Balayage spaces –
a natural setting for potential theory

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Contents

1 Introduction

The introduction of harmonic spaces by M. Brelot [8, 9, 10] and H. BAUER [1, 2, 3] was based on fundamental properties which solutions of certain linear differential equations of second order have in common. Papers of P.A. MEYER [12] and N. BOBOC – C. CONSTANTINESCU – A. CORNEA [7] showed that, by analogy with the relation between classical potential theory and Brownian motion, every harmonic space admits a corresponding Markov process with continuous paths. However, whether the paths are continuous or not plays no important role in the potential theory of Markov processes as presented in the book of R.M. BLUMENTHAL – R.K. GETOOR [6]. Moreover, even within the theory of harmonic spaces, it turned out that in some respect hyperharmonic functions and potentials are more important than harmonic functions. In fact, the monograph of C. CONSTANTINESCU – A. CORNEA [11] on potential theory of harmonic spaces contains a chapter, where basic properties of hyperharmonic functions are used in an axiomatic way to study important properties of balayage. The authors used an axiom of upper directed sets, an axiom of lower semicontinuous regularization, and an axiom of natural decomposition. On the other hand, G. MOKOBODZKI and D. SIBONY [13, 14, 15] studied cones of continuous potentials in an abstract setting, where natural decomposition and additional continuity properties play a central role.

The concept of a balayage space is obtained by combining the stated three fundamental properties of positive hyperharmonic functions with the existence of a cone of continuous potentials such that every positive hyperharmonic function is the supremum of a sequence of continuous potentials. This notion was introduced in [4], where it helped to characterize the class of Markov processes associated with harmonic spaces. It was shown that balayage spaces are closely related to sub-Markov semigroups, where every excessive function is the upper envelope of its continuous excessive minorants.

However, there are additional important reasons for the consideration of balayage spaces. First of all, harmonic spaces, Riesz potentials and Markov chains are covered, and the class of balayage spaces has remarkable permanence properties with respect to restriction on subspaces, subordination by convolution semigroups, products, and images. Moreover, balayage spaces allow us a clear and direct presentation of results on balayage of functions and balayage of measures. This can be carried out without losing known results for harmonic spaces and without more complicated proofs. In particular, different types of Dirichlet problems can be treated in an elegant manner.

The starting point of this survey will be a discussion of balayage spaces and their relationship to resolvents, semigroups, and Markov processes. It will then be pointed out, how families of harmonic kernels associated with balayage spaces may be characterized and which additional properties yield harmonic spaces. Permanence properties leading to further examples will be looked at in detail. A short treatment of the Dirichlet problem will then finish the paper. For details the reader is referred to [5].

2 Balayage spaces

In the following let X be a locally compact space with countable base. Let $\mathcal{B}(X)$ ($\mathcal{C}(X)$, respectively) denote the set of all Borel measurable numerical (continuous real, respectively) functions on X. We shall write $A \in \mathcal{B}(X)$, if $A \subset X$ and the characteristic function 1_A is Borel measurable. Let

$$\begin{aligned} \mathcal{C}_0(X) &:= \{f \in \mathcal{C}(X) \colon f \text{ vanishes at infinity}\}, \\ \mathcal{K}(X) &:= \{f \in \mathcal{C}(X) \colon f \text{ has compact support} \operatorname{supp}(f)\}. \end{aligned}$$

Finally, let $\mathcal{M}(X)$ be the set of all (positive) Radon measures on X.

Let $\mathcal{W}$ be a convex cone of positive lower semicontinuous numerical functions on X. The coarsest topology on X which is finer that the initial topology and for which all functions of $\mathcal{W}$ are continuous will be called the ($\mathcal{W}$)-*fine topology.* Topological notions with respect to the fine topology are distinguished by the term "fine(ly)" or the affix "f" from those pertaining to the initial topology on X. It can be easily be seen that every $x \in X$ has a fundamental system of fine neighborhoods which are compact in the initial topology. In particular, X endowed with the fine topology is a Baire space.

Definition 2.1. *$(X,\mathcal{W})$ is called a* balayage space*, if the following axioms are satisfied:*

(B_1) $\sup v_n \in \mathcal{W}$ *for every increasing sequence (v_n) in $\mathcal{W}$.*

(B_2) $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$ *for every non-empty subset $\mathcal{V}$ of $\mathcal{W}$.*

(B_3) *If $u, v', v'' \in \mathcal{W}$ such that $u \le v' + v''$, there exist $u', u'' \in \mathcal{W}$ such that $u = u' + u''$, $u' \le v'$, and $u'' \le v''$.*

(B_4) *$\mathcal{W}$ is linearly separating*[1]*, there exist positive $u_0, v_0 \in \mathcal{W} \cap \mathcal{C}(X)$ such that $u_0/v_0 \in \mathcal{C}_0(X)$, and, for every $v \in \mathcal{W}$,*

$$v = \sup\{u \in \mathcal{W} \cap \mathcal{C}(X) \colon u \le v\}. \tag{2.1}$$

The functions in $\mathcal{W}$ will be called positive *hyperharmonic functions* on X, the functions in

$$\mathcal{P} := \{p \in \mathcal{W} \cap \mathcal{C}(X) \colon \frac{p}{v} \in \mathcal{C}_0(X) \text{ for some } v \in \mathcal{W} \cap \mathcal{C}(X), v > 0\}$$

will be called continuous real *potentials* on X.

A convex cone $\mathcal{F} \subset \mathcal{C}^+(X)$ is called a *function cone*, if $\mathcal{F}$ is linearly separating and if, for every $f \in \mathcal{F}$, there exists $g \in \mathcal{F}$ such that $g > 0$ and $f/g \in \mathcal{C}_0(X)$. A function $f \in \mathcal{F}$ is called *strict*, if measures $\mu, \nu \in \mathcal{M}(X)$ coincide provided that $\mu(f) = \nu(f) < \infty$ and $\mu(g) \le \nu(g)$ for all $g \in \mathcal{F}$. Given a $\wedge$-stable function cone $\mathcal{F}$ and an additive, increasing functional $T \colon \mathcal{F} \to [0,\infty)$, there exists a unique measure $\mu \in \mathcal{M}(X)$ such that $T(f) = \int f\, d\mu$ for every $f \in \mathcal{F}$.

If $(X,\mathcal{W})$ is a balayage space, then $\mathcal{P}$ is a $\wedge$-stable function cone,

$$\mathcal{W} = S(\mathcal{P}) := \{\sup p_n \colon (p_n) \subset \mathcal{P} \text{ increasing}\},$$

$\mathcal{P}$ is the greatest function cone in $\mathcal{W}$, and there exist strict $p \in \mathcal{P}$ (obtained taking sums $\sum_{n=1}^{\infty} \alpha_n p_n$, where (p_n) is a sequence in $\mathcal{P}$ separating $\mathcal{M}(X)$).

Moreover, assuming that $(X,\mathcal{W})$ satisfies (B_1), (B_2), (B_3), we conclude that $(X,\mathcal{W})$ satisfies (B_4) if and only if there exists a function cone $\mathcal{F}$ such that $\mathcal{W} = S(\mathcal{F})$.

REMARKS 2.2. 1. (B_1) implies that $\sup V \in \mathcal{W}$ for every increasingly filtered subset $\mathcal{V}$ of $\mathcal{W}$.

2. Suppose (B_2) and let $\mathcal{V} \subset \mathcal{W}$. Then $\widehat{\inf \mathcal{V}} = \widehat{\inf \mathcal{V}}^f$ and the set $\{\widehat{\inf \mathcal{V}} < \inf \mathcal{V}\}$ is finely meager.

3. Suppose (B_2) and define, for every function $f \colon X \to [-\infty,\infty]$,

$$R_f := \inf\{v \in \mathcal{W} \colon v \ge f\}.$$

Clearly, $R_f = \hat{R}_f \in \mathcal{W}$, if f is finely lower semicontinuous. Moreover, (B_3) holds if and only if, for any two functions $u, v \in \mathcal{W}$, there exists $w \in \mathcal{W}$ such that $u = R_f + w$, where $f := (u-v)^+$ on $\{v < \infty\}$, $f = 0$ on $\{v = \infty\}$.

[1] That is, if, for all $x, y \in X$, $x \ne y$, and $\lambda \in \mathbb{R}_+$, there exists $v \in \mathcal{W}$ such that $v(x) \ne \lambda v(y)$.

The following results will show that many sub-Markov resolvents and sub-Markov semigroups are associated with balayage spaces.

Let us first recall that a *sub-Markov resolvent* on X is a family $\mathbb{V} = (V_\lambda)_{\lambda>0}$ of kernels V_λ on X such that $\lambda V_\lambda 1 \le 1$ and $V_\lambda - V_\mu = (\mu - \lambda) V_\lambda V_\mu$ for all $\lambda, \mu > 0$. Its *potential kernel* is given by $V_0 := \sup_{\lambda>0} \lambda V_\lambda$,

$$E_{\mathbb{V}} := \{u \in \mathcal{B}^+(X) \colon \sup_{\lambda>0} \lambda V_\lambda u = u\}$$

is the set of $\mathbb{V}$-*excessive* function, and

$$S_{\mathbb{V}} := \{u \in \mathcal{B}^+(X) \colon \sup_{\lambda>0} \lambda V_\lambda u \le u\}$$

is the set of $\mathbb{V}$-*supermedian* functions.

A *sub-Markov semigroup* $\mathbb{P}$ on X is a family $\mathbb{P} = (P_t)_{t>0}$ of kernels P_t on X such that $P_t 1 \le 1$ and $P_s P_t = P_{s+t}$ for all $s, t > 0$. The corresponding set of $\mathbb{P}$-*excessive* functions is defined by

$$E_{\mathbb{P}} := \{u \in \mathcal{B}^+(X) \colon \sup_{t>0} P_t u = u\}.$$

If $\mathbb{V}$ is the resolvent of $\mathbb{P}$, that is, for all $\lambda > 0$, $f \in \mathcal{B}^+(X)$, and $x \in X$,

$$V_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt,$$

then $E_{\mathbb{P}} = E_{\mathbb{V}}$.

Let us finally note that a *strong Feller kernel* on X is a bounded kernel V on X satisfying $V(\mathcal{B}_b(X)) \subset \mathcal{C}_b(X)$.

THEOREM 2.3. *Let $\mathbb{V} = (V_\lambda)_{\lambda>0}$ be a sub-Markov resolvent on X. Then $(X, E_{\mathbb{V}})$ is a balayage space if and only if $\lim \lambda V_\lambda f = f$ for every $f \in \mathcal{K}(X)$ and $E_{\mathbb{V}}$ satisfies (B_4).*

COROLLARY 2.4. *Let $\mathbb{V} = (V_\lambda)_{\lambda>0}$ be a sub-Markov resolvent on X such that $\lim_{\lambda\to\infty} \lambda V_\lambda f = f$ for every $f \in \mathcal{K}(X)$ and suppose that there exist $u, v \in E_{\mathbb{V}}$ such that $u, v > 0$, $u/v \in \mathcal{C}_0(X)$. Then $(X, E_{\mathbb{V}})$ is a balayage space if $E_{\mathbb{V}}$ separates the points of X or the potential kernel V_0 of $\mathbb{V}$ is proper (that is, $V_0 1_K$ is bounded for every compact K in X).*

THEOREM 2.5. *Let $\mathbb{P} = (P_t)_{t>0}$ be a sub-Markov semigroup on X. Then $(X, E_{\mathbb{P}})$ is a balayage space if and only if $\lim_{t\to0} P_t f = f$ for every $f \in \mathcal{K}(X)$ and $E_{\mathbb{P}}$ satisfies (B_4).*

COROLLARY 2.6. *Let $\mathbb{P} = (P_t)_{t>0}$ be a sub-Markov semigroup on X such that $\lim_{t\to0} P_t f = f$ for every $f \in \mathcal{K}(X)$ and the resolvent of $\mathbb{P}$ (or even $\mathbb{P}$ itself) is strong Feller. Suppose furthermore that there exist strictly positive $u, v \in E_{\mathbb{P}} \cap \mathcal{C}(X)$ such that $u/v \in \mathcal{C}_0(X)$ and that $E_{\mathbb{P}}$ separates the points of X or the potential kernel V_0 of $\mathbb{P}$ is proper. Then, for every $\alpha \ge 0$, $(X, E_{\mathbb{P}^\alpha})$ is a balayage space (where $\mathbb{P}^\alpha := (e^{-\alpha t} P_t)$).*

EXAMPLES 1. In particular, the following semigroups yield balayage spaces.

1. *Brownian semigroup.* $X = \mathbb{R}^d$, $d \ge 3$, and

$$P_t f(x) := (2\pi t)^{-d/2} \int \exp\big(-\frac{|x-y|^2}{2t}\big) f(y)\, dy.$$

2. *Semigroup* $\mathbb{T} = (T_t)_{t>0}$ *of uniform translation* (to the left). $X = \mathbb{R}$ and

$$T_t f(x) := f(x-t).$$

3. *Pseudo-Poisson semigroup.* X discrete, (at most) countable, P kernel on X such that $P1 \le 1$ and $S_P := \{u \in \mathcal{B}^+(X) \colon Pu \le u\}$ separates the points of X, and

$$P_t := e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} P^k.$$

Let us note that $E_{\mathbb{P}} = S_P$!

Suppose now that $(X, \mathcal{W})$ is an arbitrary balayage space. Given $A \subset X$ and $u \in \mathcal{W}$, we define

$$R_u^A := R_{u1_A} = \inf\{v \in \mathcal{W} \colon v \ge u \text{ on } A\} \quad \text{ and } \quad \hat{R}_u^A := \widehat{R_u^A}.$$

Let us mention some basic facts on balayage of functions and measures.

PROPOSITION 2.7. *For every subset A of X and all functions $u, v \in \mathcal{W}$,*

$$R_{u+v}^A = R_u^A + R_v^A \quad \textit{and} \quad \hat{R}_{u+v}^A = \hat{R}_u^A + \hat{R}_u^A.$$

Proof. (B_3) and (1.1.2). □

PROPOSITION 2.8. *Let (A_n) be an increasing sequence of sets in X and let (u_n) be an increasing sequence in $\mathcal{W}$, let $A := \bigcup_{n=1}^{\infty} A_n$ and $u := \sup u_n$. Then*

$$R_{u_n}^{A_n} \uparrow R_u^A \quad \textit{and} \quad \hat{R}_{u_n}^{A_n} \uparrow \hat{R}_u^A.$$

Let $\mathcal{M}(\mathcal{P})$ denote the set of all $\mu \in \mathcal{M}(X)$ such that $\int p\, d\mu < \infty$ for *some* strictly positive $p \in \mathcal{P}$ (which holds trivially if μ has compact support).

PROPOSITION 2.9. *Given $\mu \in \mathcal{M}(\mathcal{P})$ and $A \subset X$, there exists a unique measure $\mu^A \in \mathcal{M}(\mathcal{P})$ such that, for every $u \in \mathcal{W}$,*

$$\int u\, d\mu^A = \int \hat{R}_u^A\, d\mu.$$

μ^A *is called* balayage of μ on A.

The *base* of a set A in X is defined by

$$b(A) := \{x \in X \colon \varepsilon_x^A = \varepsilon_x\}$$

(where ε_x denotes the Dirac mass at x). If $p \in \mathcal{P}$ is strict, then $b(A) = \{\hat{R}_p^A = p\}$, hence $b(A)$ is a G_δ-set.

PROPOSITION 2.10. *Let $A \subset X$. Then the* fine closure *of A (that is, the closure of A with respect to the fine topology) is the set $\overline{A}^f = A \cup b(A)$ and, for every $\mu \in \mathcal{M}(\mathcal{P})$,*

$$(\mu^A)_*(X \setminus \overline{A}^f) = 0.$$

PROPOSITION 2.11. *Let $v \geq 0$ be a lower semicontinuous numerical function on X. Then $v \in \mathcal{W}$ if and only if, for every $x \in X$ and every neighborhood U of x, there exists a subset V of U such that $x \in V$, $\varepsilon_x^{V^c} \neq \varepsilon_x$, and $\int v \, d\varepsilon_x^{V^c} \leq v(x)$.*

A subset A of X is called *polar*, if $\varepsilon_x^A = 0$ for every $x \in X$. It is *totally thin* if $b(A) = 0$, and it is *semipolar*, if it is a countable union of totally thin sets.

For every $A \subset X$, the set $A \setminus b(A)$ is semipolar. A set S is semipolar if and only if there exists $\mathcal{V} \subset \mathcal{W}$ such that $S = \{\widehat{\inf \mathcal{V}} < \inf \mathcal{V}\}$.

The *essential base* of a subset A of X is the set $\beta(A)$ of all points $x \in X$ such that, for every fine neighborhood V of x, the set $A \cap V$ is not semipolar. Using the quasi-Lindelöf property of the fine topology, it is easily seen that $\beta(A)$ is the smallest finely closed subset F of X such that $A \setminus F$ is semipolar. Moreover, $\beta(\beta(A)) = b(\beta(A)) = \beta(A)$. If A is finely closed, then $\beta(A)$ is the greatest subset B of A such that $B \subset b(B)$.

3 Resolvents and semigroups on balayage spaces

In the preceding section, we stated that many resolvents and semigroups are associated with balayage spaces. We shall now see that the converse holds as well.

So let $(X, \mathcal{W})$ be a balayage space. For every $p \in \mathcal{P}$, there exists a unique kernel V such that

$$V1 = p \quad \text{and} \quad R_{Vf}^{\operatorname{supp}(f)} = Vf \in \mathcal{P} \quad \text{for every } f \in B_b^+(X).$$

V is called the *potential kernel associated with* p. We observe that $V(\mathcal{B}_b(X)) \subset \mathcal{C}(X)$, hence V is a strong Feller kernel, if p is bounded.

The following results yield the existence of many resolvents and semigroups associated with $(X, \mathcal{W})$.

THEOREM 3.1. *Let $1 \in \mathcal{W}$ and $p \in \mathcal{P}_b$. Then there exists a unique sub-Markov resolvent $\mathbb{V}$ such that $E_{\mathbb{V}} \subset \mathcal{W} \subset S_{\mathbb{V}}$ and $V_0 1 = p$. V_0 is the potential kernel of p. Moreover, $E_{\mathbb{V}} = \mathcal{W}$ if and only if p is strict.*

THEOREM 3.2. *Let $\mathbb{V} = (V_\lambda)_{\lambda>0}$ be a sub-Markov resolvent on X such that $E_{\mathbb{V}} \subset S(\mathcal{F}) \subset S_{\mathbb{V}}$ for some function cone $\mathcal{F}$. Then there exists a unique sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X such that $\mathbb{V}$ is the resolvent of $\mathbb{P}$ and $t \mapsto P_t q$ is right continuous for every $q \in S(\mathcal{F})$.*

COROLLARY 3.3. *Let $1 \in \mathcal{W}$ and let $p \in \mathcal{P}_b$ be a strict potential. Then there exists a unique sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X such that $E_{\mathbb{P}} = \mathcal{W}$ and $V_0 1 = p$. V_0 is the potential kernel associated with p. Furthermore, $P_t(\mathcal{P}) \subset \mathcal{P}$ and $P_t(\mathcal{K}(X)) \subset C_b(X)$, $t > 0$.*

In many concrete examples it is fairly easy to construct associated semigroups $\mathbb{P}$ not even satisfying $V_\lambda(\mathcal{K}(X)) \subset \mathcal{C}(X)$. Nevertheless, there will always be a corresponding Hunt process.

THEOREM 3.4. *Let $\mathbb{P}$ be a sub-Markov semigroup on X such that $E_\mathbb{P} = \mathcal{W}$. Then there exists a Hunt process $\mathfrak{X}$ on X having $\mathbb{P}$ as transition function.*

Let $\mathfrak{X} = (\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, \theta_t, P^x)$ be a Hunt process with transition function $\mathbb{P}$ such that $E_\mathbb{P} = \mathcal{W}$, and let $A \in \mathcal{B}(X)$. Then the *first hitting time* T_A, defined by

$$T_A(\omega) := \inf\{t > 0 \colon X_t(\omega) \in A\} \qquad (\omega \in \Omega),$$

is a stopping time and, for all $x \in X$ and $B \in \mathcal{B}(X)$,

$$P^x(X_{T_A} \in B) = \varepsilon_x^A(B).$$

The set A is polar if and only if $T_A = \infty$ almost surely. Moreover,

$$b(A) = \{x \in X \colon T_A = 0 \ P^x\text{-a.s.}\}.$$

In particular, $b(A)$ is totally thin if and only if $T_A > 0$ almost surely.

The *penetration time* τ_A is defined by

$$\tau_A(\omega) := \inf\{t > 0 \colon \{s \in [0,t] \colon X_s(\omega) \in A\} \text{ is uncountable}\}.$$

It can be shown that

$$\tau_A = T_{\beta(A)} \qquad \text{a.s.}$$

In particular,

$$\beta(A) = \{s \in X \colon \tau_A = 0 \ P^x\text{-a.s.}\}.$$

Moreover, A is semipolar if and only if $\tau_A > 0$ a.s. or, equivalently, if and only if $\tau_A = \infty$ a.s.

4 Balayage spaces and harmonic kernels

Since the axioms of a harmonic space may be expressed in terms of harmonic measures, it will be interesting to note that a similar description can be given for balayage spaces. The main difference will be the fact that the harmonic measures are not necessarily supported by the boundary of the open set, their support may be the entire complement.

Let $\mathcal{U}$ be a base of relatively compact open sets in X. For all $x \in X$ and open sets U in X, let

$$\mathcal{U}(x) := \{U \in \mathcal{U} \colon x \in U\}, \qquad \mathcal{U}(U) := \{V \in \mathcal{U} \colon \overline{V} \subset U\}.$$

Let $(H_U)_{U \in \mathcal{U}}$ be a family of *sweeping kernels* on X, that is, for each $U \in \mathcal{U}$, we have a kernel H_U on X satisfying $H_U(x, U) = 0$ for every $x \in X$ and $H_U(x, \cdot) = \varepsilon_x$ for every $x \in U^c$. For every open set V in X, we define

$$\begin{aligned}
{}^*\mathcal{H}^+(V) &:= \{v \in \mathcal{B}^+ \colon v|_V \text{ l.s.c., } H_U v \le v \text{ for every } U \in \mathcal{U}(V)\},\\
\mathcal{S}^+(V) &:= \{s \in {}^*\mathcal{H}^+(V) \colon H_U s \text{ continuous on } U \text{ for every } U \in \mathcal{U}(V)\},\\
\mathcal{H}^+(V) &:= \{h \in \mathcal{S}^+(V) \colon H_U h = h \text{ for every } U \in \mathcal{U}(V)\}.
\end{aligned}$$

${}^*\mathcal{H}^+(V)$ ($\mathcal{S}^+(V)$, $\mathcal{H}^+(V)$, respectively) is the set of all positive Borel functions which are *hyperharmonic* (*superharmonic, harmonic, respectively*) *on* V. If $V = X$, we may simply write ${}^*\mathcal{H}^+$, $\mathcal{S}^+$, and $\mathcal{H}^+$.

Given $U \in \mathcal{U}$, we shall say that a sequence (x_n) in U converging to a point $z \in \partial U$ is *purely irregular in* U, if there is no subsequence (y_n) of (x_n) satisfying $\lim_{n\to\infty} H_U(y_n, \cdot) = \varepsilon_z$.

$(H_U)_{U\in\mathcal{U}}$ is called a *family of harmonic kernels* if the following axioms are satisfied (where $U, V \in \mathcal{U}$).

(H_1) For every $x \in X$, $\lim_{U,\mathcal{U}_x} H_U(x, \cdot) = \varepsilon_x$ or $R_1^{\{x\}}$ is l.s.c. at x.

(H_2) $H_V H_U = H_U$, whenever $\overline{V} \subset U$.

(H_3) If $f \in \mathcal{B}_b$ has compact support, then $H_U f$ is continuous and bounded on U.

(H_4) For every $x \in U$, there exists $w \in {}^*\mathcal{H}^+(U)$ such that $w(x) < \infty$ and $w(x_n) \to \infty$ for every purely irregular sequence (x_n) in U.

(H_5) ${}^*\mathcal{H}^+$ is linearly separating and there exists $s \in \mathcal{S}^+ \cap C$, $s > 0$.

EXAMPLES 2. It is easy to recover our first standard examples.

1. *Classical theory.* $\mathcal{U}$ family of all open balls in $\mathbb{R}^d$, H_U given by the Poisson integral.

2. *Uniform translation on* $\mathbb{R}$. $U := \{(\alpha, \beta)\colon -\infty < \alpha < \beta < \infty\}$,

$$H_{(\alpha,\beta)}(x, \cdot) = \varepsilon_\alpha \qquad \text{for } \alpha < x < \beta$$

(take $w(x) := (\beta - x)^{-1}$).

3. *Discrete theory.* Let P be a sub-Markov kernel on a countable, discrete space X, $\mathcal{U} := \big\{\{x\}\colon x \in X\big\}$ and define

$$H_{\{x\}}(x, A) := \begin{cases} \dfrac{P(x, A \setminus \{x\})}{1 - P(x, \{x\})}, & P(x, \{x\}) < 1, \\ 0, & P(x, \{x\}) = 1. \end{cases}$$

Then ${}^*\mathcal{H}^+ = S_P$ and (H_1) – (H_4) are trivially satisfied. (H_5) holds if S_P separates the points of X.

THEOREM 4.1. *Let* $(X, \mathcal{W})$ *be a balayage space, let* $\mathcal{U}$ *be the family of all relatively compact open sets in* X*, and define sweeping kernels* H_U*,* $U \in \mathcal{U}$*, by*

$$H_U(x, \cdot) := \varepsilon_x^{U^c}, \qquad x \in U,$$

and $H_U(x, \cdot) := \varepsilon_x$*, if* $x \in U^c$*.*

Then $(H_U)_{U\in\mathcal{U}}$ *is a family of harmonic kernels,*

$$\mathcal{W} = {}^*\mathcal{H}^+(X), \qquad W \cap C = \mathcal{S}^+ \cap C,$$

and

$$\begin{aligned} \mathcal{P} &= \big\{p \in \mathcal{W} \cap C\colon \inf\{R_p^{K^c}\colon K \text{ compact in } X\} = 0\big\} \\ &= \big\{p \in \mathcal{W} \cap C\colon h \in \mathcal{H}^+(X),\ 0 \le h \le p \Rightarrow h = 0\big\}. \end{aligned}$$

THEOREM 4.2. *Let* $(H_U)_{U\in\mathcal{U}}$ *be a family of harmonic kernels on* X. *Then* $(X, {}^*\mathcal{H}^+)$ *is a balayage space and* $H_U(x,\cdot) = \varepsilon_x^{U^c}$ *for every* $x \in X$ *and* $x \in U$.

Moreover, let us note that we have the following sheaf properties.

PROPOSITION 4.3. *Let* $(H_U)_{U\in\mathcal{U}}$ *be a family of harmonic kernels on* X *and let* $(V_i)_{i\in I}$ *be family of open sets in* X. *Then*

$$ {}^*\mathcal{H}^+(\bigcup_{i\in I} V_i) = \bigcap_{i\in I} {}^*\mathcal{H}^+(V_i), \qquad \mathcal{H}^+(\bigcup_{i\in I} V_i) = \bigcap_{i\in I} \mathcal{H}^+(V_i). $$

5 Harmonic spaces

If $(X,\mathcal{H})$ is a $\mathcal{P}$-harmonic space, then $(X, {}^*\mathcal{H}^+)$ is a balayage space and the sheaf $\mathcal{H}$ is uniquely determined by ${}^*\mathcal{H}^+$. Hence we may identify $\mathcal{P}$-harmonic spaces with a subclass of balayage spaces. Having several equivalent ways to describe balayage spaces, we have different possibilities to characterize the subclass of $\mathcal{P}$-harmonic spaces.

Let $(X,\mathcal{W})$ be a balayage space and let $(H_U)_{U\in\mathcal{U}}$ be a family of harmonic kernels such that ${}^*\mathcal{H}^+ = \mathcal{W}$. We shall say that $\mathcal{W}$ has the *local truncation property*, if, for every open set U in X and all $u, v \in \mathcal{W}$ satisfying $u \geq v$ on ∂U, the function w, defined by $w := u \wedge v$ on U and $w := v$ on U^c, is contained in $\mathcal{W}$.

THEOREM 5.1. *The following statements are equivalent:*

1. $(X,\mathcal{W})$ *is a harmonic space.*

2. $\mathcal{W}$ *has the local truncation property and there are no finely isolated points.*

3. *For all* $U \in \mathcal{U}$ *and* $x \in U$, $H_U(x,\cdot)$ *is supported by* ∂U. *For every* $x \in X$, *there exists* $U \in \mathcal{U}_x$ *such that* $H_U(x, \partial U) > 0$.

If $1 \in \mathcal{W}$, *we may add:*

4. *There exists an associated sub-Markov semigroup* $\mathbb{P}$ *on* X *having no absorbing points (that is, points* x *with* $P_t(x,\{x\}^c) = 0$, $t > 0$*) and such that, for every* $f \in \mathcal{K}$, *the function* $(1/t)P_t f$ *tends to zero locally uniformly on the complement of* $\operatorname{supp}(f)$ *as* t *tends to zero.*

5. *There exists an associated Hunt process with continuous paths which has no absorbing points.*

6 Permanence properties, further examples

Balayage spaces have nice permanence properties which may serve to construct many new examples. In particular, the list of standard examples will be completed by balayage spaces corresponding to Riesz potentials and the heat equation.

6.1 Restriction on subsets

Two entirely different types of restriction are possible: restriction on an open subset by restriction of the harmonic kernels, restriction on a closed basic subset by simply restricting the global positive hyperharmonic functions. Given a $\mathcal{P}$-harmonic space, the first kind of restriction yields, of course, again a $\mathcal{P}$-harmonic space, whereas the second one usually produces a balayage space which is not a harmonic space. We note, however, that restriction on an absorbing subset belongs to both types.

Let $(X, \mathcal{W})$ be a balayage space and let $(H_U)_{U\in\mathcal{U}}$ be a corresponding family of harmonic kernels.

PROPOSITION 6.1. *For every open set U in X, $(U, {}^*\mathcal{H}^+(U)|_U)$ is a balayage space, $(H_V|_U)_{V\in\mathcal{U}(U)}$ is a corresponding family of harmonic kernels.*

PROPOSITION 6.2. *Let A be an* absorbing set *in X, that is, $A = \{u = 0\}$ for some $u \in \mathcal{W}$. Then $(A, \mathcal{W}|_A)$ is a balayage space, a corresponding family of harmonic kernels is given by ${}^AH_U(x, B) := H_U(x, B) = \varepsilon_x^{U^c}(B)$, $U \in \mathcal{U}$ with $A \cap U \neq \emptyset$, $x \in A \cap U$, $B \in \mathcal{B}(A)$.*

PROPOSITION 6.3. *Let A be a closed set in X such that $b(A) = A$. Then $(A, \mathcal{W}|_A)$ is a balayage space, a corresponding family of harmonic kernels is given by ${}^AH_U(x, B) = \varepsilon_x^{A\setminus U}(B)$, $U \in \mathcal{U}$ with $A \cap U \neq \emptyset$, $x \in A \cap U$, $B \in \mathcal{B}(A)$.*

If we apply Proposition 6.3 to the harmonic space $(\mathbb{R}^d, {}^*\mathcal{H}^+(\mathbb{R}^d))$, $d \geq 3$, of classical potential theory restricting to the hyperplane $\mathbb{R}^{d-1} \times \{0\}$, we obtain the balayage space of Riesz potentials of index 1 on $\mathbb{R}^{d-1}$ (see Subsection 6.2). This is a consequence of the trivial observation $|x - y|^{2-d} = |x - y|^{1-(d-1)}$.

6.2 Subordination by convolution semigroups, Riesz potentials

A family $(\mu_t)_{t>0}$ of measures on $(0, \infty)$ is called a (weakly continuous) *convolution semigroup* on $(0, \infty)$, provided $\mu_t(1) \leq 1$, $\mu_{s+t} = \mu_s * \mu_t$ for all $s, t \in (0, \infty)$, and $\lim_{t\to 0} \mu_t = \varepsilon_0$ (weakly, that is, $\lim_{t\to 0} \mu_t(f) = f(0)$ for all $f \in \mathcal{K}(X)$).

Given a convolution semigroup $(\mu_t)_{t>0}$ on $(0, \infty)$ and a measurable sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X^2, we obtain a sub-Markov semigroup $\mathbb{P}^\mu = (P_t^\mu)_{t>0}$ on X defining

$$P_t^\mu f(x) := \int P_s f(x)\, d\mu_t(s) \qquad (f \in \mathcal{B}^+(X)).$$

$\mathbb{P}^\mu$ is called the sub-Markov *semigroup subordinated to* $\mathbb{P}$ *by means of* $(\mu_t)_{t>0}$. Interesting examples are furnished by the *one-sided stable semigroup* $(\eta_t^\alpha)_{t>0}$, $0 < \alpha < 2$, having Laplace-transform

$$\mathcal{L}\eta_t^\alpha(s) = \exp(-ts^{\alpha/2}) \qquad (s > 0).$$

If $\mathbb{P}$ is the Brownian semigroup on $\mathbb{R}^d$ and $0 < \alpha < 2$, the sub-Markov semigroup subordinated to $\mathbb{P}$ by means of $(\eta_t^\alpha)_{t>0}$ is the *symmetric stable semigroup* $\mathbb{P}^{\eta^\alpha} = (P_t^{\eta^\alpha})_{t>0}$ *of index α on $\mathbb{R}^d$.*

[2]Here *measurable* means that, for every $B \in \mathcal{B}(X)$, the function $(t, x) \mapsto P_t(x, B)$ is measurable.

Let us suppose that $\alpha < 1$, if $d = 1$. It is easily verified that the potential kernel V^α of $\mathbb{P}^{\eta^\alpha}$ is given by

$$V^\alpha f = c_d^\alpha k_\alpha * f, \qquad \text{where } k_\alpha(x) := |x|^{\alpha-d}$$

(and c_d^α is a constant). Therefore the excessive functions of $\mathbb{P}^{\eta^\alpha}$ are the Riesz potentials of order α.

The following general result is an immediate consequence of Corollary 2.6.

PROPOSITION 6.4. *Let $\mathbb{P}$ be a strong Feller semigroup on X such that $(X, E_{\mathbb{P}})$ is a balayage space, and let $(\mu_t)_{t>0}$ be a convolution semigroup on $(0,\infty)$. Then $\mathbb{P}^\mu$ is a strong Feller semigroup, $(X, E_{\mathbb{P}^\mu})$ is a balayage space, and $E_{\mathbb{P}} \subset E_{\mathbb{P}^\mu}$.*

In particular, we obtain that, for each $\alpha \in (0,2)$ and natural $d \geq \alpha$, the Riesz potentials of order α on $\mathbb{R}^d$ form a balayage space $(\mathbb{R}^d, E_{P^{\eta^\alpha}})$ (if $d \leq 2$, we have to go back to Corollary 2.6).

The preceding proposition states that any convolution semigroup will do, if we have a balayage space $(X, E_{\mathbb{P}})$ given by a strong Feller semigroup. Let us now ask which convolution semigroups will admit a subordination on *any* balayage space, where the positive constants are hyperharmonic. To that end we recall from Corollary 3.3 that, for every balayage space $(X, \mathcal{W})$ with $1 \in \mathcal{W}$, there exist many sub-Markov semigroups $\mathbb{P}$ such that $E_{\mathbb{P}} = \mathcal{W}$ and $\mathbb{P}$ has a strong Feller resolvent. So the following result is an answer to our question.

THEOREM 6.5. *Let $(\mu_t)_{t>0}$ be a convolution semigroup on $(0,\infty)$, $\kappa := \int_0^\infty \mu_t\, dt$. Then the following statements are equivalent:*

1. *$(\mathbb{R}, E_{\mathbb{T}^\mu})$ is a balayage space.*
2. *κ is absolutely continuous with respect to Lebesgue measure $\lambda_{(0,\infty)}$ on $(0,\infty)$.*

We observe that the semigroup $\mathbb{T}$ of uniform translation on $\mathbb{R}$ has a strong Feller resolvent. So the balayage space $(\mathbb{R}, E_{\mathbb{T}})$ may serve as a test for subordination. We shall see in Theorem 6.7 that $(\mathbb{R}, E_{\mathbb{T}})$ plays the same crucial role for products of balayage spaces.

Let us finally mention that, given any $\alpha \in (0,2)$, the measure $\int \eta_t^\alpha\, dt$ has the density

$$t \mapsto (\Gamma(\frac{\alpha}{2}))^{-1}\, t^{\frac{\alpha}{2}-1}$$

with respect to $\lambda_{(0,\infty)}$. Hence $(\mathbb{R}, E_{\mathbb{T}^{\eta^\alpha}})$ is a balayage space with $T_t^{\eta^\alpha} = f * \eta_t^\alpha$.

6.3 Products, heat semigroup

Let $\mathbb{P} = (P_t)_{t>0}$ be a sub-Markov semigroup on X and let $\tilde{\mathbb{P}} = (\tilde{P}_t)_{t>0}$ be a sub-Markov semigroup on a space $\tilde{X}$. Defining

$$(P_t \otimes \tilde{P}_t) f(x, \tilde{x}) := \iint f(y, \tilde{y})\, P_t(x, dy)\, \tilde{P}_t(\tilde{x}, d\tilde{y})$$

we obtain a sub-Markov semigroup $\mathbb{P} \otimes \tilde{\mathbb{P}} = (P_t \otimes \tilde{P}_t)_{t>0}$ on $X \times \tilde{X}$.

Will $(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ be a balayage space provided that $(X, E_{\mathbb{P}})$ and $(\tilde{X}, E_{\tilde{\mathbb{P}}})$ are balayage spaces? It suffices to consider $X = \tilde{X} = \mathbb{R}$ and $\mathbb{P} = \tilde{\mathbb{P}} = \mathbb{T}$ in order to see that further conditions are necessary. A first result based on Corollary 2.6 is the following.

PROPOSITION 6.6. *Let $(X, \mathbb{P})$ and $(\tilde{X}, \tilde{\mathbb{P}})$ be balayage spaces. Then the product space $(X\times\tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ is a balayage space if and only if $E_{\mathbb{P}\otimes\tilde{\mathbb{P}}} = S(E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})\cap C(X\times\tilde{X})$. In particular, $(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ is a balayage space, if $\mathbb{P}\otimes\tilde{\mathbb{P}}$ has a strong Feller resolvent.*

The next theorem is of the same type as Theorem 6.5.

THEOREM 6.7. *Let $(X, E_{\mathbb{P}})$ be a balayage space. Then the following statements are equivalent:*

1. *$(X \times \tilde{X}, E_{\mathbb{P}\otimes\tilde{\mathbb{P}}})$ is a balayage space for every balayage space $(\tilde{X}, E_{\tilde{\mathbb{P}}})$ such that $\tilde{\mathbb{P}}$ is a sub-Markov semigroup having a strong Feller resolvent.*
2. *$(X \times \mathbb{R}, E_{\mathbb{P}\otimes\mathbb{T}})$ is a balayage space.*
3. *$\mathbb{P}$ is a strong Feller semigroup.*

Taking the Brownian semigroup $\mathbb{P}$ on $\mathbb{R}^d$, $d \geq 1$, we obtain the *heat semigroup* $\mathbb{P}\otimes\mathbb{T}$ on $\mathbb{R}^{d+1}$. The preceding result implies that $(\mathbb{R}^{d+1}, E_{\mathbb{P}\otimes\mathbb{T}})$ is a balayage space, if $d \geq 3$. Going back to Corollary 2.6 and using the fact that $\mathbb{P}\otimes\mathbb{T}$ has a strong Feller resolvent, we conclude that $(\mathbb{R}^{d+1}, E_{\mathbb{P}\otimes\mathbb{T}})$ is a balayage space for any $d \geq 1$. By Theorem 5.1, it is even a harmonic space.

6.4 Brownian semigroups on the infinite dimensional torus

The Brownian semigroup $\overline{\mathbb{P}}$ on the torus $T := \mathbb{R}/2\pi\mathbb{Z}$ is obtained by considering the Brownian semigroup $\mathbb{P}$ on $\mathbb{R}$ modulo 2π, that is, defining

$$\overline{P}_t f(\overline{x}) := P_t(f\circ j)(x),$$

where $j\colon x \mapsto \overline{x}$ denotes the quotient map from $\mathbb{R}$ to $T := \mathbb{R}/2\pi\mathbb{Z}$. Using Fourier transforms it is easily verified that

$$\overline{P}_t f(\overline{x}) = \int_0^{2\pi} \overline{g}_t(x-y)(f\circ j)(y)\,dy,$$

where the density $\overline{g}_t$ is given by

$$\overline{g}_t(x) := \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \exp\bigl(-\frac{t}{2}n^2\bigr)\cos nx.$$

Given an infinite sequence $\mathcal{A} = (a_k)$ in $(0,\infty)$, we define probability measures $P_t^{\mathcal{A}}(x,\cdot)$, $t > 0$, $x = (x_k) \in T^\infty$, by

$$P_t^{\mathcal{A}}(x,\cdot) := \bigotimes_{k=1}^{\infty} \overline{P}_{a_k t}(x_k,\cdot).$$

Using the fact that

$$\left|\overline{g}_t - \frac{1}{2\pi}\right| \le \frac{1}{\pi}\sum_{n=1}^{\infty} \exp\left(-\frac{t}{2}n^2\right)$$

and applying Corollary 2.6, Theorem 5.1, and Theorem 6.7, we obtain the following result.

THEOREM 6.8. *Let $\mathcal{A} = (a_k)$ be an infinite sequence in $(0,\infty)$ such that, for every $t > 0$, the series $\sum_{k=1}^{\infty} \exp(-a_k t)$ converges (for example, $\sum_{k=1}^{\infty} a_k^{-m} < \infty$ for some $m \in \mathbb{N}$). Then $\mathbb{P}^{\mathcal{A}} := (P_t^{\mathcal{A}})_{t>0}$ is a strong Feller semigroup on T^∞ such that, for every $f \in \mathcal{K}(T^\infty)$, $\lim_{t\to 0}(1/t)P_t^{\mathcal{A}} f = 0$ locally uniformly on the complement of* $\operatorname{supp}(f)$.

In particular, $(T^\infty, E_{e^{-\alpha t}P_t^{\mathcal{A}}})$ is a harmonic space for every $\alpha > 0$. Moreover, $(T^\infty \times \mathbb{R}, E_{\mathbb{P}^{\mathcal{A}}\otimes\mathbb{T}})$ is a harmonic space.

6.5 Images

Little is needed to guarantee that the image of a balayage space is again a balayage space.

THEOREM 6.9. *Let $(X,\mathcal{W})$ be a balayage space and let π be an open continuous mapping of X onto a locally compact space X_π with countable base such that, given a lower semicontinuous function $f\colon X_\pi \to [0,\infty]$, there exists $g\colon X_\pi \to [0,\infty]$ with $R_{f\circ\pi} = g\circ\pi$. Let*

$$\mathcal{W}_\pi := \{w \in \mathcal{B}^+(X_\pi)\colon w\circ\pi \in \mathcal{W}\}.$$

Then $(X_\pi, \mathcal{W}_\pi)$ is a balayage space if and only if $\mathcal{W}_\pi$ is linearly separating and there exists a strictly positive function in $\mathcal{W}_\pi \cap \mathcal{C}(X_\pi)$.

Clearly, $1 \in \mathcal{W}_\pi \cap \mathcal{C}(X_\pi)$ if $1 \in \mathcal{W}$, and then $\mathcal{W}_\pi$ is linearly separating if it is separating. If $(X,\mathcal{W})$ is a $\mathcal{P}$-harmonic space, then the additional assumption that none of the sets $\pi^{-1}(x)$, $x \in \mathcal{W}$, is an absorbing set in X is necessary and sufficient to have a $\mathcal{P}$-harmonic space $(X_\pi, \mathcal{W}_\pi)$.

We shall say that a homeomorphism $\sigma\colon X \to X$ is an *automorphism of the balayage space* $(X,\mathcal{W})$ if $W\circ\sigma = \mathcal{W}$.

COROLLARY 6.10. *Let G be a group of automorphisms of a balayage space $(X,\mathcal{W})$ and let $\pi\colon X \to X/G =: X_\pi$ denote the quotient mapping. Suppose that X_π is a Hausdorff space, that points lying on different orbits X are linearly separated by the set $\mathcal{W}^G$ of all G-invariant functions in $\mathcal{W}$, and that there exists a strictly positive function in $\mathcal{W}^G \cap \mathcal{C}(X)$.*

Then $w \mapsto w\circ\pi$ is a one-to-one correspondence between $\mathcal{W}_\pi$ and $\mathcal{W}^G$, and $(X_\pi, \mathcal{W}_\pi)$ is a balayage space.

Evidently, the preceding result allows us the construction of many new examples from the classical harmonic space, the space of Riesz potentials, and the harmonic space associated with the heat equation using reflections, translation, and rotations.

7 The Dirichlet problem

Let us finish this overview discussing how the Perron-Wiener-Brelot method for the solution of the Dirichlet problem can be adapted to balayage spaces.

In the following, let $(X, \mathcal{W})$ be a balayage space and let $\mathcal{U}$ be the set of all relatively compact open sets in X. Given an open set U in X, we define the set ${}^*\mathcal{H}(U)$ of all *hyperharmonic functions on* U by

$$ {}^*\mathcal{H}(U) := \{u \in \mathcal{B}(X) \colon u|_U \text{ l.s.c }, -\infty < H_V u(x) \le u(x) \text{ for all } x \in V \in \mathcal{U}(U)\}, $$

the set of all *harmonic functions on* U by

$$ \mathcal{H}(U) := {}^*\mathcal{H}(U) \cap (-{}^*\mathcal{H}(U)) = \{h \in \mathcal{B}(X) \colon h|_U \in \mathcal{C}(U), H_V h = h \text{ for all } V \in \mathcal{U}(U)\}, $$

and the set $\mathcal{S}(U)$ of all *superharmonic functions on* U by

$$ \begin{aligned} \mathcal{S}(U) &:= \{s \in {}^*\mathcal{H}(U) \colon H_V s|_V \in \mathcal{C}(V) \text{ for all } V \in \mathcal{U}(U)\} \\ &= \{s \in {}^*\mathcal{H}(U) \colon H_V s \in \mathcal{H}(V) \text{ for all } V \in \mathcal{U}(U)\}. \end{aligned} $$

Let us stress that all these functions are *functions on* X and that the values on U^c are important, unless $H_V 1_{U^c} = 0$ for all $V \in \mathcal{U}(U)$.

A numerical function f on X is called *$\mathcal{P}$-bounded* (*lower $\mathcal{P}$-bounded, upper $\mathcal{P}$-bounded*, respectively), if there exists a potential $p \in \mathcal{P}$ such that $|f| \le p$ ($f \ge -p$, $f \le p$, respectively).

7.1 Generalized Dirichlet problem

Let U be an open set in X (not necessarily relatively compact). For every $f \colon X \mapsto [-\infty, \infty]$, we define the set

$$ \mathcal{U}_f^U := \{u \in {}^*\mathcal{H}(U) \colon u \text{ l.s.c. and lower } \mathcal{P}\text{-bounded on } X,\ u \ge f \text{ on } U^c\} $$

of all *upper functions* of f. The set of all *lower functions* of f is defined by

$$ \mathcal{L}_f^U := -\mathcal{U}_{-f}^U. $$

Then

$$ \overline{H}_f^U := \inf \mathcal{U}_f^U \quad \text{and} \quad \underline{H}_f^U := \sup \mathcal{L}_f^U $$

are called the *upper* and *lower solution of the generalized Dirichlet problem for f on U*, respectively. The function f is called *resolutive* (for U) provided

$$ \overline{H}_f^U = \underline{H}_f^U =: H_U f \quad \text{and} \quad H_U f \in \mathcal{H}(U). $$

REMARKS 7.1. 1. $\overline{H}_f^U, \underline{H}_f^U$ depend only on $f|_{U^c}$ and $\overline{H}_f^U = \underline{H}_f^U = f$ on U^c.

2. If $(X, \mathcal{W})$ is a $\mathcal{P}$-harmonic space, then $\mathcal{U}_f^U|_U$ is the set of all lower semicontinuous, lower $\mathcal{P}$-bounded numerical functions u on U such that $H_V u(x) \le u(x)$ for all $u \in V \in \mathcal{U}(U)$ and $\liminf_{x \to z} u(x) \ge f(z)$ for every $z \in \partial U$, hence $\overline{H}_f^U|_U$ and $\underline{H}_f^U|_U$ are the upper and lower solution, respectively, which are familiar for harmonic spaces, for the function $f|_{\partial U}$.

THEOREM 7.2. *For every $f\colon X \to \overline{\mathbb{R}}$ and every $x \in U$,*

$$\overline{H}_f^U(x) = \big(\varepsilon_x^{U^c}\big)^*(f) \quad \text{and} \quad \underline{H}_f^U(x) = \big(\varepsilon_x^{U^c}\big)_*(f).$$

In particular, a $\mathcal{P}$-bounded function $f\colon X \to \mathbb{R}$ is resolutive if and only if it is $\varepsilon_x^{U^c}$-integrable for every $x \in U$.

So we may write $H_U f$ instead of H_f^U, whenever $f \in \mathcal{B}$ and $\varepsilon_x^{U^c}(f)$ is defined for every $x \in U$.

A boundary point $z \in \partial U$ is called *regular* (with respect to U), if

$$\lim_{x \to z, x \in U} H_U f(x) = f(z) \qquad \text{for every } f \in \mathcal{K}(X).$$

Exactly as for harmonic spaces we obtain the following result.

PROPOSITION 7.3. *For every $z \in \partial U$, the following properties are equivalent:*

1. *z is regular.*
2. *U^c is not thin at z, that is, $\varepsilon_z^{U^c} = \varepsilon_z$.*
3. *z admits a* barrier*, that is, there exists an open neighborhood V of z and a function $w \in {}^*\mathcal{H}^+(U \cap V)$ such that $w > 0$ on $U \cap V$ and $\lim_{x \to z, x \in U} w(x) = 0$.*

COROLLARY 7.4. *The set of irregular boundary points is semipolar.*

7.2 The weak Dirichlet problem

As before, let U be an open set in a balayage space $(X, \mathcal{W})$. Let $\mathcal{C}_\mathcal{P}$ denote the set of all $\mathcal{P}$-bounded continuous real functions on X and define

$$H(U) := \mathcal{H}(U) \cap \mathcal{C}_\mathcal{P}, \qquad S(U) := \mathcal{S}(U) \cap \mathcal{C}_\mathcal{P}.$$

Obviously, $H(U) + \mathcal{P} \subset S(U)$ and $S(U) \cap (-S(U)) = H(U)$. For every $x \in X$, let

$$\mathcal{M}_x(S(U)) := \{\mu \in \mathcal{M}(X)\colon \mu(s) \le s(x) \text{ for all } s \in S(U)\}.$$

The *Choquet boundary* of X with respect to $S(U)$ is defined by

$$\mathrm{Ch}_{S(U)} X := \big\{x \in X\colon \mathcal{M}_x(S(U)) = \{\varepsilon_x\}\big\}.$$

Clearly, $\mathrm{Ch}_{S(U)} X \subset U^c$, since $\varepsilon_x^{V^c} \in \mathcal{M}_x(S(U))$ for every $x \in V \in \mathcal{U}(U)$.

By a general minimum principle, a function $s \in S(U)$ is positive, if $s \ge 0$ on $\mathrm{Ch}_{S(U)} X$. In particular, each function $h \in H(U)$ is uniquely determined by its restriction to $\mathrm{Ch}_{S(U)} X$.

The *weak Dirichlet problem* is now the following: Given a closed subset A_0 of $\mathrm{Ch}_{S(U)} X$ and a continuous $\mathcal{P}$-bounded function f on A_0, is there a continuous extension f to a function $h \in H(U)$?

We shall see that we may solve this weak Dirichlet problem even in an additive and increasing way using $\mathcal{P}$-dilations obtained by a *smearing* of balayage on suitable families of closed sets.

By definition, a *dilation* is a kernel D on X such that $Dp \le p$ for every $p \in \mathcal{P}$, and a dilation D is called a $\mathcal{P}$-dilation provided $D(\mathcal{P}) \subset \mathcal{P}$. In order to obtain suitable dilations we use the following strong result involving the essential base.

THEOREM 7.5. *Let $A \in \mathcal{B}$ and let A_0 be a closed set in $A \cap \beta(A)$. Then there exists an increasing family $(A_t)_{t>0}$, of closed sets contained in $A \cap \beta(A)$ such that $A_0 = \bigcap_{0<t<1} A_t$, $\beta(A) = \beta(\bigcup_{0<t<1} A_t)$, and $A_s \subset \beta(A_t)$ for all $0 < s < t < 1$.*

Suppose now that $(A_t)_{a \le t \le b}$, $a, b \in \mathbb{R}$, $a < b$, is a family of closed sets in X such that $A_s \subset \beta(A_t)$ for all $a \le s < t \le b$. Then it is easily verified that

$$D\colon (x, B) \mapsto \frac{1}{b-a} \int_a^b \varepsilon_x^{A_t}(B)\, dt$$

is a $\mathcal{P}$-dilation such that $Dp = p$ on A_a and $Dp \in \mathcal{H}(A_b^c)$ for every $p \in \mathcal{P}$. This leads to the following result:

THEOREM 7.6. *$Ch_{S(U)}X = \beta(U^c)$ and, for every closed subset A_0 of $\beta(U^c)$, there exists a $\mathcal{P}$-dilation D such that $Df \in H(U)$, $Df = f$ on A_0 for every $f \in \mathcal{C}_\mathcal{P}$, and all the measures $D(x, \cdot)$, $x \in X$, are supported by a closed set A_1 in $\beta(U^c)$.*

Moreover, for every $x \in X$, the measure $D_U(x, \cdot) := \varepsilon_x^{\beta(U^c)}$ is the only measure $D_U(x, \cdot) \in \mathcal{M}_x(S(U))$ such that $D_U(x, X \setminus Ch_{S(U)}X) = 0$. In particular, $S(U)$ is a simplicial cone.

REMARK 7.7. *A dilation K is called a* Keldych operator*, if $K(\mathcal{C}_\mathcal{P}) \subset \mathcal{H}(U)$ and $Kh = h$ for every $h \in H(U)$. The kernels D_U and H_U are Keldych operators and $D_U p \le Kp \le H_U p$ for every Keldych operator K and every $p \in \mathcal{P}$.*

Let us finally note that the preceding considerations on the weak Dirichlet problem can be extended to functions in $\mathcal{C}_\mathcal{P}$ which are finely harmonic (finely superharmonic, respectively) on a finely open set G. Sets $H(G)$ ($S(G)$, respectively), where G is finely open, but in general not open, arise in a natural way, if we are interested in functions in $\mathcal{C}_\mathcal{P}$ which are harmonic (superharmonic, respectively) in a neighborhood of a given closed set F. The set of these functions is dense in $H(G)$ ($S(G)$, respectively), where G is the fine interior of F.

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