarrow t} P_t\varphi = \varphi$ locally uniformly for all $\varphi \in \mathcal{K}(X)$ and $t > 0$. This follows immediately from Lemma 3.2.4 using the identity $P_s\varphi = P_{s-t}P_t\varphi$.

PROPOSITION 3.2.6. *The following properties are equivalent:*

1. *$\mathbb{P}$ is strong Feller and, for all $\varphi \in \mathcal{K}(X)$ and $t > 0$, $\lim_{s\downarrow t} P_s\varphi = P_t\varphi$ locally uniformly.*

2. *For every $f \in \mathcal{B}_b(X)$, the function $(x, t) \mapsto P_t f(x)$ is continuous on $X \times (0, \infty)$.*

Proof. (1) ⇒ (2): Let $f \in \mathcal{B}(X)$, $0 \leq f \leq 1$, $t \in (0, \infty)$, K compact in X, and $\varepsilon > 0$. Let us fix $0 < t_0 < t$. There exists $\varphi \in \mathcal{K}(X)$ such that $0 \leq \varphi \leq 1$ and $P_{t_0}(1 - \varphi) < \varepsilon$ on K. By Lemma 3.2.4, there exists $0 < \delta_0 < t - t_0$ such that, for every $t_0 < r \leq t_0 + \delta_0$,

$$P_r(1 - \varphi) < \varepsilon \qquad \text{on } K.$$

Let $t_1 := t - t_0 - \delta_0$ and $g := P_{t_1} f$. Then $g \in \mathcal{C}_b(X)$. So, by Lemma 3.2.4, there exists $0 < \delta_1 < \delta_0$ such that, for every $0 < r < \delta_1$,

$$|P_r g - g| < \varepsilon \qquad \text{on } S(\varphi).$$

Let $t < s < t + \delta_1$. Since $1 \le 1 - \varphi + 1_{S(\varphi)}$ and $|P_{s-t}g - g| \le g \vee P_{s-t}g \le 1$, we have

$$\begin{aligned} &|P_s f - P_t f| = |P_{t_0+\delta_0}(P_{s-t}g - g)| \\ \le\ & P_{t_0+\delta_0}(1-\varphi) + P_{t_0+\delta_0}(1_{S(\varphi)}|P_{s-t}g - g|) < \varepsilon + P_{t_0+\delta_0}\varepsilon \le 2\varepsilon \qquad \text{on } K. \end{aligned}$$

In a similar way, for every $t - \delta_1 < s < t$,

$$|P_s f - P_t f| = |P_{s-t_1}(g - P_{t-s}g)| < 2\varepsilon \qquad \text{on } K$$

(since $t_0 < s - t_1 < t_0 + \delta_0$). So $P_s f$ converges locally uniformly to $P_t f$ as s tends to t. Since $P_t f \in \mathcal{C}_b(X)$, we conclude that the function $(x,t) \mapsto P_t f(x)$ is continuous on $X \times (0,\infty)$.

(2) ⇒ (1): Trivial. □

PROPOSITION 3.2.7. *Let us suppose that $\mathbb{P}$ is strong Feller and $\lim_{s\downarrow t} P_s\varphi = P_t\varphi$ locally uniformly for all $\varphi \in \mathcal{K}(X)$ and $t > 0$. Then $\mathbb{P} \otimes \mathbb{T}$ has a strong Feller resolvent $\mathbb{W} = (W_\lambda)_{\lambda>0}$.*

Proof. Let us fix $\lambda > 0$ and $g \in \mathcal{B}_b(X \times \mathbb{R})$. Then, for every $(x,t) \in X \times \mathbb{R}$,

$$W_\lambda g(x,t) = \int_0^\infty e^{-\lambda s} P_s g(\cdot, t-s)(x)\, ds = \int_t^\infty e^{-\lambda(s-t)} P_{s-t} g(\cdot, s)(x)\, ds.$$

By Lebesgue's convergence theorem and Proposition 3.2.6, we obtain that $W_\lambda g$ is continuous on $X \times (0,\infty)$. □

COROLLARY 3.2.8. *Let $(X, E_{\mathbb{P}})$ let a balayage space. Then the following statements are equivalent:*

1. *$\mathbb{P}$ is a strong Feller semigroup.*
2. *$(X \times \mathbb{R}, E_{\mathbb{P}\otimes\mathbb{T}})$ is a balayage space.*

Proof. (1)⇒(2): Proposition 3.2.7 and Proposition 3.2.1.

(2)⇒(1): We fix $f \in \mathcal{B}(X)$ such that $0 \le f \le 1$ and define $F\colon X \times \mathbb{R} \to [0,1]$ by

$$F(x,s) := \begin{cases} P_s f(x), & x \in X,\ s > 0, \\ 0, & x \in X,\ s \le 0. \end{cases}$$

Let $t > 0$ and $(x,s) \in X \times \mathbb{R}$. If $t < s$, then

$$(P_t \otimes T_t)F(x,s) = P_t F(\cdot, s-t)(x) = P_t P_{s-t} f(x) = P_s f(x) = F(x,s).$$

If $t \ge s$, then

$$(P_t \otimes T_t)F(x,s) = P_t F(\cdot, s-t)(x) = P_t 0(x) = 0.$$

Moreover,

$$(P_t \otimes T_t)1_{X\times(0,\infty)}(x,s) = \begin{cases} P_t 0(x) = 0, & \text{if } t \geq s, \\ P_t 1(x), & \text{if } t < s, \end{cases}$$

where $\sup_{t>0} P_t 1(x) = \sup_{t>0}(P_t \otimes T_t)1(x,0) = 1$.

Therefore F and $1_{X\times(0,\infty)} - F$ are functions in $E_{\mathbb{P}\otimes\tilde{\mathbb{P}}}$. In particular, they are lower semicontinuous. Since their sum is $1_{X\times(0,\infty)}$, we conclude that both are continuous on $X \times (0,\infty)$. Thus $P_s f \in \mathcal{C}(X)$ for every $s > 0$, that is, $\mathbb{P}$ is a strong Feller semigroup. □

EXAMPLE 3.2.9. Let $\mathbb{P}$ be the Brownian semigroup on $\mathbb{R}^d$, $d \geq 1$. Then $\mathbb{P} \otimes \mathbb{T}$ is called the *heat semigroup* on $\mathbb{R}^{d+1}$. We claim that $(\mathbb{R}^{d+1}, E_{\mathbb{P}\otimes\mathbb{T}})$ is a balayage space.

Proof. This follows at once from Corollary 3.2.8, if $d \geq 3$. For the general case, we go back to Corollary 2.3.8. We define strictly positive functions $u, v \in C(\mathbb{R}^{d+1})$ by

$$u(x,s) := \exp(-\max_{1\leq i\leq d} |x_i| + \tfrac{1}{2}s) \quad \text{and} \quad v(x,s) := e^s \vee 1.$$

Clearly, $u/v \in \mathcal{C}_0(\mathbb{R}^{d+1})$. To finish the proof it suffices to show that $u, v \in E_{\mathbb{P}\otimes\mathbb{T}}$. Since $(P_t \otimes T_t)v(x,s) = e^{s-t} \vee 1$, we see that $v \in E_{\mathbb{P}\otimes\mathbb{T}}$. Defining

$$f^i_{\pm}(x,s) := \exp\bigl(\pm x_i + \tfrac{1}{2}s\bigr), \qquad 1 \leq i \leq d,\ (x,s) \in \mathbb{R}^{d+1}.$$

we have

$$u = f^1_+ \wedge \cdots \wedge f^d_+ \wedge f^1_- \wedge \cdots \wedge f^d_-,$$

where, for all $t > 0$,

$$\begin{aligned} (P_t \otimes T_t) f^i_{\pm}(x,s) &= (2\pi t)^{-d/2} \int \exp\bigl(-\frac{|y-x|^2}{2t}\bigr) \exp\bigl(\pm y_i + \tfrac{1}{2}(s-t)\bigr)\, dy \\ &= (2\pi t)^{-1} \int \exp\bigl(-\frac{(y_i - x_i)^2}{2t}\bigr) \exp\bigl(\pm y_i + \tfrac{1}{2}(s-t)\bigr)\, dy_i \\ &= \exp\bigl(\pm x_i + \tfrac{1}{2}s\bigr)\,(2\pi t)^{-1} \int \exp\bigl(-\frac{(y_i - (x_i \pm t))^2}{2t}\bigr)\, dy_i \\ &= f^i_{\pm}(x,s). \end{aligned}$$

Thus $f^1_{\pm}, \dots, f^d_{\pm} \in E_{\mathbb{P}\otimes\mathbb{T}}$, $u \in S_{\mathbb{P}\otimes\mathbb{T}}$. Since u is continuous and $\lim_{t\to 0}(P_t \otimes T_t)\psi = \psi$ for every $\psi \in \mathcal{K}(\mathbb{R}^{d+1})$ (see Lemma 3.2.2), we finally conclude that $u \in E_{\mathbb{P}\otimes\mathbb{T}}$. □

Let us finish this section by the following result which shows that $\mathbb{T}$ is also the test case for forming products (cf. Theorem 3.1.9).

THEOREM 3.2.10. *Let $(X, E_{\mathbb{P}})$ be a balayage space. Then the following statements are equivalent:*

1. $(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ *is a balayage space for every balayage space* $(\tilde{X}, E_{\tilde{\mathbb{P}}})$ *such that* $\tilde{\mathbb{P}}$ *is a sub-Markov semigroup having a strong Feller resolvent.*

2. $(X \times \mathbb{R}, E_{\mathbb{P}\otimes\mathbb{T}})$ *is a balayage space.*

3. $\mathbb{P}$ *is a strong Feller semigroup.*

Proof. (1)⇒(2): Trivial, since $\mathbb{T}$ has a strong Feller resolvent.

(2)⇒(3): Corollary 3.2.8.

(3)⇒(1): Combining Lemma 2.3.3, Lemma 3.2.4, and Remark 3.2.5 we see that, for all $\varphi \in \mathcal{K}(X)$ and $t > 0$, $\lim_{s \downarrow t} P_s\varphi = P_t\varphi$ locally uniformly. The proof is finished by the following Proposition 3.2.11 (which generalizes Proposition 3.2.7). □

PROPOSITION 3.2.11. *Suppose that $\mathbb{P}$ is strong Feller and, for all $\varphi \in \mathcal{K}(X)$ and $t > 0$, $\lim_{s\downarrow t} P_s\varphi = P_t\varphi$ locally uniformly. Moreover, assume that $\tilde{\mathbb{P}}$ has a strong Feller resolvent $(\tilde{V}_\lambda)_{\lambda>0}$. Then $\mathbb{P} \otimes \tilde{\mathbb{P}}$ has a strong Feller resolvent $(W_\lambda)_{\lambda>0}$.*

Proof. Let $\lambda > 0$ and $g \in \mathcal{B}_b(X \times \tilde{X})$. By Proposition 3.2.6, for every $\tilde{x} \in \tilde{X}$,

$$(x,t) \mapsto P_t g(\cdot,\tilde{x})(x) \text{ is continuous on } X \times (0,\infty). \tag{2.1}$$

Let us fix a sequence $(x_n, \tilde{x}_n)$ in $X \times \tilde{X}$ converging to a point $(x_0, \tilde{x}_0) \in X \times \tilde{X}$, and, for all $n \in \mathbb{N} \cup \{0\}$ and $(\tilde{x}, t) \in \tilde{X} \times (0,\infty)$, let

$$f_n(\tilde{x}, t) := P_t g(\cdot, \tilde{x})(x_n).$$

Then, for all $n, m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} W_\lambda g(x_n, \tilde{x}_m) &= \int_0^\infty \int_{\tilde{X}} \int_X e^{-\lambda t} g(x,\tilde{x})\, P_t(x_n, dx)\, P_t(\tilde{x}_m, d\tilde{x})\, dt \\ = & \int_0^\infty \int_{\tilde{X}} e^{-\lambda t} P_t g(\cdot,\tilde{x})(x_n)\, \tilde{P}_t(\tilde{x}_m, d\tilde{x})\, dt = \int_0^\infty e^{-\lambda t} \tilde{P}_t f_n(\cdot, t)(\tilde{x}_m)\, dt. \end{aligned}$$

By (2.1), for every $\tilde{x} \in \tilde{X}$, the function $f_0(\tilde{x}, \cdot)$ is continuous on $(0, \infty)$. Hence, by Lemma 6.4.3, applied to $\tilde{\mathbb{P}}$ and the function f_0, we obtain that

$$\lim_{m\to\infty} W_\lambda g(x_0, \tilde{x}_m) = W_\lambda g(x_0, \tilde{x}_0).$$

Next we fix $\varepsilon > 0$, choose $0 < r < s < \infty$ such that $2\|g\|_\infty(1 - e^{-\lambda r} + e^{-\lambda s}) < \lambda\varepsilon$, and define

$$g_n(\tilde{x}) := \sup_{r \le t \le s} |f_n(\tilde{x}, t) - f_0(\tilde{x}, t)| \qquad (\tilde{x} \in \tilde{X}).$$

Clearly, $0 \le g_n \le 2\|g\|_\infty$, $n \in \mathbb{N}$. By (2.1), $(g_n) \subset \mathcal{B}(\tilde{X})$ and $\lim_{n\to\infty} g_n = 0$. Since

$$\begin{aligned} \big|W_\lambda g(x_n, \tilde{x}_n) - W_\lambda g(x_0, \tilde{x}_n)\big| &\le \int_0^\infty e^{-\lambda t} \tilde{P}_t |f_n(\cdot,t) - f_0(\cdot,t)|(\tilde{x}_n)\, dt \\ &\le \varepsilon + \int_r^s e^{-\lambda t} \tilde{P}_t g_n(\tilde{x}_n)\, dt \le \varepsilon + \tilde{V}_\lambda g_n(\tilde{x}_n) \end{aligned}$$

and $\lim_{n\to\infty} \tilde{V}_\lambda g_n(\tilde{x}_n) = 0$ by Proposition 6.4.1, we see that

$$\lim_{n\to\infty} \big(W_\lambda g(x_n, \tilde{x}_n) - W_\lambda g(x_0, \tilde{x}_n)\big) = 0.$$

Thus $\lim_{n\to\infty} W_\lambda g(x_n, \tilde{x}_n) = W_\lambda g(x_0, \tilde{x}_0)$. □

Chapter 4

More on balayage spaces

4.1 Balayage of functions and measures

Let $(X, \mathcal{W})$ be a balayage space. For all $u \in \mathcal{W}$ and $A \subset X$, let

$$R_u^A := R_{1_A u} = \inf\{v \in \mathcal{W} \colon v \geq u \text{ on } A\} \quad \text{and} \quad \hat{R}_u^A := \widehat{R_u^A} \tag{1.1}$$

so that $\hat{R}_u^A \in \mathcal{W}$. We note that $R_u^A = \hat{R}_u^A \in \mathcal{W}$, if A is finely open.

We shall say that R_u^A and $\hat{R}_u^A$ are obtained *reducing u on A, sweeping u on A*, respectively.

PROPOSITION 4.1.1. *For all $p, q \in \mathcal{P}$,*

$$R_{p+q}^A = R_p^A + R_q^A \quad \textit{and} \quad \hat{R}_{p+q}^A = \hat{R}_p^A + \hat{R}_q^A. \tag{1.2}$$

Proof. If $u, v \in \mathcal{W}$ such that $u \geq p$ on A and $v \geq q$ on A, then $u + v \in \mathcal{W}$ and $u + v \geq p + q$ on A. Therefore $R_{p+q}^A \leq R_p^A + R_q^A$. To prove the reverse inequality

$$R_{p+q}^A \geq R_p^A + R_q^A \tag{1.3}$$

let $w \in \mathcal{W}$ such that $w \geq p + q$ on A. Since $\mathcal{W}$ is $\wedge$-stable, we may assume that $w \leq p + q$ on X (replace w by $w \wedge (p + q)$), and hence $w = p + q$ on A. By (B_3), there exist $u, v \in \mathcal{W}$ such that $w = u + v$, $u \leq p$ and $v \leq q$. Then, of course, $u = p$ on A and $v = q$ on A. Thus $w = u + v \geq R_p^A + R_q^A$, and (1.3) follows.

The second identity in (1.2) follows from the first one by Proposition 1.1.2 and Proposition 1.1.5, since every function $\hat{R}_u^A$, $u \in \mathcal{W}$, is finely continuous and the finely meager set $\{\hat{R}_u^A < R_u^A\}$ has no finely interior points. $\square$

EXERCISE 4.1.2. Show that, for all $u, v \in \mathcal{W}$,

$$R_{u+v}^A = R_u^A + R_v^A \quad \text{and} \quad \hat{R}_{u+v}^A = \hat{R}_u^A + \hat{R}_v^A.$$

Next we shall define *reduced* and *swept* measures. Let A be an arbitrary subset of X. By Proposition 1.2.1, Proposition 6.2.3, and Proposition 4.1.1, there exist unique measures $\varepsilon_x^A, \hat{\varepsilon}_x^A \in \mathcal{M}(X)$, $x \in X$, such that, for every $p \in \mathcal{P}$,

$$\int p \, d\varepsilon_x^A = R_p^A(x), \qquad \int p \, d\hat{\varepsilon}_x^A = \hat{R}_p^A(x). \tag{1.4}$$

We say that ε_x^A and $\hat{\varepsilon}_x^A$ are obtained by *reducing* and *sweeping* (*balayage*), respectively, of the Dirac measure ε_x on A.

If $f \in \mathcal{K}^+(X)$ and $p, q, q_0 \in \mathcal{P}$, $\varepsilon > 0$ such that $0 \le p - q \le f \le p - q + \varepsilon q_0$, then, for every $x \in X$,

$$0 \le \hat{R}_p^A(x) - \hat{R}_q^A(x) \le \hat{\varepsilon}_x^A(f) \le \hat{R}_p^A(x) - \hat{R}_q^A(x) + \varepsilon q_0(x). \tag{1.5}$$

Since the functions $\hat{R}_p^A$ and $\hat{R}_q^A$ are lower semicontinuous (and hence Borel measurable), we obtain from Proposition 6.2.2 that, for every $f \in \mathcal{K}^+(X)$, the function $x \mapsto \hat{\varepsilon}_x^A(f)$ is Borel measurable (and finely continuous). Thus $(x, B) \mapsto \hat{\varepsilon}_x^A(B)$ is a kernel on X.

Let $\mathcal{M}(\mathcal{P})$ denote the set of all measures μ on X such that $\mu(p) < \infty$ for *some* strictly positive $p \in \mathcal{P}$ and hence

$$\int \hat{\varepsilon}_x^A(p)\, d\mu(x) \le \int p(x)\, d\mu(x) < \infty. \tag{1.6}$$

Of course, every $\mu \in \mathcal{M}(X)$ with compact support is contained in $\mathcal{M}(\mathcal{P})$.

Given $\mu \in \mathcal{M}(\mathcal{P})$, we obtain from (1.6) that

$$\hat{\mu}^A(f) := \int \hat{\varepsilon}_x^A(f)\, d\mu(x) < \infty \qquad \text{for every } f \in \mathcal{K}^+(X). \tag{1.7}$$

Therefore (1.7) defines a measure $\hat{\mu}^A \in \mathcal{M}(X)$, and

$$\hat{\mu}^A(p) = \int \hat{R}_p^A\, d\mu \le \mu(p) \qquad \text{for every } p \in \mathcal{P}. \tag{1.8}$$

In particular, $\hat{\mu}^A \in \mathcal{M}(\mathcal{P})$.

REMARK 4.1.3. If U is an open set in X, then $\hat{\varepsilon}_x^U = \varepsilon_x^U$ for every $x \in X$, since $\hat{R}_p^U = R_p^U$ for every $p \in \mathcal{P}$. Moreover, for all $x \in X$ and $v \in \mathcal{W}$,

$$\int v\, d\varepsilon_x^U = R_v^U(x). \tag{1.9}$$

Indeed, if (p_n) is a sequence in $\mathcal{P}$ such that $p_n \uparrow v$, then the sequence $(R_{p_n}^U)$ is increasing to a function $w \in \mathcal{W}$ such that $w = v$ on U and hence $w \ge R_v^U$. Trivially, $w \le R_v^U$. Hence $w = R_v^U$, and (1.9) follows from (1.4). (In fact, (1.9) holds with any set A in place of U; see [1, VI.1.7].)

PROPOSITION 4.1.4. *Let $A \subset X$. Then, for all $x \in X$ and $\mu \in \mathcal{M}(P)$, the measures ε_x^A and $\hat{\mu}^A$ are supported by the closure of A.*

Proof. Indeed, let $f \in \mathcal{K}^+(X)$ such that $f = 0$ on A. Let $q_0 \in \mathcal{P}$, $q_0 > 0$, and $\varepsilon > 0$. Choosing $p, q \in \mathcal{P}$ as above, we then have $p = q$ on A, hence $R_p^A = R_q^A$ and $\hat{R}_p^A = \hat{R}_q^A$. Let $x \in X$. By (1.5), $\varepsilon_x^A(f), \hat{\varepsilon}_x^A(f) \in [0, \varepsilon q_0(x)]$. So $\varepsilon_x^A(f) = \hat{\varepsilon}_x^A(f) = 0$. Integrating with respect to $\mu \in \mathcal{M}(\mathcal{P})$, we finally see that $\hat{\mu}^A(f) = 0$. □

Next we shall use a general minimum principle which holds for functions in convex cones $\mathcal{F}$ of lower $\mathcal{P}$-bounded, lower semicontinuous functions on X containing $\mathcal{P}$ (cf. Section 6.3). They are positive, if they are positive on the Choquet boundary $Ch_{\mathcal{F}}X$ (where $Ch_{\mathcal{F}}X$ is the set of all points $x \in X$ such that ε_x is the only $\mu \in \mathcal{M}(X)$ satisfying $\mu(f) \le f(x)$ for all $f \in \mathcal{F}$). First the following definition.

DEFINITIONS 4.1.5. *1. The* fine support *of $p \in \mathcal{P}$ is the set*

$$\delta(p) := Ch_{\mathcal{P}-\mathbb{R}^+p} X,$$

that is, $\delta(p)$ is the set of all $x \in X$ such that ε_x is the only measure $\mu \in \mathcal{M}(X)$ satisfying $\mu(p) = p(x)$ and $\mu(q) \le q(x)$ for all $q \in \mathcal{P}$.

2. The carrier *or* superharmonic support *$C(p)$ of $p \in \mathcal{P}$ is the closure of $\delta(p)$.*

3. The support *$S(f)$ of a real function f on X is the closure of the set $\{f \neq 0\}$.*

PROPOSITION 4.1.6. *Let $p \in \mathcal{P}$. Then $R_p^{\delta(p)} = p$, and $C(p)$ is the smallest closed set A in X such that $R_p^A = p$.*

Moreover, for every $f \in \mathcal{C}_{\mathcal{P}}(X)$, $C(R_f) \subset S(f) \cap \{R_f = f\}$.

Proof. If $u \in \mathcal{W}$ and $u \ge p$ on $\delta(p)$, then $u \ge p$ by Theorem 6.3.2. Hence $R_p^{\delta(p)} = p$ and, obviously, $R_p^{C(p)} = p$ as well. Now let A be any closed set in X such that $R_p^A = p$. Then, for every $x \in A^c$, $\varepsilon_x^A(p) = R_p^A(x) = p(x)$, $\varepsilon_x^A(q) \le q(x)$ for all $q \in \mathcal{P}$, and $\varepsilon_x^A \neq \varepsilon_x$, since ε_x^A is supported by A (see Proposition 4.1.4), whence $x \notin \delta(p)$. Therefore $\delta(p) \subset A$ and $C(p) = \overline{\delta(p)} \subset A$.

Finally, let $f \in \mathcal{C}_{\mathcal{P}}(X)$. We know, by Corollary 1.2.2, that $R_f \in \mathcal{P}$. Of course, $R_{R_f}^{S(f)} = R_f$ and hence $C(R_f) \subset S(f)$. By Corollary 6.3.5, $\delta(R_f) \subset \{R_f = f\}$ and hence $C(R_f) \subset \{R_f = f\}$. $\square$

The following characterization of functions in $\mathcal{W}$ will be very useful.

PROPOSITION 4.1.7. *For every lower semicontinuous numerical function u on X which is lower $\mathcal{P}$-bounded (that is, $u \ge -p$ for some $p \in \mathcal{P}$), the following statements are equivalent.*

1. *$u \in \mathcal{W}$.*

2. *For all $x \in X$ and open neighborhoods U of x,*

$$\int u \, d\varepsilon_x^{U^c} \le u(x).$$

3. *For every $x \in X$ and every open neighborhood U of x, there exists a set V in U such that*

$$\varepsilon_x^{V^c} \neq \varepsilon_x \quad \text{and} \quad \int u \, d\varepsilon_x^{V^c} \le u(x).$$

Proof. (1) $\Rightarrow$ (2): By Proposition 1.2.1, there exists a sequence (p_n) in $\mathcal{P}$ which is increasing to u. Hence, for all $x \in X$ and open neighborhoods U of x,

$$\int u \, d\varepsilon_x^{U^c} = \sup \int p_n \, d\varepsilon_x^{U^c} = \sup_n R_{p_n}^{U^c}(x) \le \sup_n p_n(x) = u(x).$$

(2) $\Rightarrow$ (3): Let $x \in X$ and let U be an open neighborhood of x. Then $\varepsilon_x^{U^c} \neq \varepsilon_x$, since $\varepsilon_x^{U^c}$ is supported by U^c. So we may take $V := U$.

(3) $\Rightarrow$ (1): Let $f \in \mathcal{C}_{\mathcal{P}}(X)$ such that $-p \le f \le u$. By Proposition 1.2.2, $R_f \in \mathcal{P}$. Let $A := \{R_f = f\}$. Then, of course, $u - R_f \ge f - R_f = 0$ on A. We intend to use

our general minimum principle (Theorem 6.3.2) to show that $u - R_f \geq 0$ on X. To that end we consider the convex cone

$$\mathcal{F} := \mathbb{R}^+ u + \mathcal{P} - \mathbb{R}^+ R_f$$

of lower semicontinuous functions on X. Obviously, $\mathcal{P} \subset \mathcal{F}$ and every function in $\mathcal{F}$ is lower $\mathcal{P}$-bounded.

Let $x \in A^c$. By assumption, there exists a set V in A^c such that $\varepsilon_x^{V^c} \neq \varepsilon_x$ and $\varepsilon_x^{V^c}(u) \leq u(x)$. Of course, $\varepsilon_x^{V^c}(q) \leq q(x)$ for every $p \in \mathcal{P}$. Moreover, by Proposition 4.1.6, $C(R_f) \subset A \subset V^c$ and hence $\varepsilon_x^{V^c}(R_f) = R_f(x)$. So $\varepsilon_x^{V^c} \in \mathcal{M}_{\mathcal{F}}(X) \setminus \{\varepsilon_x\}$, $x \notin Ch_{\mathcal{F}} X$.

Thus $Ch_{\mathcal{F}} X \subset A$, and Theorem 6.3.2 implies that $u - R_f \geq 0$ on X, that is, $u \geq R_f \geq f$. Since u is the limit of an increasing sequence (f_n) of positive functions in $\mathcal{C}_{\mathcal{P}}(X)$, this shows that $u = \lim_{n\to\infty} R_{f_n} \in \mathcal{W}$. □

4.2 Harmonic kernels on balayage spaces

Again, let $(X, \mathcal{W})$ be a balayage space. Given any open set U in X, let

(2.1) $$H_U(x, \cdot) := \varepsilon_x^{U^c} \qquad (x \in X).$$

In other words, H_U is the unique kernel on X such that

(2.2) $$H_U q = R_q^{U^c} \qquad \text{for every } q \in \mathcal{P}.$$

By Proposition 4.1.4, the measures $H_U(x, \cdot)$, $x \in X$, are supported by U^c. Of course, $H_U(x, \cdot) = \varepsilon_x$, if $x \in U^c$.

DEFINITION 4.2.1. *Let W be an open set in X.*

1. *Let $\mathcal{U}(W)$ be the set of all open V in W such that $\overline{V}$ is a compact set in W.*

2. *Let ${}^*\mathcal{H}(W)$ denote the set of all* hyperharmonic functions on W, *that is, ${}^*\mathcal{H}(W)$ is the set of all $v \in \mathcal{B}(X)$ such that v is lower semicontinuous on W and*
$$-\infty < H_U v(x) \leq v(x) \qquad \text{for all } x \in U \in \mathcal{U}(W).$$

3. *Let $\mathcal{S}(W)$ denote the set of all* superharmonic functions on W, *that is,*
$$\mathcal{S}(W) := \big\{ v \in {}^*\mathcal{H}(W) \ : \ H_U v|_U \in \mathcal{C}(U) \text{ for all } U \in \mathcal{U}(W),\ v(x) < \infty \text{ if } x \text{ is finely isolated} \big\}.$$

4. *Let $\mathcal{H}(W)$ denote the set of all* harmonic functions on W, *that is,*
$$\begin{aligned} \mathcal{H}(W) &:= {}^*\mathcal{H}(W) \cap (-{}^*\mathcal{H}(W)) \\ &= \big\{ h \in \mathcal{B}(X) \colon h|_W \in \mathcal{C}(W) \text{ and } H_U h = h \text{ for all } U \in \mathcal{U}(W) \big\}. \end{aligned}$$

Let us stress that, given a function $v \in \mathcal{B}(X)$, it may depend very much on the values of v on W^c, if v is *hyperharmonic on* W (*harmonic on* W, respectively), since the support of the measures $H_U(x, \cdot)$, $x \in U \in \mathcal{U}(W)$, may contain W^c.

The following statements are simple consequences of the definitions.

LEMMA 4.2.2. *Let W, W_n be open sets in X.*

1. *${}^*\mathcal{H}(W)$ is a convex cone which is both $\wedge$-stable and σ-stable.*
2. *$\mathcal{H}(W)$ is a linear space.*
3. *If $W_n \uparrow W$, then ${}^*\mathcal{H}(W_n) \downarrow {}^*\mathcal{H}(W)$ and $\mathcal{H}(W_n) \downarrow \mathcal{H}(W)$.*

The following property of $\mathcal{W}$ is an immediate consequence of Proposition 4.1.7 (for a stronger statement see Corollary 5.2.8).

PROPOSITION 4.2.3. *If $(W_i)_{i\in I}$ is an open covering of X, then*

$$\bigcap_{i\in I} {}^*\mathcal{H}^+(W_i) = \mathcal{W}.$$

In particular, ${}^*\mathcal{H}^+(X) = \mathcal{W}$.

Moreover, we immediately have the following result.

PROPOSITION 4.2.4. *Every potential $p \in \mathcal{P}$ is harmonic on $X \setminus C(p)$, and $C(p)$ is the smallest closed set in X having this property.*

In particular, $R_f \in \mathcal{H}\big((X \setminus S(f)) \cup \{R_f > f\}\big)$ for every $f \in \mathcal{C}_{\mathcal{P}}(X)$.

Proof. Consequence of Proposition 4.1.6. □

To obtain convergence properties of harmonic functions we shall use (B_2) in a decisive way. To that end we first establish the following (weak) lifting.

LEMMA 4.2.5. *Let W be an open set in X, $V \in \mathcal{U}(W)$, and $c \in (0,\infty)$. Then there exists $q \in \mathcal{P}$ and an increasing mapping $v \mapsto v'$ from the set of all $v \in {}^*\mathcal{H}^+(W)$, which satisfy $v|_W \in \mathcal{C}(W)$ and $v|_V \le c$, into $\mathcal{P}$ such that*

$$v' = v + q \qquad \text{on } V.$$

Proof. Let K be a compact neighborhood of $\overline{V}$ in W. By Proposition 6.2.2, there exist $p, q \in \mathcal{P}$ such that $p - q \ge 0$ on X, $p - q \ge c$ on $\overline{V}$, and $p = q$ on K^c.

For every $v \in {}^*\mathcal{H}^+(W)$ with $v|_V \le c$, we define

$$v' := p \wedge (v + q).$$

Evidently, $v' \in {}^*\mathcal{H}^+(W)$. Moreover, $v' \le p$ on X, $v' = p$ in K^c, and therefore $v' \in {}^*\mathcal{H}^+(K^c)$. Since $W \cup K^c = X$, we conclude from Proposition 4.2.3 that $v' \in \mathcal{W}$. If v is continuous on W, then v' is continuous on X, and therefore $v' \in \mathcal{P}$.

The proof is finished since, obviously, the map $v \mapsto v'$ is increasing. □

PROPOSITION 4.2.6. *Let W be an open set in X and let (h_n) be a monotone sequence in $\mathcal{H}(W)$ which is bounded by some function $w \in \mathcal{W} \cap \mathcal{C}(X)$. Then $\lim_{n\to\infty} h_n \in \mathcal{H}(W)$.*

Proof. We may assume without loss of generality that (h_n) is an increasing sequence in $\mathcal{H}^+(W)$. Indeed, if (h_n) is increasing, we may replace (h_n) by $(h_n - h_1)$ and use at the end that $\lim h_n = \lim_{n\to\infty}(h_n - h_1) + h_1$. If (h_n) is decreasing, we replace (h_n) by $(h_1 - h_n)$.

Let $h = \lim_{n\to\infty} h_n$. Then h is lower semicontinuous on W and, for all $U \in \mathcal{U}(W)$,

$$H_U h = \lim_{n\to\infty} H_U h_n = \lim_{n\to\infty} h_n = h.$$

Let $V \in \mathcal{U}(W)$. It remains to prove that h is upper semicontinuous on V. Defining $c := \sup w(\overline{V})$ we choose $q \in \mathcal{P}$ and a mapping $v \mapsto v'$ according to Proposition 4.2.5. We obtain an increasing sequence (h'_n) in $\mathcal{P}$ such that, for every $n \in \mathbb{N}$,

$$h_n + q = h'_n \quad \text{on } V.$$

Then $h + q = \sup_n h'_n$ on V, where $\sup_n h'_n \in \mathcal{W}$. Hence h is finely continuous on V.

Further, for every $n \in \mathbb{N}$, $v_n := w - h_n \in {}^*\mathcal{H}^+(W)$, and $v_n|_V$ is continuous and bounded by c. Therefore we also have

$$v_n + q = v'_n \quad \text{on } V,$$

and hence

$$w - h + q = \inf v_n + q = \inf v'_n \quad \text{on } V. \tag{2.3}$$

Since $w - h + q$ is finely continuous on V, we see that

$$\widehat{\inf v'_n}^f = \inf v'_n \quad \text{on } V.$$

On the other hand, we know from (B_2) that $\widehat{\inf v_n}^f$ is lower semicontinuous. Thus, by (2.3), the function $w - h + q$ is lower semicontinuous on V. Since w and q are continuous, we finally conclude that h is upper semicontinuous on V. □

PROPOSITION 4.2.7. *Let f be a positive function on X which is lower semicontinuous or upper semicontinuous and suppose that $f \le w$ for some $w \in \mathcal{W} \cap \mathcal{C}(X)$. Then R_f is harmonic on $X \setminus S(f)$.*

Proof. Let us assume first that f is lower semicontinuous. There exists a sequence (φ_n) in $\mathcal{K}^+(X)$ such that $\varphi_n \uparrow f$. By Proposition 4.2.4, $R_{\varphi_n} \in \mathcal{H}(X \setminus S(f))$, $n \in \mathbb{N}$. By Lemma 1.1.6, $R_{\varphi_n} \uparrow R_f$. Since $R_f \le w$, we obtain, by Proposition 4.2.6, that $R_f \in \mathcal{H}(X \setminus S(f))$.

Now let us assume that f is upper semicontinuous. We choose continuous $f_n \le w$, $n \in \mathbb{N}$, such that $f_n \downarrow f$ and $S(f_n) \downarrow S(f)$. By the first part of the proof, each R_{f_n} is harmonic on $X \setminus S(f_n)$. Moreover, by Proposition 1.2.2, $R_{f_n} \downarrow R_f$. Hence Proposition 4.2.6 implies that $R_f \in \mathcal{H}(X \setminus S(f_n))$, $n \in \mathbb{N}$. Thus, by Lemma 4.2.2, $R_f \in \mathcal{H}(X \setminus S(f))$. □

COROLLARY 4.2.8. *Let $w \in \mathcal{W} \cap \mathcal{C}(X)$.*

1. *For every closed set A in X, the reduced function R_w^A is harmonic on $X \setminus A$.*

2. *If V is an open set in X and $u \in \mathcal{W}$ such that $u \le w$, then the reduced function R_u^V is harmonic on $X \setminus \overline{V}$.*

PROPOSITION 4.2.9. *Let U be an open set in X. Then $H_V H_U = H_U$ for every $V \in \mathcal{U}(U)$. Moreover, $H_U g \in \mathcal{H}(U)$ for every $g \in \mathcal{B}(X)$ such that $|g| \le w$ for some $w \in \mathcal{W} \cap \mathcal{C}(X)$.*

Proof. Let $V \in \mathcal{U}(U)$ and $q \in \mathcal{P}$. By Corollary 4.2.8, $H_U q \in \mathcal{H}(U)$ and hence $H_V H_U q = H_U q$. This shows that $H_V H_U = H_U$.

Now let $g \in \mathcal{B}(X)$ and $w \in \mathcal{W} \cap \mathcal{C}(X)$ such that $|g| \le w$. In view of the preceding considerations, it suffices to show that $H_U g$ is continuous on U. To that end let us consider the set

$$\mathcal{F} := \{f \colon f \text{ lower semicontinuous}, \ -w \le f \le g\}.$$

Obviously, $\mathcal{F}$ is $\vee$-stable and $\sup\{H_U f \colon f \in \mathcal{F}\} = H_U g$. Hence, by Lemma 6.1.1, there exists an increasing sequence (f_n) in $\mathcal{F}$ such that $\sup H_U f_n = H_U g$ on U. Since $\sup H_U f_n \in \mathcal{H}(U)$ by Proposition 4.2.6, we see that $H_U g$ is continuous on U. □

PROPOSITION 4.2.10. $\mathcal{W} \cap \mathcal{C}(X) = \mathcal{S}^+(X) \cap \mathcal{C}(X)$ *and*

$$\begin{aligned} \mathcal{P} &= \big\{p \in \mathcal{W} \cap \mathcal{C}(X) \colon \inf\{R_p^{X \setminus K} \colon K \text{ compact in } X\} = 0\big\} \\ &= \big\{p \in \mathcal{W} \cap \mathcal{C}(X) \colon h \in \mathcal{H}^+(X),\, h \le p \Rightarrow h = 0\big\}. \end{aligned}$$

Proof. The first equality follows from Proposition 4.2.9.

Let us fix $p \in \mathcal{W} \cap \mathcal{C}(X)$. If $p \in \mathcal{P}$ and $h \in \mathcal{H}^+(X)$ such that $h \le p$, then $h \in \mathcal{P}$ and $C(h) = \emptyset$, hence $h = R_h^{\emptyset} = 0$.

Next let us choose a decreasing sequence $(f_n) \in \mathcal{C}(X)$ such that, for each $n \in \mathbb{N}$, $0 \le f_n \le p$, $f_n = p$ outside a compact set in X, and the interiors of the sets $\{f_n = 0\}$ cover X. By Corollary 1.2.2, (R_{f_n}) is a decreasing sequence in $\mathcal{P}$ such that

$$g := \lim R_{f_n} \in \mathcal{H}^+(X) \quad \text{and} \quad g \le p.$$

It is immediately seen that

$$g = \inf\{R_p^{X \setminus K} \colon K \text{ compact in } X\}. \tag{2.4}$$

If the greatest harmonic minorant of p is the constant function 0, then, of course, $g = 0$. So let us finally assume that $g = 0$. Then, by Dini's lemma, $R_{f_n} \downarrow 0$ locally uniformly on X. So there exists a subsequence (q_n) of (R_{f_n}) such that the series $q := \sum_{n=1}^{\infty} q_n$ converges locally uniformly on X. Therefore $q \in \mathcal{W} \cap C(X)$ and $p/q \in \mathcal{C}_0(X)$ (see the proof of Proposition 1.2.1). Thus $p \in \mathcal{P}$. □

DEFINITION 4.2.11. *Given an open set U in X, a sequence (x_n) in U which converges to a point $z \in \partial U$ is* regular *provided*

$$\lim_{n \to \infty} H_U(x_n, \cdot) = \varepsilon_z.$$

It is called purely irregular*, if it does not contain any regular subsequence.*

An Evans function *on U is a function $w \in {}^*\mathcal{H}^+(U)$ such that $\lim_{n\to\infty} w(x_n) = \infty$ for every purely irregular sequence (x_n) in U (converging to some $z \in \partial U$).*

A point $z \in \partial U$ is a regular boundary point*, if every sequence in U, which converges to z, is regular.*

The set U is a regular set*, if every boundary point of U is regular.*

Let us note that a potential $p \in \mathcal{P}$ is called *strict* if any two $\mu, \nu \in \mathcal{M}(X)$ coincide provided that $\mu(p) = \nu(p) < \infty$ and $\mu(q) \le \nu(q)$ for all $q \in \mathcal{P}$ (cf. Definition 6.2.4). By Proposition 1.2.1 and Proposition 6.2.5, we know that there exist strict potentials.

LEMMA 4.2.12. *Let U be an open set in X and $x \in U$. Then there exists an Evans function w on U such that $w(x) < \infty$.*

Proof. Let $p \in \mathcal{P}$ be strict and let $f_n \in \mathcal{C}(X)$ such that $0 \le f_n \le p$ on X, $f_n = p$ in a neighborhood of U^c and $f_n \downarrow 1_{U^c} p$. By Corollary 1.2.2, $p_n := R_{f_n} \in \mathcal{P}$ and $p_n \downarrow H_U p$. Let $x \in U$. We may assume that $p_n(x) - H_U p(x) < 2^{-n}$. Then

$$w := \sum_{n=1}^{\infty} (p_n - H_U p) \in {}^*\mathcal{H}^+(U) \quad \text{and} \quad w(x) < 1.$$

Let (x_m) be a sequence in U which converges to a point $z \in \partial U$ and is purely irregular. Since $p_n = p$ in a neighborhood of U^c, we shall obtain that $\lim_{m\to\infty} w(x_m) = \infty$ provided we establish that

$$\liminf_{m\to\infty} (p - H_U p)(x_m) > 0. \tag{2.5}$$

To that end let us assume that there exists a subsequence (y_m) of (x_m) such that $\lim_{m\to\infty}(p - H_U p)(y_m) = 0$. If $q \in \mathcal{P}$, then $H_U q(y_m) \le q(y_m)$, $m \in \mathbb{N}$, where the sequence $(q(y_m))$ is bounded, since $\lim_{m\to\infty} q(y_m) = q(z)$. By Proposition 1.2.3, we may suppose that the measures $\mu_m := H_U(y_m, \cdot)$, $m \in \mathbb{N}$, converge weakly to a measure μ as $m \to \infty$ and that

$$\mu(q) = \lim \mu_m(q) = \lim H_U q(y_m) \qquad \text{for every } q \in \mathcal{P}.$$

In particular, $\mu(p) = p(z)$ and $\mu(q) \le q(z)$ for every $q \in \mathcal{P}$. Since p is strict, we obtain that $\mu = \varepsilon_z$. This, however, is impossible, since the sequence (x_n) is purely irregular. Thus (2.5) holds and the proof is finished. □

LEMMA 4.2.13. *For every $x \in X$ the following holds.*

1. $\lim_{U \downarrow x} H_U(x, \cdot) = \varepsilon_x^{X\setminus\{x\}}$.

2. *If x is not finely isolated, then $\lim_{U\downarrow x} H_U(x, \cdot) = \varepsilon_x$.*

3. *If x is finely isolated, then $R_1^{\{x\}} \in \mathcal{P}$.*

Proof. Let (U_n) be a sequence of open sets in X such that $\bigcap_{n=1}^{\infty} U_n = \{x\}$ and $\overline{U}_{n+1} \subset U_n$ for every $n \in \mathbb{N}$. Let $p \in \mathcal{P}$, $p(x) = 1$. Then $(R_p^{X\setminus\overline{U}_n})$ is an increasing sequence in $\mathcal{W}$, hence $w := \sup R_p^{X\setminus\overline{U}_n} \in \mathcal{W}$ such that $w \ge p$ on $X \setminus \{x\}$. Therefore $\sup R_p^{X\setminus U_n} = \sup R_p^{X\setminus\overline{U}_n} = R_p^{X\setminus\{x\}}$, that is, $\lim_{U\downarrow x} R_p^{X\setminus U} = R_p^{X\setminus\{x\}}$. If x is not finely isolated, then, of course, $R_p^{X\setminus\{x\}} = p$. Hence (1) and (2) follow by Proposition 6.2.2.

Finally, let us suppose that x is finely isolated. Then $q := R_1^{\{x\}} \in \mathcal{W}$, since $1_{\{x\}}$ is finely continuous. Moreover, $q \in \mathcal{H}^+(X \setminus \{x\})$ by Corollary 4.2.8. In particular, q is continuous on $X \setminus \{x\}$. Further, $q \le p$ and hence $\limsup_{y\to x} q(y) \le \limsup_{y\to x} p(y) = p(x) = 1 = q(x)$. So q is continuous on X, $q \in \mathcal{P}$. □

Chapter 5

Families of harmonic kernels

5.1 Balayage spaces and harmonic kernels

As before, let X be a locally compact space with countable base. Let $(H_U)_{U\in\mathcal{U}_0}$ be a family of *sweeping kernels* on X, that is, $\mathcal{U}_0$ is a base of in X consisting of relatively compact open sets and, for each $U \in \mathcal{U}_0$, we have a kernel H_U on X satisfying $H_U(x, U) = 0$ for every $x \in X$ and $H_U(x,\cdot) = \varepsilon_x$ for every $x \in U^c$. For every open set W in X, we define ${}^*\mathcal{H}^+(W)$, $\mathcal{S}^+(W)$, and $\mathcal{H}^+(W)$ as in Definition 4.2.1, but using the set

$$\mathcal{U}_0(W) := \mathcal{U}(W) \cap \mathcal{U}_0$$

instead of $\mathcal{U}(W)$. As we shall see in Corollary 5.2.9, this does not make any difference, once we have shown that $(X, {}^*\mathcal{H}^+(X))$ is a balayage space. By definition, the corresponding fine topology is the ${}^*\mathcal{H}^+(X)$-topology. Regularity for sets $U \in \mathcal{U}_0$ is to be understood as in Definition 4.2.11.

DEFINITION 5.1.1. *$(H_U)_{U\in\mathcal{U}_0}$ is called a* family of harmonic kernels *if the following axioms are satisfied (where $U, V \in \mathcal{U}$).*

(H_1) *For every $x \in X$, $\lim_{U\downarrow x} H_U(x,\cdot) = \varepsilon_x$ or $R_1^{\{x\}}$ is l.s.c. at x.*

(H_2) *$H_V H_U = H_U$, whenever $\overline{V} \subset U$.*

(H_3) *If $f \in \mathcal{B}_b(X)$ has compact support, then $H_U f$ is continuous and bounded on U.*

(H_4) *For every $x \in U$, there exists $w \in {}^*\mathcal{H}^+(U)$ such that $w(x) < \infty$ and, for every purely irregular sequence (x_n) in U, $\lim_{n\to\infty} w(x_n) = \infty$.*

(H_5) *${}^*\mathcal{H}^+(X)$ is linearly separating and there exists $s \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$, $s > 0$.*

The following result is an immediate consequence of Lemma 4.2.13, Proposition 4.2.9, Lemma 4.2.12, Proposition 1.2.1, and Proposition 4.2.10.

THEOREM 5.1.2. *If $(X, \mathcal{W})$ is a balayage space, then the corresponding kernels given by $H_U(x,\cdot) := \varepsilon_x^{U^c}$, $x \in U$, U relatively compact open in X, form a family of harmonic kernels.*

EXAMPLES 5.1.3. It is easy to recover our first standard examples in terms of harmonic kernels.

1. *Classical theory.* X open set in $\mathbb{R}^d$, $d \geq 1$, such that $X \neq \mathbb{R}$ for $d = 1$, $\overline{X} \neq \mathbb{R}^2$ for $d = 2$; $\mathcal{U}_0$ family of all open balls U such that $\overline{U} \subset X$, H_U given by the Poisson integral.

2. *Uniform translation on* $\mathbb{R}$. $\mathcal{U}_0 := \{(\alpha, \beta)\colon -\infty < \alpha < \beta < \infty\}$,

$$H_{(\alpha,\beta)}(x, \cdot) = \varepsilon_\alpha \qquad \text{for } \alpha < x < \beta$$

(take $w(x) := (\beta - x)^{-1}$).

3. *Discrete theory.* Let P be a sub-Markov kernel on a countable, discrete space X, $\mathcal{U}_0 := \big\{\{x\}\colon x \in X\big\}$ and define

$$H_{\{x\}}(x, A) := \begin{cases} \dfrac{P(x, A \setminus \{x\})}{1 - P(x, \{x\})}, & P(x, \{x\}) < 1, \\ 0, & P(x, \{x\}) = 1. \end{cases}$$

Then ${}^*\mathcal{H}^+(X) = S_P$ and (H_1) – (H_4) are trivially satisfied. (H_5) holds if S_P separates the points of X.

In Section 5.3 we shall prove the following converse to Theorem 5.1.2 (see Theorem 5.3.11 and Proposition 5.3.12).

THEOREM 5.1.4. *Let $(H_U)_{\mathcal{U}\in\mathcal{U}_0}$ be a family of harmonic kernels on X. Then $(X, {}^*\mathcal{H}^+(X))$ is a balayage space and $H_U(x, \cdot) = \varepsilon_x^{U^c}$ for all $x \in U \in \mathcal{U}_0$.*

5.2 Minimum principle and sheaf properties

Let $(H_U)_{U\in\mathcal{U}_0}$ be a family of harmonic kernels on X. We first note some simple consequences of the definitions.

LEMMA 5.2.1. *Let $U \in \mathcal{U}_0$ and $f \in \mathcal{B}^+(X)$. Then $H_U f \in {}^*\mathcal{H}^+(U)$. If there exists a function $g \in \mathcal{B}^+(X)$ such that $f \leq g$ and $(H_U g)|_U \in \mathcal{C}(U)$, then $H_U f \in \mathcal{H}^+(U)$.*

Proof. We may choose $f_n \in \mathcal{B}_b^+(X)$ with compact support such that $f_n \uparrow f$. Hence $H_U f = \sup H_U f_n$ is lower semicontinuous on X by (H_3). Thus $H_U f \in {}^*\mathcal{H}^+(U)$ by (H_2).

Finally, let $g \in \mathcal{B}^+(X)$ such that $f \leq g$ and $H_U g|_U \in \mathcal{C}(U)$. There exists $f' \in \mathcal{B}^+(X)$ with $f + f' = g$. Hence

$$H_U f + H_U f' = H_U g,$$

where $H_U f$ and $H_U f'$ are lower semicontinuous on U. Therefore $(H_U f)|_U \in \mathcal{C}(U)$ and even $H_U f \in \mathcal{H}^+(U)$ by (H_2). □

PROPOSITION 5.2.2. *Let W be an open set in X. Then the following holds.*

1. *${}^*\mathcal{H}^+(W)$ is a convex cone which is $\wedge$-stable and σ-stable.*

2. *$\mathcal{S}^+(W)$ is a convex subcone of ${}^*\mathcal{H}^+(W)$ such that $s \wedge v \in \mathcal{S}^+(W)$, whenever $s \in \mathcal{S}^+(W)$ and $v \in {}^*\mathcal{H}^+(W)$.*

3. *If W_n are open sets in W such that $W_n \uparrow W$, then ${}^*\mathcal{H}^+(W_n) \downarrow {}^*\mathcal{H}^+(W)$ and $\mathcal{H}^+(W_n) \downarrow \mathcal{H}^+(W)$.*

Proof. The second statement in (2) follows from Lemma 5.2.1. The other statements are immediate consequences of the definitions. □

PROPOSITION 5.2.3. *Let W be a open set in X and let (h_n) be a decreasing sequence in $\mathcal{H}^+(W)$. Then $\inf h_n \in \mathcal{H}^+(W)$.*

Proof. Let $h := \inf h_n$ and $U \in \mathcal{U}_0(W)$. Then $H_U h = \inf H_U h_n = \inf h_n = h$ and, by Lemma 5.2.1, $H_U h$ is continuous on U. □

Let W be an open set in X and, for every $x \in W$, let $\mathcal{V}(x)$ be a fundamental system of neighborhoods $U \in \mathcal{U}_0(W)$ of x. We define

$$ {}^*\mathcal{H}_\mathcal{V}(W) := \{v \in \mathcal{B}(X) \colon v \text{ l.s.c. on } W,\ -\infty < H_U v(x) \le v(x) \quad \text{for all } x \in W \text{ and } U \in \mathcal{V}(x)\}, $$

$$ \mathcal{H}_\mathcal{V}(W) = {}^*\mathcal{H}_\mathcal{V}(W) \cap (-{}^*\mathcal{H}_V(W)). $$

Of course, ${}^*\mathcal{H}(W) \subset {}^*\mathcal{H}_\mathcal{V}(W)$ and $\mathcal{H}(W) \subset \mathcal{H}_\mathcal{V}(W)$.

PROPOSITION 5.2.4. *If W is relatively compact, $v \in {}^*\mathcal{H}_\mathcal{V}(W)$, $v \ge 0$ on W^c, and $\liminf_{x\to z} v(x) \ge 0$ for every $z \in \partial W$, then $v \ge 0$.*

Proof. Let $s \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$, $s > 0$, $\varepsilon > 0$, and $w := v + \varepsilon s$. Clearly, w is lower semicontinuous on W, $w \ge 0$ on W^c, and, for every $z \in \partial W$,

$$ \liminf_{x\to z} w(x) = \liminf_{x\to z} v(x) + \varepsilon s(z) \ge \varepsilon s(z) > 0. $$

Hence $K := \{x \in X \colon w(x) \le 0\}$ is a compact set in W. We define

$$ \mathcal{F} := ({}^*\mathcal{H}^+(X) + \mathbb{R}^+ w)|_K. $$

Let $x \in K$, $U \in \mathcal{U}_0(W)$ such that $-\infty < H_U v(x) \le v(x)$, and $\mu := H_U(x,\cdot)|_K$. Then $\mu \ne \varepsilon_x$, since $H_U(x,U) = 0$. Since $w \ge 0$ on K^c, we see that

$$ \mu(w|_K) \le H_U w(x) = H_U v(x) + \varepsilon H_U s(x) \le v(x) + \varepsilon s(x) = w(x). $$

Obviously, $\mu(u|_K) \le H_U u(x) \le u(x)$ for every $u \in {}^*\mathcal{H}^+(X)$. So $Ch_\mathcal{F} K = \emptyset$. By (H_5) and Proposition 6.3.1, we hence see that $w \ge 0$ on K. Thus $v \ge 0$. □

LEMMA 5.2.5. *Let $U \in \mathcal{U}_0$, $z \in \partial U$, and let (x_n) be a regular sequence in U which converges to z. Moreover, let $v \in \mathcal{B}(X)$ be lower semicontinuous at z and $v \ge \varphi$ for some $\varphi \in \mathcal{K}(X)$. Then $\liminf_{n\to\infty} H_U v(x_n) \ge v(z)$.*

Proof. Let $a < v(z)$. We may assume that $\varphi(z) > a$. Then $a < \lim_{n\to\infty} H_U\varphi(x_n) \le \liminf_{n\to\infty} H_U v(x_n)$. □

LEMMA 5.2.6. *Let g be a numerical function on $U \in \mathcal{U}_0$, $z \in \partial U$, and $\alpha \in \mathbb{R}$ such that $\liminf_{x\to z, x\in U} g(x) > -\infty$ and $\liminf_{n\to\infty} g(x_n) \ge \alpha$ for every regular sequence (x_n) in U converging to z. Then $\liminf_{x\to z, x\in U}(g+w)(x) \ge \alpha$ for every Evans function w on U.*

Proof. Let w be an Evans function on U. There exists a sequence (x_n) in U such that $\lim_{n\to\infty} x_n = z$ and

$$\beta := \liminf_{x\to z, x\in U} (g+w)(x) = \lim_{n\to\infty} (g+w)(x_n).$$

If (x_n) contains a regular subsequence, then $\beta \geq \alpha$ by assumption. If not, then $\lim_{n\to\infty} w(x_n) = \infty$ and hence $\beta = \infty$. □

COROLLARY 5.2.7. *If* $W \in \mathcal{U}_0$ *and* $v \in {}^*\mathcal{H}_{\mathcal{V}}(W)$ *such that* $\inf v(W) > -\infty$, $v \geq 0$ *on* W^c, *and* $\liminf_{n\to\infty} v(x_n) \geq 0$ *for every regular sequence* (x_n). *Then* $v \geq 0$.

Proof. Lemma 5.2.6 and Proposition 5.2.4. □

COROLLARY 5.2.8. ${}^*\mathcal{H}_{\mathcal{V}}^+(W) = {}^*\mathcal{H}^+(W)$ *and* $\mathcal{H}_{\mathcal{V}}^+(W) = \mathcal{H}^+(W)$.

Proof. Let $v \in {}^*\mathcal{H}_{\mathcal{V}}^+(W)$, $U \in \mathcal{U}_0(W)$, and $f \in \mathcal{B}_b^+(X)$ with compact support such that $f \leq v$ and f is continuous on W. To prove that $v \in {}^*\mathcal{H}^+(W)$ it suffices to show that $H_U f \leq v$ (since v is the limit of an increasing sequence of such functions f). By (H_2) and (H_3), $H_U f \in \mathcal{H}(U)$. Moreover, $H_U f = f \leq v$ on U^c. If (x_n) is a regular sequence in U, then $\lim_{n\to\infty} H_U f(x_n) = f(z)$ by Lemma 5.2.5, and hence

$$\liminf_{n\to\infty} (v - H_U f)(x_n) \geq v(z) - f(z) \geq 0.$$

So, by Corollary 5.2.7, $v - H_U f \geq 0$. Thus ${}^*\mathcal{H}_{\mathcal{V}}^+(W) = {}^*\mathcal{H}^+(W)$.

Next we fix $h \in \mathcal{H}_{\mathcal{V}}^+(W)$ and $U \in \mathcal{U}_0$. Then $H_U h \leq h$, since $h \in {}^*\mathcal{H}_{\mathcal{V}}^+(W) = {}^*\mathcal{H}^+(W)$. Let $g \in \mathcal{B}(X)$ such that $g = 0$ on U^c and $g = H_U h - h$ on U. By Lemma 5.2.1, $g \in {}^*\mathcal{H}_{\mathcal{V}}(U)$ (and $g \leq 0$). Let (x_n) be a regular sequence in U converging to $z \in \partial U$. Then, by Lemma 5.2.5,

$$\liminf_{n\to\infty} g(x_n) = \liminf_{n\to\infty} H_U h(x_n) - h(z) \geq 0.$$

By Corollary 5.2.7, $g \geq 0$, that is, $h \leq H_U h$ whence $h = H_U h$. Thus $h \in \mathcal{H}^+(W)$. □

COROLLARY 5.2.9. *If* $(\tilde{H}_U)_{U\in\tilde{\mathcal{U}}_0}$ *is a family of harmonic kernels on* X *such that* $\tilde{\mathcal{U}}_0 \cap \mathcal{U}_0$ *is a base of* X *and* $\tilde{H}_U = H_U$ *for every* $U \in \tilde{\mathcal{U}}_0 \cap \mathcal{U}_0$, *then, for every open set* W *in* X, ${}^*\mathcal{H}^+(W)$ $(\mathcal{H}^+(W)$, *respectively*) *is the set of all positive functions on* X *which are hyperharmonic on* W (*harmonic on* W, *respectively*) *with respect to* $(\tilde{H}_U)_{U\in\tilde{\mathcal{U}}_0}$.

COROLLARY 5.2.10. *For every family* $(W_i)_{i\in I}$ *of open sets in* X,

$${}^*\mathcal{H}^+(\bigcup_{i\in I} W_i) = \bigcap_{i\in I} {}^*\mathcal{H}^+(W_i) \quad \text{and} \quad \mathcal{H}^+(\bigcup_{i\in I} W_i) = \bigcap_{i\in I} \mathcal{H}^+(W_i).$$

COROLLARY 5.2.11. *Let* V, W *be open sets in* X, $V \subset W$. *Let* $v \in {}^*\mathcal{H}^+(V)$ *and* $w \in {}^*\mathcal{H}^+(W)$ *such that* $v \geq w$ *on* V^c *and* $\liminf_{x\to z, x\in V} v(x) \geq w(x)$ *for every* $z \in \partial V \cap W$. *Then* $u := v \wedge w \in {}^*\mathcal{H}^+(W)$.

Proof. Clearly, u is lower semicontinuous on W. If $x \in V$ and $U \in \mathcal{U}_0(V)$, then $H_U v(x) \leq v(x)$ and $H_U w(x) \leq w(x)$, hence $H_U u(x) \leq u(x)$. If $x \in W \setminus U$ and $U \in \mathcal{U}_0(W)$, then $H_U u(x) \leq H_U w(x) \leq w(x) = u(x)$. Thus $u \in {}^*\mathcal{H}^+(W)$, by Corollary 5.2.8. □

5.3 Harmonic kernels and balayage spaces

Again let $(H_U)_{U\in\mathcal{U}_0}$ be a family of harmonic kernels on X. We define

$$X_0 := \{x \in X \colon \lim_{U\downarrow x} H_U(x,\cdot) = \varepsilon_x\}$$

(cf. Lemma 4.2.13). For every numerical function f on X, let

$$R_f := \inf\{v \in {}^*\mathcal{H}^+(X) \colon v \ge f\}$$

(which coincides with (1.2), if $(X,\mathcal{W})$ is a balayage space and $\mathcal{W} = {}^*\mathcal{H}^+(X)$).

PROPOSITION 5.3.1. *Let $\mathcal{V} \subset {}^*\mathcal{H}^+(X)$ and $v := \inf \mathcal{V}$. Then $\hat{v} \in {}^*\mathcal{H}^+(X)$, $\hat{v} = v$ on $X \setminus X_0$.*

In particular, $\hat{R}_f \in {}^*\mathcal{H}^+(X)$ for every $f \colon X \to [-\infty,\infty]$. Moreover, $\hat{R}_f = R_f$, if f is lower semicontinuous.

Proof. Let $U \in \mathcal{U}_0$. Then

$$H_U\hat{v} \le \inf_{w\in\mathcal{V}} H_U w \le \inf_{w\in\mathcal{V}} w = v,$$

where $H_U\hat{v}$ is lower semicontinuous on U, by Lemma 5.2.1. Hence $H_U\hat{v} \le \hat{v}$. Thus $\hat{v} \in {}^*\mathcal{H}^+(X)$.

Let us now consider $x \in X \setminus X_0$. Then $w \ge v(x)R_1^{\{x\}}$ for every $w \in \mathcal{V}$ and hence $v \ge v(x)R_1^{\{x\}}$. By (H_1), $\hat{R}_1^{\{x\}}(x) \ge 1$ and hence $\hat{v}(x) \ge v(x)$, that is, $\hat{v}(x) = v(x)$.

Finally, let $f \colon X \to [-\infty,\infty]$. Then $\hat{R}_f \in {}^*\mathcal{H}^+(X)$ by the previous considerations. If f is lower semicontinuous, then $R_f \ge f$ implies that $\hat{R}_f \ge f$ and hence $\hat{R}_f \ge R_f$. The converse inequality is trivial. □

COROLLARY 5.3.2. *Let $v \in {}^*\mathcal{H}^+(X)$ and $U \in \mathcal{U}_0$. Then $\widehat{H_U v} \in {}^*\mathcal{H}^+(X)$.*

Proof. Let w be an Evans function on U and $u := (H_U v + w) \wedge v$. By Lemma 5.2.6 and Corollary 5.2.11, $u \in {}^*\mathcal{H}^+(X)$. Let $\mathcal{V}$ be the set of all functions obtained this way, varying the Evans function w. By (H_4), $\inf \mathcal{V} = H_U v$. Thus $\widehat{H_U v} \in {}^*\mathcal{H}^+(X)$, by Proposition 5.3.1.

□

PROPOSITION 5.3.3. *Let $f \in \mathcal{C}^+(X)$ such that $f \le s_0$ for some $s_0 \in \mathcal{S}^+(X)$. Then $R_f \in \mathcal{S}^+(X) \cap \mathcal{H}(X \setminus S(f))$. If $U \in \mathcal{U}_0$ such that $H_U f \ge f$, then $H_U R_f = R_f$.*

Proof. By Proposition 5.3.1 and Proposition 5.2.2,

$$s := R_f = \hat{R}_f \in \mathcal{S}^+(X).$$

If $U \in \mathcal{U}_0$ such that $H_U f \ge f$, then $H_U s \ge H_U f \ge f$ and hence $\widehat{H_U s} \ge f$, where, by Corollary 5.3.2, $\widehat{H_U s} \in \mathcal{S}^+(X)$. Thus $\widehat{H_U s} \ge s$, that is, $H_U s = s$. It remains to show that s is upper semicontinuous and real.

Let us fix $x \in X$, $\varepsilon > 0$, and $a, b \in \mathbb{R}$ such that $a \le b$, $a \le \sup_{x\in U\in\mathcal{U}_0} H_U s(x)$, and $f(x) \le b \le s(x)$. By (H_5), there exists $t \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that

$$b - a < t(x) < b - a + \varepsilon.$$

Then $f(x) - t(x) \le b - t(x) < a$, hence there exists $U \in \mathcal{U}_0$ such that $x \in U$ and

$$f(x) - t(x) < H_U s(x).$$

So there exists $V \in \mathcal{U}_0(U)$ such that $x \in V$ and $f - t < H_U s$ on V, and hence

$$f \le t + \widehat{H_V s},$$

since $H_U s = H_V H_U s \le H_V s$ and $H_V s = s \ge f$ on V^c. Therefore $s = R_f \le t + \widehat{H_V s}$. In particular, $s(x) < \infty$. Choosing $a := \sup_{x \in U \in \mathcal{U}_0} H_U s(x)$, $b := s(x)$ and using the continuity of $t + \widehat{H_V s}$ on V, we finally conclude that

$$\limsup_{y\to x} s(y) \le t(x) + \widehat{H_V s}(x) < s(x) - a + \varepsilon + \widehat{H_V s}(x) \le s(x) + \varepsilon.$$

Thus s is upper semicontinuous at x. □

COROLLARY 5.3.4. *For every $v \in {}^*\mathcal{H}^+(X)$, there exists an increasing sequence (s_n) in $\mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that $v = \sup s_n$.*

Proof. Let (φ_n) be an increasing sequence in $\mathcal{K}^+(X)$, $\varphi_n \uparrow v \in {}^*\mathcal{H}^+(X)$. We define $s_n := R_{\varphi_n}$, $n \in \mathbb{N}$. By (H_5) and Proposition 5.3.3, (s_n) is an increasing sequence in $\mathcal{S}^+(X) \cap \mathcal{C}(X)$. Since $\varphi_n \le s_n \le v$ and $\sup \varphi_n = v$, we finally see that $\sup s_n = v$. □

LEMMA 5.3.5. *For all $x \in X_0$ and $v \in {}^*\mathcal{H}^+(X)$, $\lim_{U \downarrow x} H_U v(x) = v(x)$.*

Proof. It suffices to consider the case $v(x) > 0$. Given $0 \le a < v(x)$, there exists $\varphi \in \mathcal{K}(X)$ such that $0 \le \varphi \le v$ and $\varphi(x) = a$. Then

$$a = \lim_{U\downarrow x} H_U \varphi(x) \le \liminf_{U\downarrow x} H_U v(x) \le \limsup_{U\downarrow x} H_U v(x) \le v(x).$$

□

LEMMA 5.3.6. *Let K be a compact fine neighborhood of a point $x \in X_0$. Then $\lim_{U\downarrow x} H_U(x, K) = 1$.*

Proof. By Lemma 1.1.1, there exists a relatively compact open neighborhood V of x, a function $u \in {}^*\mathcal{H}^+(X)$, and $\alpha \in \mathbb{R}$ such that $u(x) < \alpha$ and

$$V \cap \{u < \alpha\} \subset K.$$

Let $s \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that $s(x) = u(x)$ and define $u' := u \wedge s$. Moreover, let $\beta \in (u(x), \alpha)$ and $W := V \cap \{s < \beta\}$. Then W is an open neighborhood of x. Further,

$$u - u' \ge \alpha - \beta > 0 \quad \text{on } W \setminus K,$$

and hence, using Lemma 5.3.5,

$$(\alpha - \beta) \limsup_{U\downarrow x} H_U(x, W \setminus K) \le \lim_{U\downarrow x} H_U(u - u')(x) = u(x) - u'(x) = 0.$$

So $\lim_{U\downarrow x} H_U(x, W \setminus K) = 0$. Since $\lim_{U\downarrow x} H_U(x, W) = 1$ by definition of X_0, the statement follows. □

PROPOSITION 5.3.7. *For all* $\mathcal{V} \subset {}^*\mathcal{H}^+(X)$, $\widehat{\inf \mathcal{V}} = \widehat{\inf \mathcal{V}}^f$.

Proof. Let $\mathcal{V} \subset {}^*\mathcal{H}^+(X)$, $v := \inf \mathcal{V}$. Obviously, $\hat{v} \le \hat{v}^f \le v$. By Proposition 5.3.1, we know that $\hat{v} = v$ on $X \setminus X_0$.

So let $x \in X_0$ and $\alpha < \hat{v}^f(x)$. There exists a compact fine neighborhood K of x such that $\alpha \le v$ on K. Let $U \in \mathcal{U}_0$ containing x. For every $w \in \mathcal{V}$, $\alpha 1_K \le v \le w$ and hence $\alpha H_U 1_K \le H_U w \le w$. So $\alpha H_U 1_K \le v$. By (H_3), $H_U 1_K$ is continuous on U. Therefore $\alpha H_U 1_K \le \hat{v}$ on U. By Lemma 5.3.6, we finally see that $\alpha \le \hat{v}(x)$. □

PROPOSITION 5.3.8. ${}^*\mathcal{H}^+(X)$ *is a potential cone.*

Proof. Let $u, v \in {}^*\mathcal{H}^+(X)$ and $f := 1_{\{v<\infty\}}(u-v)^+$. Then f is finely lower semicontinuous and hence, by Proposition 5.3.1 and Proposition 5.3.7, $R_f \in {}^*\mathcal{H}^+(X)$. By Definition 1.1.7, we have to show that there exists $w \in {}^*\mathcal{H}^+(X)$ such that $u = w + R_f$.

Let us first suppose that $u, v \in \mathcal{C}(X)$ and hence $f = (u-v)^+$. Then, by Proposition 5.3.3,

$$s := R_f \in \mathcal{S}^+(X) \cap \mathcal{C}(X).$$

So $u - s \in \mathcal{C}^+(X)$. We have to prove that $u - s \in {}^*\mathcal{H}^+(X)$. To that end we fix $U \in \mathcal{U}_0$, an Evans function w for U, $\varphi \in \mathcal{K}(X)$ such that $0 \le \varphi \le u$, and define

$$s' := s \wedge (H_U s + u - H_U\varphi + w).$$

Since $H_U\varphi \le H_U u \le u$, we have $H_U s + u - H_U\varphi + w \in {}^*\mathcal{H}^+(U)$. Moreover,

$$H_U s + u - H_U\varphi + w = s + u - \varphi + w \ge s \quad \text{on } X \setminus U.$$

If $z \in \partial U$ and (x_n) is a regular sequence converging to z, then

$$\liminf_{n\to\infty} (H_U s + u - H_U\varphi)(x_n) \ge s(z) + u(z) - \varphi(z) \ge s(z).$$

By Lemma 5.2.6 and Corollary 5.2.11, we hence obtain that $s' \in {}^*\mathcal{H}^+(X)$. We have $H_U\varphi \le H_U u \le H_U v + H_U s \le v + H_U s$,

$$u = H_U\varphi + u - H_U\varphi \le v + H_U s + u - H_U\varphi,$$

and hence $u - v \le s'$. Therefore $s = R_{u-v} \le s' \le H_U s + u - \mathcal{H}_U\varphi + w$. This implies that $s \le H_U s + u - H_U u$, that is, $H_U(u-s) \le u - s$. Thus $u - s \in {}^*\mathcal{H}^+(X)$ (and even $u - s \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$).

Next let us suppose that $u = R_\varphi$ for some $\varphi \in \mathcal{K}^+(X)$. By Corollary 5.3.4, we may choose $v_n \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that $v_n \uparrow v$. By the previous part,

$$w_n := u - R_{u-v_n} \in {}^*\mathcal{H}^+(X) \qquad (n \in \mathbb{N}).$$

Clearly, the sequence (w_n) is increasing to a function $w \in {}^*\mathcal{H}^+(X)$ such that

$$u = w + \inf R_{u-v_n}. \tag{3.1}$$

Obviously, $\inf R_{u-v_n} \ge R_f$. To prove the reverse inequality let $w \in {}^*\mathcal{H}^+(X)$ such that $w > u - v$, that is, $w + v > u \ge \varphi$. Then there exists $n \in \mathbb{N}$ such that

$w + v_n > \varphi$ on $S(\varphi)$, and hence $w + v_n \geq R_\varphi = u$, $w \geq R_{u-v_n} \geq \inf R_{u-v_n}$. Therefore $R_f \geq \inf R_{u-v_n}$. So (3.1) implies that $u = w + R_f$.

Finally, let $u, v \in {}^*\mathcal{H}^+(X)$ and $\varphi_n \in \mathcal{K}^+(X)$, $\varphi_n \uparrow u$. Then $u_n := R_{\varphi_n} \uparrow u$ and $f_n := 1_{\{v<\infty\}}(u_n - v)^+) \uparrow f$. By the preceding part, $R_{f_n} \in {}^*\mathcal{H}^+(X)$ and there exist $w'_n \in {}^*\mathcal{H}^+(X)$ such that

$$u_n = w'_n + R_{f_n} \qquad (n \in \mathbb{N}). \tag{3.2}$$

The sequence (R_{f_n}) is increasing to a function $w' \in {}^*\mathcal{H}^+(X)$ such that $w' \geq f$. Therefore $\lim_{n\to\infty} R_{f_n} = w' = R_f$. For every $n \in \mathbb{N}$, let

$$w_n := \inf_{m\geq n} w'_m.$$

The sequence (w_n) is increasing and, by (3.2),

$$u = \sup w_n + R_f.$$

where we use the fact that $u = \infty$ on $\{R_f = \infty\}$. For every $n \in \mathbb{N}$, $\hat{w}_n \in {}^*\mathcal{H}^+(X)$ by Proposition 5.3.1 and the set $\{\hat{w}_n < w_n\}$ is finely meager by Proposition 1.1.5. Therefore $w := \sup \hat{w}_n \in {}^*\mathcal{H}^+(X)$ satisfies $u = w + R_f$. □

PROPOSITION 5.3.9. *For every $u \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X)$, the following properties are equivalent.*

1. *The greatest harmonic minorant of u is zero.*

2. *There exists $v \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X)$, $v > 0$, such that $u/v \in \mathcal{C}_0(X)$.*

For every $u \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X)$, there exists a greatest harmonic minorant h of u. If (f_n) is a decreasing sequence in $\mathcal{C}^+(X)$ such that the sets $\{f_n < u\}$, $n \in \mathbb{N}$, are relatively compact and the interiors of the sets $\{f_n = 0\}$ cover X, then $h = \inf R_{f_n}$.

Proof. Let (f_n) be as indicated. Then, by Proposition 5.2.3, $h := \inf R_{f_n} \in \mathcal{H}^+(X)$. Next, let $g \in \mathcal{H}^+(X)$ such that $g \leq u$. Then, for every $n \in \mathbb{N}$, $R_{f_n} \geq g$ outside the relatively compact open set $\{f_n < u\}$ and hence, by the minimum principle (Proposition 5.2.4), $R_{f_n} \geq g$ on X. Therefore $h \geq g$, that is, h is the greatest harmonic minorant of u.

If $h = 0$, then there is a subsequence (q_n) of (R_{f_n}) such that the series $v := \sum_{n=1}^\infty q_n$ is locally uniformly convergent. Since $q_n = u$ outside a compact set in X, we obtain that $u/v \in \mathcal{C}_0(X)$ (cf. the last lines of the proof of Proposition 1.2.1.

Finally, let us suppose that there exists $v \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X)$, $v > 0$, such that $u/v \in \mathcal{C}_0(X)$. Then, given $\varepsilon > 0$, there exists a compact set K in X, such that $u \leq \varepsilon v$ on K^c. If $n \in \mathbb{N}$, such that $f_n = 0$ on K then $\varepsilon v \geq R_{f_n} \geq h$. Thus $h = 0$. □

PROPOSITION 5.3.10. *There exist strictly positive $u, v \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X)$ such that $u/v \in \mathcal{C}_0(X)$.*

Proof. Let us fix a strictly positive $s_0 \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$.

1. Let $x \in U \in \mathcal{U}_0$. We claim that there exists $s \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X)$ such that $s \le s_0$ and $H_U s(x) < s(x)$. Indeed, if $H_U(x,\cdot) = 0$, then $H_U s_0(x) = 0 < s_0(x)$. So let us assume that $H_U(x,\cdot) \neq 0$ and let y be a point in the support of $H_U(x,\cdot)$. By (H_5) and Proposition 5.3.4, there exist $s_1, s_2 \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that $s_j \le s_0$, $j = 1, 2$, and $s_1(x)s_2(y) < s_1(y)s_2(x)$. Defining

$$s := \min\{s_1(x)s_2, s_2(x)s_1\} \quad \text{and} \quad t := s_2(x)s_1$$

we have $s, t \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$, $s \le t$, $s(y) < t(y)$, and $s(x) = t(x)$. Thus

$$H_U s(x) < H_U t(x) \le t(x) = s(x).$$

By continuity, $H_U s < s$ in a neighborhood of x.

2. By the previous part of the proof we may choose a sequence (U_n) in $\mathcal{U}_0$ and a sequence (s_n) in ${}^*\mathcal{H}^+(X) \cap \mathcal{C}(X)$ such that $s_n \le s_0$, $n \in \mathbb{N}$, and the sets $\{H_{U_n} s_n < s_n\}$ cover X. Then

$$s := \sum_{n=1}^{\infty} 2^{-n} s_n \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X).$$

Let h be the greatest harmonic minorant of s and

$$u := s - h.$$

Then $u \in {}^*\mathcal{H}^+(X)$. Since $H_{U_n} h = h$, we see that the sets $\{H_{U_n} u < u\}$, $n \in \mathbb{N}$, cover X. In particular, $u > 0$ on X. By Proposition 5.3.9, there exists $v \in {}^*\mathcal{H}^+(X)$, $v > 0$, such that $u/v \in \mathcal{C}_0(X)$. □

Combining the previous results we obtain the following.

THEOREM 5.3.11. *Let $(H_U)_{\mathcal{U} \in \mathcal{U}_0}$ be a family of harmonic kernels on X. Then $(X, {}^*\mathcal{H}^+(X))$ is a balayage space.*

As before let $\mathcal{P}$ denote the set of all continuous real potentials on X, that is,

$$\mathcal{P} := \{u \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X) \colon u/v \in \mathcal{C}_0(X) \text{ for some } v \in {}^*\mathcal{H}^+(X) \cap \mathcal{C}(X),\ v > 0\}.$$

To finish the proof of Theorem 5.1.4 it remains to prove the following.

PROPOSITION 5.3.12. *For all $x \in U \in \mathcal{U}_0$, $H_U(x,\cdot) = \varepsilon_x^{U^c}$.*

Proof. Let $U \in \mathcal{U}_0$, $p \in \mathcal{P}$. If $w \in {}^*\mathcal{H}^+(X)$ such that $w \ge p$ on U^c, then, obviously, $w \ge H_U w \ge H_U p$. Hence $R_p^{U^c} \ge H_U p$. To prove the reverse inequality, we take an Evans function w_0 for U. Then $u := H_U p + w_0 \in {}^*\mathcal{H}^+(U)$, $\liminf_{x \to z} u(x) \ge p(z)$ for every $z \in \partial U$. Therefore, by Corollary 5.2.11.

$$w := (H_U p + w_0) \wedge p \in {}^*\mathcal{H}^+(X).$$

Since $w \ge p$ on U^c, we see that $w \ge R_p^{U^c}$. This implies that $H_U p \ge R_p^{U^c}$ on U. □

5.4 Restriction on subsets

Let $(X, \mathcal{W})$ be a balayage space and let $(H_U)_{U \in \mathcal{U}}$ be a corresponding family of harmonic kernels. We shall consider restrictions on open sets in X (Proposition 5.4.1) and restrictions on absorbing sets in X (Proposition 5.4.3; cf. also Proposition 5.4.4).

PROPOSITION 5.4.1. *For every open set U in X, $(U, {}^*\mathcal{H}^+(U)|_U)$ is a balayage space, $(H_V|_U)_{V \in \mathcal{U}(U)}$ is a corresponding family of harmonic kernels.*

Proof. For every $U \in \mathcal{U}(U)$, let

$$^U H_V := H_V|_U.$$

To prove that $(^U H_V)_{V \in \mathcal{U}(U)}$ is a family of harmonic kernels, we shall follow the convention that numerical functions on U are identified in an obvious way with functions on X vanishing outside U. (This does not create a problem with ${}^*\mathcal{H}^+(U)|_U$, since trivially $1_U v \in {}^*\mathcal{H}^+(U)$ for every $v \in {}^*\mathcal{H}^+(U)$.)

(H_1): Let $x \in U$. If x is not finely isolated, then, by Lemma 4.2.13,

$$\varepsilon_x = \lim_{V \downarrow x} H_V(x, \cdot) = \lim_{V \downarrow x} {}^U H_V(x, \cdot).$$

If x is finely isolated, then $R_1^{\{x\}}$ is lower semicontinuous at x. Let $V \in \mathcal{U}(U)$ such that $x \in V$. Using the extension result Proposition 4.2.5, we see that there exists $q \in \mathcal{P}$ such that

$$R_{1+q(x)}^{\{x\}} \le {}^U R_1^{\{x\}} + q \qquad \text{on } V,$$

where ${}^U R_1^{\{x\}} := \inf\{u \colon u \in {}^*\mathcal{H}^+(U)|_U, u(x) \ge 1\}$. Hence in this case

$$\liminf_{y \to x} {}^U R_1^{\{x\}}(y) \ge \liminf_{y \to x, y \in V} \big(R_{1+q(x)}^{\{x\}}(y) - q(y)\big) = 1 = {}^U R_1^{\{x\}}(x).$$

(H_2): If $V, W \in \mathcal{U}(U)$ and $\overline{V} \subset W$, then, for every $f \in \mathcal{K}(U)$,

$${}^U H_V {}^U H_W f = {}^U H_V H_W f = H_V H_W f = H_W f = {}^U H_W f.$$

(H_3) – (H_5) follow immediately from the definitions. □

A closed set A in X is called *absorbing*, if

$$\varepsilon_x^{A^c} = 0 \qquad \text{for every } x \in A.$$

Let us note some equivalent properties.

PROPOSITION 5.4.2. *Let A be a subset of X. Then the following properties are equivalent:*

(i) *A is an absorbing set.*

(ii) *For every $u \in \mathcal{W}$, $1_{A^c} u \in \mathcal{W}$.*

(iii) *A is closed and $H_U(x, A^c) = 0$ for all $x \in A$ and $x \in U \in \mathcal{U}$.*

(iv) *If* $u \in \mathcal{W}$*, then* $u_A := 1_A u + \infty \cdot 1_{A^c} \in \mathcal{W}$*.*

(v) *There exists* $u \in \mathcal{W}$ *such that* $A = \{u = 0\}$*.*

(vi) *There exists* $u \in \mathcal{W}$ *such that* $A = \overline{\{u < \infty\}}$*.*

Proof. Let $u_0 \in \mathcal{W} \cap \mathcal{C}(X)$, $u_0 > 0$.

(i) $\Rightarrow$ (ii): Let $u \in \mathcal{W}$. Then $R_u^{A^c} \in \mathcal{W}$, since A^c is open. Moreover, $R_u^{A^c} = u$ on A^c, whereas $R_u^{A^c}(x) = \varepsilon_x^{A^c}(u) = 0$ for every $x \in A$. Therefore $1_{A^c} u = R_u^{A^c} \in \mathcal{W}$.

(ii) $\Rightarrow$ (iii): Let $v := 1_{A^c} u_0$. Then $v \in \mathcal{W}$, $A = \{v \leq 0\}$ is closed, and, for all $x \in A$ and $U \in \mathcal{U}$ containing x, the inequality $H_U v(x) \leq v(x) = 0$ implies that $H_U(x, A^c) = 0$.

(iii) $\Rightarrow$ (iv): Let $u \in \mathcal{W}$. Clearly, u_A is lower semicontinuous, since A is closed. Let $x \in U \in \mathcal{U}$. If $x \in A$, then $H_U u_A(x) = H_U u(x) \leq u(x) = u_A(x)$. If $x \in A^c$, then $H_U u_A(x) \leq \infty = u_A(x)$. Thus $u \in {}^*\mathcal{H}^+(X) = \mathcal{W}$.

(iv) $\Rightarrow$ (v): Trivial, since $0 \in \mathcal{W}$.

(v) $\Rightarrow$ (vi): Obviously, $v := \sup_{n \in \mathbb{N}} (nu) \in \mathcal{W}$, $v = \infty$ on A^c and $v = 0$ on the closed set $A = \{u \leq 0\}$.

(vi) $\Rightarrow$ (i): Let $v := \widehat{\inf_{n \in \mathbb{N}} (u/n)}$. Then $v \in \mathcal{W}$, $v = 0$ on A, and $v = \infty$ on A^c. In particular, A is closed. Let $x \in A$. Then, for every $n \in \mathbb{N}$, $n\varepsilon_x^{A^c}(u_0) = R_{nu_0}^{A^c}(x) \leq R_v^{A^c}(x) \leq v(x) = 0$. Therefore $\varepsilon_x^{A^c}(u_0) = 0$, that is, $\varepsilon_x^{A^c} = 0$. □

If A is an absorbing set, then A is finely open, since $v := 1_{A^c}\infty \in \mathcal{W}$ and $A = \{v < \infty\}$.

PROPOSITION 5.4.3. *Let* A *be an* absorbing set *in* X*. Then* $(A, \mathcal{W}|_A)$ *is a balayage space, a corresponding family of harmonic kernels is given by* ${}^A H_U(x, B) := H_U(x, B) = \varepsilon_x^{U^c}(B)$*,* $U \in \mathcal{U}$ *with* $A \cap U \neq \emptyset$*,* $x \in A \cap U$*,* $B \in \mathcal{B}(A)$*.*

Proof. Since A is closed, finely open, and, for every $u \in \mathcal{W}$, $1_A u + \infty \cdot 1_{A^c} \in \mathcal{W}$, it is easily seen that $(A, \mathcal{W}|_A)$ satisfies (B_1) – (B_4) (see Definition 1.1.3). Moreover, we obtain that, for all $U \in \mathcal{U}$, $x \in A \cap U$, and $v \in \mathcal{W}$,

$$\inf\{v(x) \colon v \in \mathcal{W}|_A, v \geq u \text{ on } A \setminus U\} = R_v^{U^c}(x).$$

This implies the additional statement about corresponding harmonic kernels. □

Finally, let us note that, more generally, the following holds (for a proof see [1, VI.6.16]).

PROPOSITION 5.4.4. *Let* A *be any closed set in* X *such that* $\hat{\varepsilon}_x^A = \varepsilon_x$ *for every* $x \in A$*. Then* $(A, \mathcal{W}|_A)$ *is a balayage space and, for all* $x \in A$ *and sets* B *in* A*, the measure obtained by sweeping* ε_x *on* B *with respect to* $(A, \mathcal{W}|_A)$ *is the measure* $\hat{\varepsilon}_x^B$ *(obtained sweeping* ε_x *on* B *with respect to* $(X, \mathcal{W})$*).*

Chapter 6

Appendix

6.1 Two topological lemmas

LEMMA 6.1.1. *Let $\mathcal{F}$ be a family of lower semicontinuous numerical functions on X. Then there exists a countable family $\mathcal{F}_0 \subset \mathcal{F}$ such that* $\sup \mathcal{F}_0 = \sup \mathcal{F}$*.*

Proof. Since X has a countable base, every union of open sets is a countable union. So there exist countable families $\mathcal{G}_s \subset \mathcal{F}$, $s \in \mathbb{Q}$, such that

$$\bigcup_{f \in \mathcal{G}_s} \{f > s\} = \bigcup_{f \in \mathcal{F}} \{f > s\},$$

that is, $\{\sup \mathcal{G}_s > s\} = \{\sup \mathcal{F} > s\}$. Let $\mathcal{F}_0 := \bigcup_{s \in \mathbb{Q}} \mathcal{G}_s$. Then $\mathcal{F}_0 \subset \mathcal{F}$ is countable. If $x \in X$ and $a < \sup \mathcal{F}(x)$, then there exists $s \in \mathbb{Q}$ such that $a < s < \sup \mathcal{F}(x)$ and hence $\sup \mathcal{F}_0(x) \geq \sup \mathcal{G}_s(x) > s > a$. Thus $\sup \mathcal{F}_0 = \sup \mathcal{F}$. □

LEMMA 6.1.2. *Let $\mathcal{F}$ be a family of numerical functions on X. Then there exists a countable family $\mathcal{F}_0$ in $\mathcal{F}$ such that* $\widehat{\inf \mathcal{F}_0} = \widehat{\inf \mathcal{F}}$*.*

Proof. Let $\{U_n \colon n \in \mathbb{N}\}$ be a base of X. For every $n \in \mathbb{N}$, there exists a countable family $\mathcal{F}_n$ in $\mathcal{F}$ such that

$$\inf\{f(y) \colon y \in U_n, f \in \mathcal{F}_n\} = \inf\{f(y) \colon y \in U_n, f \in \mathcal{F}\}.$$

Then $\mathcal{F}_0 := \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ has the desired property. Indeed, it suffices to show that

$$\widehat{\inf \mathcal{F}_0}(x) \leq \widehat{\inf \mathcal{F}}(x)$$

for every $x \in X$ such that $\widehat{\inf \mathcal{F}_0}(x) > -\infty$. So let us fix $x \in X$, $a \in \mathbb{R}$, and assume that $a < \widehat{\inf \mathcal{F}_0}(x)$. There exists $n \in \mathbb{N}$ such that $x \in U_n$ and $a < \inf \mathcal{F}_0(y)$ for every $y \in U_n$, and therefore

$$a \leq \inf\{f(y) \colon y \in U_n, f \in \mathcal{F}_n\} = \inf\{f(y) \colon y \in U_n, f \in \mathcal{F}\},$$

which yields that $a \leq \inf \mathcal{F}$ on U_n. Thus $a \leq \widehat{\inf \mathcal{F}}(x)$. □

6.2 Function cones

DEFINITION 6.2.1. *A* function cone *(on X) is a convex cone S in $\mathcal{C}(X)$ having the following two properties.*

1. *S^+ is linearly separating.*
2. *S is* adapted*, that is, for every $u \in S$, there exists $v \in S$ such that $v > 0$ and $u/v \in \mathcal{C}_0(X)$.*

Given a function cone S, let

$$W(S) := \{u_1 \wedge \cdots \wedge u_k \colon k \in \mathbb{N},\, u_1, \ldots, u_k \in S\}.$$

Then $W(S)$ is a $\wedge$-stable function cone, the smallest one containing S. Moreover, S^+, $W(S^+)$, $W(S^+)-W(S^+)$, and the linear space $\mathcal{C}_S(X)$ of all S-bounded functions in $\mathcal{C}(X)$ are function cones.

PROPOSITION 6.2.2. *Let S be a $\wedge$-stable function cone in $\mathcal{C}^+(X)$ and $v_0 \in S$, $v_0 > 0$. Then, for all $f \in \mathcal{K}^+(X)$ and $\varepsilon > 0$, there exist $u, v \in S$ such that*

$$0 \le u - v \le f \le u - v + \varepsilon v_0. \tag{2.1}$$

Proof. Let $w_0 \in S$, $w_0 > 0$ such that $v_0/w_0 \in \mathcal{C}_0(X)$ and $w_0 \le v_0$ on $S(f)$. Then

$$\mathcal{F} := \{\frac{u}{w_0} \colon u \in S,\ u \le \alpha v_0 \text{ for some } \alpha > 0\}$$

is a $\wedge$-stable, linearly separating convex cone in $\mathcal{C}_0(X)$. By a version of the approximation theorem of Stone-Weierstrass, $\mathcal{F} - \mathcal{F}$ is dense in $\mathcal{C}_0(X)$ with respect to uniform convergence. In particular, there exist $u, u' \in \mathcal{F}$ such that

$$|\frac{f}{w_0} - \frac{u - u'}{w_0}| \le \frac{\varepsilon}{2}.$$

Then $v := u \wedge (u' + (\varepsilon/2)w_0) \in S$ and $0 \le u - v \le f \le u - v + \varepsilon w_0$. Since $w_0 \le v_0$ on $S(f)$, (2.1) follows. □

PROPOSITION 6.2.3. *Let S be a $\wedge$-stable function cone in $\mathcal{C}^+(X)$ and let T be a positive linear form on $S - S$. Then there exists a unique $\mu \in \mathcal{M}(X)$ such that $T(u) = \int u\, d\mu$ for every $u \in S - S$.*

Proof. By Proposition 6.2.2, for every $f \in \mathcal{K}^+(X)$,

$$Lf := \sup\{T(v) \colon v \in S - S,\ v \le f\} = \inf\{T(v) \colon v \in S - S,\ v \ge f\}.$$

Therefore L defines a positively homogeneous, additive functional on $\mathcal{K}^+(X)$, that is, a measure $\mu \in \mathcal{M}(X)$. Let $u \in S$. Then

$$\begin{aligned}\int u\, d\mu &= \sup\{\int f\, d\mu \colon f \in \mathcal{K}^+(X), f \le u\} \\ &= \sup\{L(f) \colon f \in \mathcal{K}^+(X), f \le u\} \le T(u).\end{aligned}$$

For the reverse inequality, we choose $v \in S^+$ such that $v > 0$ and $u/v \in \mathcal{C}_0(X)$. Given $\varepsilon > 0$, we have $f := (u - \varepsilon v)^+ \in \mathcal{K}^+(X)$ and $u - \varepsilon v \le f \le u$, hence

$$T(u) - \varepsilon T(v) = T(u - \varepsilon v) \le L(f) = \int f\,d\mu \le \int u\,d\mu.$$

Therefore $T(u) = \int u\,d\mu$.

By Proposition 6.2.2, the measure $\mu \in \mathcal{M}(X)$ is uniquely determined. □

Let S be a $\wedge$- stable function cone in $\mathcal{C}^+(X)$ and

$$S_\sigma := \Big\{\sum_{n=1}^{\infty} u_n \colon (u_n) \subset S\Big\}.$$

DEFINITION 6.2.4. *A function $u \in S_\sigma$ is* strict *if any two $\mu, \nu \in \mathcal{M}(X)$ coincide provided that $\mu(u) = \nu(u) < \infty$ and $\mu(v) \le \nu(v)$ for all $v \in \mathcal{P}$.*

Obviously, every strict element u in S_σ is strictly positive (if $u(x) = 0$, consider the measures $\mu = 0$ and $\nu = \varepsilon_x$).

PROPOSITION 6.2.5. *There exist strict $u \in S_\sigma \cap \mathcal{C}(X)$. More precisely, for every strictly positive $w \in S$, there exists a strict $u \in \mathcal{S}_\sigma \cap \mathcal{C}(X)$ such that $u \le w$.*

Proof. Let $w \in S$, $w > 0$. There exists a sequence (f_n) in $\mathcal{K}^+(X)$ separating $\mathcal{M}(X)$. We may suppose that $f_n \le w$ for every $n \in \mathbb{N}$. By Proposition 6.2.2, there exist sequences (u_m), (v_m) in S such that, for all $k, n \in \mathbb{N}$,

$$|f_n - (u_{n_k} - v_{n_k})| \le \frac{1}{k} w \tag{2.2}$$

for some $n_k \in \mathbb{N}$. Since S is $\wedge$-stable, we may suppose that $u_m \le w$ and $v_m \le w$, $m \in \mathbb{N}$. Then

$$u := \frac{1}{3}\Big(w + \sum_{m=1}^{\infty} 2^{-m}(u_m + v_m)\Big) \in S_\sigma \cap \mathcal{C}(X)$$

and $u \le w$. If $\mu, \nu \in \mathcal{M}(X)$ such that $\mu(u) = \nu(u) < \infty$ and $\mu(v) \le \nu(v)$ for every $v \in S$, then $\mu(w) = \nu(w) < \infty$ and $\mu(u_m) = \nu(u_m)$, $\mu(v_m) = \nu(v_m)$ for every $m \in \mathbb{N}$. By (2.2), we see that $\mu(f_n) = \nu(f_n)$ for all $n \in \mathbb{N}$. Thus $\mu = \nu$. □

6.3 A general minimum principle

Let $\mathcal{F}$ be a convex cone of lower semicontinuous functions $f \colon X \to (-\infty, \infty]$. For every $x \in X$, let $\mathcal{M}_x(\mathcal{F})$ denote the set of all $\mu \in \mathcal{M}(X)$ such that

$$-\infty < \mu(f) \le f(x) \qquad \text{for every } f \in \mathcal{F}.$$

Of course, $\varepsilon_x \in \mathcal{M}_x(\mathcal{F})$ for every $x \in X$. The *Choquet boundary* $Ch_{\mathcal{F}}X$ is the set of all points $x \in X$, where ε_x is the only measure in $\mathcal{M}_x(\mathcal{F})$, that is,

$$Ch_{\mathcal{F}}X := \big\{x \in X \colon \mathcal{M}_x(\mathcal{F}) = \{\varepsilon_x\}\big\}.$$

Here is a first minimum principle (for the general minimum principle, Theorem 6.3.2, we shall suppose that $\mathcal{F}$ contains a function cone $\mathcal{P}$ and that all functions in $\mathcal{F}$ are lower $\mathcal{P}$-bounded).

PROPOSITION 6.3.1. *Let us suppose that X is compact, $\mathcal{F}$ is linearly separating, and there exists a strictly positive function $f_0 \in \mathcal{F} \cap \mathcal{C}(X)$.*

Then every function $f \in \mathcal{F}$, which is positive on $Ch_{\mathcal{F}}X$, is positive on X.

Proof. (a) Let us first assume that $1 \in \mathcal{F}$ and let $\mathcal{A}$ be the set of all non-empty compact sets A in X such that

$$\mu(X \setminus A) = 0 \qquad \text{for all } x \in A \text{ and } \mu \in \mathcal{M}_x(\mathcal{F}).$$

Of course, $X \in \mathcal{A}$. Moreover, the set $\mathcal{A}$ is inductively ordered by the reverse inclusion relation. So, by Zorn's lemma, every set $A \in \mathcal{A}$ contains a minimal set $A' \in \mathcal{A}$.

Let us now fix $f \in \mathcal{F}$ and assume that $\alpha := \inf f(X) < 0$. We intend to prove that $f(x) < 0$ for some $x \in Ch_{\mathcal{F}}X$. To that end we consider the non-empty compact set $A := \{f = \alpha\}$. Let $y \in A$ and $\mu \in \mathcal{M}_y(\mathcal{F})$. Since $\mu(1) \leq 1$ and $\alpha < 0$, we obtain that

$$\alpha \leq \int \alpha \, d\mu \leq \int f \, d\mu \leq f(y) = \alpha,$$

hence $\mu(X) = 1$ and $\int (f - \alpha) \, d\mu = 0$, that is, $\mu(X \setminus A) = 0$. Thus $A \in \mathcal{A}$.

Let $A' \in \mathcal{A}$ be minimal such that $A' \subset A$. We intend to show that A' consists of one point only. Let us suppose the contrary. Then there exists a function $g \in \mathcal{F}$ such that $g|_{A'}$ is non-constant. Let $\beta := \inf g(A')$ and

$$A'' := A' \cap \{g = \beta\}.$$

Then A'' is a non-empty compact set, $A'' \neq A'$. If $y \in A''$ and $\mu \in \mathcal{M}_y(\mathcal{F})$, then

$$\beta = \int_{A'} \beta \, d\mu \leq \int_{A'} g \, d\mu = \int g \, d\mu \leq g(y) = \beta,$$

hence $\int_{A'} (g - \beta) \, d\mu = 0$ and therefore

$$\mu(X \setminus A'') = \mu(X \setminus A') + \mu(A' \setminus A'') = 0.$$

Thus $A'' \in \mathcal{A}$, contradicting the minimality of A'. So A' reduces to a singleton $\{x\}$. Knowing that $\mu(X) = 1$ for every $\mu \in \mathcal{M}_x(\mathcal{F})$, we hence obtain that $\mathcal{M}_x(\mathcal{F}) = \{\varepsilon_x\}$, that is, $x \in Ch_{\mathcal{F}}X$. Of course, $f(x) = \alpha < 0$.

(b) Let us now consider the general case. Let $f_0 \in \mathcal{F} \cap \mathcal{C}(X)$, $f_0 > 0$, and

$$\mathcal{F}_0 = \Big\{\frac{f}{f_0} : f \in \mathcal{F}\Big\}.$$

Then $\mathcal{F}_0$ satisfies the assumptions of (a) and, for every $x \in X$,

$$\mathcal{M}_x(\mathcal{F}_0) = \Big\{\frac{f_0 \mu}{f_0(x)} : \mu \in \mathcal{M}_x(\mathcal{F})\Big\},$$

hence $Ch_{\mathcal{F}_0}X = Ch_{\mathcal{F}}X$. Finally, let $f \in \mathcal{F}$, $f \geq 0$ on $Ch_{\mathcal{F}}X$. Then $f/f_0 \in \mathcal{F}_0$ and $f/f_0 \geq 0$ on $Ch_{\mathcal{F}_0}X$. By (a), $f/f_0 \geq 0$ on X, that is, $f \geq 0$ on X. □

From now on let us suppose that $\mathcal{F}$ contains a function cone $\mathcal{P}$ such that all functions in $\mathcal{F}$ are lower $\mathcal{P}$-bounded.

THEOREM 6.3.2. *Every function $f \in \mathcal{F}$, which is positive on $Ch_{\mathcal{F}}X$, is positive on X.*

Proof. Let us assume that there exists a function $f \in \mathcal{F}$ such that f is not positive, but $f \geq 0$ on $Ch_{\mathcal{F}}X$. Then there are $p, q \in \mathcal{P}$ such that $-p \leq f$, $q > 0$, $p/q \in \mathcal{C}_0(X)$, and $g := f + q$ is not positive. We define

$$K := \{g \leq 0\} \quad \text{and} \quad \mathcal{G} := (\mathcal{P} + \mathbb{R}^+ g)|_K.$$

Then K is a non-empty compact subset of $X \setminus Ch_{\mathcal{F}}X$. So, fixing $x \in K$, there exists a measure $\mu \in \mathcal{M}_x(\mathcal{F})$ such that $\mu \neq \varepsilon_x$.

Let $\nu := 1_K \mu$. Of course, for every $u \in \mathcal{P}$, $\int u \, d\nu \leq \int u \, d\mu \leq u(x)$. In addition, since $g > 0$ outside K,

$$\int g \, d\nu \leq \int g \, d\nu + \int_{X \setminus K} g \, d\mu = \int g \, d\mu \leq g(x). \tag{3.1}$$

So $\nu \in \mathcal{M}_x(\mathcal{G})$. If $\mu(X \setminus K) > 0$, then the first inequality in (3.1) is strict, hence $\nu \neq \varepsilon_x$. If, however, $\mu(X \setminus K) = 0$, then $\nu = \mu \neq \varepsilon_x$.

Thus $x \notin Ch_{\mathcal{G}}K$, that is, $Ch_{\mathcal{G}}K = \emptyset$. By Proposition 6.3.1, this implies that $g \geq 0$ on K and hence $g \geq 0$ on X, a contradiction. □

For applications of the minimum principle, we shall need an easy consequence of the theorem of Hahn-Banach. Let us first define

$$\overline{\varphi} := \inf\{f \in \mathcal{F} \colon f \geq \varphi\} \qquad (\varphi \in \mathcal{C}_{\mathcal{P}}(X)).$$

Since $\mathcal{F}$ is a convex cone, the mapping $\varphi \mapsto \overline{\varphi}$ is positively homogeneous and subadditive.

LEMMA 6.3.3. *For all $x \in X$ and $\varphi \in \mathcal{C}_{\mathcal{P}}(X)$,*

$$\overline{\varphi}(x) = \max\{\mu(\varphi) \colon \mu \in \mathcal{M}_x(\mathcal{F})\} \tag{3.2}$$

Proof. Let $x \in X$ and $\varphi \in \mathcal{C}_{\mathcal{P}}(X)$. If $\mu \in \mathcal{M}_x(\mathcal{F})$, then $\mu(\varphi) \leq \mu(f) \leq f(x)$, whenever $f \in \mathcal{F}$ and $f \geq \varphi$, and hence $\mu(\varphi) \leq \overline{\varphi}(x)$.

Moreover, by the theorem of Hahn-Banach, there exists a linear functional l on $\mathcal{C}_{\mathcal{P}}(X)$ such that $l(\varphi) = \overline{\varphi}(x)$ and, for every $\psi \in \mathcal{C}_{\mathcal{P}}(X)$, $l(\psi) \leq \overline{\psi}(x)$. If $\psi \leq 0$, then $l(\psi) \leq \overline{\psi}(x) \leq 0$. So l is positive. By Proposition 6.2.3, there exists $\mu \in \mathcal{M}(X)$ such that $\mu(\psi) = l(\psi)$ for all $\psi \in \mathcal{C}_{\mathcal{P}}(X)$. In particular, $\mu(\varphi) = \overline{\varphi}(x)$.

Finally, let $f \in \mathcal{F}$. If $\psi \in \mathcal{C}_{\mathcal{P}}(X)$ and $\psi \leq f$, then $\mu(\psi) = l(\psi) \leq \overline{\psi}(x) \leq f(x)$. Since f is lower semicontinuous and lower $\mathcal{P}$-bounded, we conclude that $\mu(f) \leq f(x)$. Thus $\mu \in \mathcal{M}_x(\mathcal{F})$ finishing the proof. □

PROPOSITION 6.3.4. *For every $\varphi \in \mathcal{C}_{\mathcal{P}}(X)$, $\overline{\varphi} = \varphi$ on $Ch_{\mathcal{F}}X$. If $p \in \mathcal{P}$ is strict, then $Ch_{\mathcal{F}}(X) = \{\overline{(-p)} = -p\}$.*

Proof. The first statement follows immediately from Proposition 6.3.3.

So let p be a strict element in $\mathcal{P}$, $x \in X$ with $\overline{(-p)}(x) = -p(x)$, and $\mu \in \mathcal{M}_x(\mathcal{F})$. Then $\mu(-p) \leq \overline{(-p)}(x) = -p(x)$. Moreover, $\mu(q) \leq q(x)$ for every $q \in \mathcal{P}$, since $\mathcal{P} \subset \mathcal{F}$. In particular, $\mu(p) = p(x)$, and we conclude that $\mu = \varepsilon_x$, since p is strict. Thus $x \in Ch_{\mathcal{F}}X$. □

COROLLARY 6.3.5. *Let $\varphi, \psi \in \mathcal{C}_{\mathcal{P}}(X)$ such that $\varphi \le \psi \le \overline{\varphi}$. Then*

$$\overline{\varphi} = \inf\{f \in \mathcal{F} - \mathbb{R}^+\psi \colon f \ge \varphi\}. \tag{3.3}$$

In particular, $Ch_{\mathcal{F}-\mathbb{R}^+\psi} X \subset \{\overline{\varphi} = \varphi\}$.

Proof. Since $\mathcal{F} \subset \mathcal{F} - \mathbb{R}^+\psi$, the right side in (3.3) is at most $\overline{\varphi}$. To prove the reverse inequality, we fix $f \in \mathcal{F}$ and $a \ge 0$ such that $g := f - a\psi \ge \varphi$. Then $f \ge \varphi + a\psi \ge (a+1)\varphi$ and hence

$$f \ge (a+1)\overline{\varphi} \ge (a+1)\psi. \tag{3.4}$$

So $af \ge (a+1)a\psi = (a+1)(f-g)$, that is, $(a+1)g \ge f$. Combining this estimate with the first inequality in (3.4) we see that $g \ge \overline{\varphi}$. So (3.3) holds. The proof is finished by Proposition 6.3.4. □

6.4 Some properties of strong Feller kernels

Let us recall that a kernel V on X is a strong Feller kernel, if $V(B_b(X)) \subset \mathcal{C}_b(X)$. The following results are used in Sections 3.1 and 3.2.

PROPOSITION 6.4.1. *Let V be a strong Feller kernel on X. Let (f_n) be a sequence in $\mathcal{B}_b(X)$ which is uniformly bounded and converges pointwise to a function f on X, and let (x_n) be a sequence in X converging to a point $x \in X$. Then $\lim_{n\to\infty} Vf_n(x_n) = Vf(x)$.*

Proof. Let $g_m := \sup_{n\ge m} f_n$, $m \in \mathbb{N}$. If $n, m \in \mathbb{N}$, $n \ge m$, then $g_m \ge f_n$, hence $Vg_m \ge Vf_n$. So, for every $m \in \mathbb{N}$,

$$Vg_m(x) = \lim_{n\to\infty} Vg_m(x_n) \ge \limsup_{n\to\infty} Vf_n(x_n).$$

Since the sequence (g_m) is decreasing to f, we see that

$$Vf(x) = \lim_{m\to\infty} Vg_m(x) \ge \limsup_{n\to\infty} Vf_n(x_n).$$

An application to the sequence $(-f_n)$ yields that $Vf(x) \le \liminf_{n\to\infty} Vf_n(x_n)$. Thus $\lim_{n\to\infty} Vf_n(x_n) = Vf(x)$. □

COROLLARY 6.4.2. *Let $V, \tilde{V}$ be strong Feller kernels on X, $\tilde{X}$, respectively ($\tilde{X}$ a locally compact space with countable base). Then $V \otimes \tilde{V}$ is a strong Feller kernel on $X \times \tilde{X}$.*

Proof. Let $\in \mathcal{B}_b(X \times \tilde{X})$ and let $\big((x_n, \tilde{x}_n)\big)$ be a sequence in $X \times \tilde{X}$ converging to $(x_0, \tilde{x}_0)$. For every $n \in \mathbb{N} \cup \{0\}$, we define $f_n \colon X \to \mathbb{R}$ by

$$f_n(x) := \int g(x, \tilde{y}) \tilde{V}(\tilde{x}_n, d\tilde{y}) = \tilde{V} g_x(\tilde{x}_n),$$

where $g_x(y) := g(x, y)$. Then (f_n) is a uniformly bounded sequence in $\mathcal{B}_b(X)$ and, for every $x \in X$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} Vg_x(\tilde{x}_n) = Vg_x(\tilde{x}_0) = f_0(x).$$

So we conclude, by Proposition 6.4.1, that

$$\lim_{n\to\infty}(V\otimes\tilde{V})g(x_n,\tilde{x}_n)=\lim_{n\to\infty}Vf_n(x_n)=Vf_0(x_0)=(V\otimes\tilde{V})g(x_0,\tilde{x}_0).$$

□

LEMMA 6.4.3. *Let $\mathbb{P}=(P_t)_{t>0}$ be a sub-Markov semigroup on X having a strong Feller resolvent $(V_\lambda)_{\lambda>0}$. Let $f\in\mathcal{B}_b(X\times(0,\infty))$ such that all functions $f(x,\cdot)$, $x\in X$, are continuous on $(0,\infty)$. Then, for every $\lambda>0$, the function $\int_0^\infty P_tf(\cdot,t)\,dt$ is continuous on X.*

Proof. Let us fix $\lambda>0$. We note first that, for all $g\in\mathcal{B}_b(X)$ and $0<r<s<\infty$ the functions

$$\int_s^\infty e^{-\lambda t}P_tg\,dt=e^{-\lambda s}V_\lambda P_sg\quad\text{and}\quad\int_r^s e^{-\lambda t}P_tg\,dt=e^{-\lambda r}V_\lambda P_rg-e^{-\lambda s}V_\lambda P_sg$$

are continuous. For all $n\in\mathbb{N}$, $0\le i\le n2^n$, and $x\in X$, we define

$$g_{n,i}(x,t):=\sup\{f(x,t)\colon t\in I_{n,i}\},\quad\text{where } I_{n,i}:=\begin{cases}(i2^{-n},(i+1)2^{-n}], & \text{if } i<n2^n,\\ (n,\infty), & \text{if } i=n2^n.\end{cases}$$

Then the functions $g_n\in\mathcal{B}_b(X\times(0,\infty))$, $n\in\mathbb{N}$, defined by

$$g_n(x):=\sum_{i=0}^{n2^n}I_{n,i}(t)g_{n,i}(x),\qquad(x,t)\in X\times(0,\infty),$$

are decreasing to f. By our considerations above, the functions

$$F_n:=\int_0^\infty e^{-\lambda t}P_tg_n(\cdot,t)\,dt=\sum_{i=0}^{n2^n}\int_{I_{n,i}}e^{-\lambda t}P_tg_{n,i}\,dt,\qquad n\in\mathbb{N},$$

are continuous on X. So the function

$$F:=\int_0^\infty e^{-\lambda t}P_tf(\cdot,t)\,dt=\inf F_n$$

is upper semicontinuous on X. Replacing f by $-f$ we conclude that F is lower semicontinuous as well. Thus F is continuous on X. □

COROLLARY 6.4.4. *Let $\mathbb{P}=(P_t)_{t>0}$ be a sub-Markov semigroup on X having a strong Feller resolvent $(V_\lambda)_{\lambda>0}$. Let $f\in\mathcal{B}_b(X)$ and let g be an integrable function on $(0,\infty)$. Then the function $x\mapsto\int_0^\infty P_tf(x)g(t)\,dt$ is continuous on X.*

Proof. Let $\varepsilon>0$. There exists $\varphi\in\mathcal{K}((0,\infty))$ such that $\int_0^\infty|g-\varphi|\,dt<\varepsilon$ and hence, for every $x\in X$,

$$|\int_0^\infty P_tf(x)g(t)\,dt-\int_0^\infty P_tf(x)\varphi(t)\,dt|\le\|f\|_\infty\int_0^\infty|g-\varphi|\,dt<\varepsilon\|f\|_\infty.$$

For every $(x,t)\in X\times(0,\infty)$, let $\tilde{f}(x,t):=f(x)e^t\varphi(t)$. By Lemma 6.4.3, the function $x\mapsto\int_0^\infty P_tf(x)\varphi(t)\,dt=\int_0^\infty e^{-t}P_t\tilde{f}(\cdot,t)(x)\,dt$ is continuous on X. Hence the assertion follows. □

6.5 A simple example

Let us discuss a simple example of a convex cone of lower semicontinuous functions, such that properties (B_1), (B_3), (B_4) of a balayage space hold, but (B_2) is not satisfied in spite of the following property:

(B_2') $\widehat{\inf \mathcal{V}} \in \mathcal{W}$ for every subset $\mathcal{V}$ of $\mathcal{W}$.

Let $X := \mathbb{N} \cup \{\infty\} \subset [0, \infty]$ (with the induced topology) and let $\mathcal{W}$ be the convex cone of all positive numerical functions u on X which are lower semicontinuous, that is, which satisfy

$$\liminf_{n\to\infty} u(n) \geq u(\infty). \tag{5.1}$$

It is immediately seen that (B_1), (B_2'), and (B_4) hold. The fine topology is the discrete topology, since $1_{\mathbb{N}} \in \mathcal{W}$ and hence $\{\infty\} = \{1_{\mathbb{N}} < 1\}$ is finely open.

However, (B_2) does not hold. Indeed, since $1_{\mathbb{N}} \in \mathcal{W}$, the point ∞ is finely isolated. Clearly, $v_m := 1_{\{n\in X\colon m\leq n\leq\infty\}} \in \mathcal{W}$, $m \in \mathbb{N}$, whereas $v := \inf v_m = 1_{\{\infty\}}$ and hence $\hat{v}^f = 1_{\{\infty\}} \notin \mathcal{W}$.

It remains to show that (B_3) is satisfied. Let $u, v', v'' \in \mathcal{W}$ such that $u \leq v' + v''$ and, without loss of generality, $v' \leq u$, $v'' \leq u$. If $v'(\infty) = \infty$, it suffices to take $u' := 1_{\{v''<\infty\}}(u - v'')$ and $u'' := v''$. So let us assume that $a := v''(\infty) < \infty$. Then $v'' \wedge a$ is a continuous real function on X and hence

$$w := u - v'' \wedge a \in \mathcal{W}, \qquad u' := w \wedge v' \in \mathcal{W}, \qquad u' \leq v'.$$

We define

$$u'' := \begin{cases} u - u', & \text{on } \{u' < \infty\}, \\ v'' \wedge a, & \text{on } \{u' = \infty\}. \end{cases}$$

Then

$$u' + u'' = u \quad \text{ and } \quad v'' \wedge a \leq u'' \leq v'',$$

since, on the set $\{u' < \infty\} \cap \{u = \infty\}$,

$$w = \infty, \quad u' = v' < \infty, \quad u'' = \infty, \quad v'' \geq u - v' = \infty,$$

whereas, on the set $\{u' < \infty\} \cap \{u < \infty\}$,

$$w < \infty, \quad v' < \infty, \quad u - u' = (u - w) \vee (u - v'), \quad u - w = v'' \wedge a, \quad u - v' \leq v''.$$

In particular, $\liminf_{n\to\infty} u''(n) \geq a = u''(\infty)$, since $\liminf_{n\to\infty} v''(n) \geq v''(\infty) = a$. Thus $u'' \in \mathcal{W}$.

Bibliography

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