Perturbation of the Laplace operator

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1 Introduction

In this short course, we shall discuss perturbations of the Laplace operator

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

on $\mathbb{R}^d$, $d \geq 1$. We recall that a function h on an open set W is called *harmonic*, if $h \in \mathcal{C}^2(W)$ and $\Delta h = 0$ or – equivalently – if $h \in \mathcal{C}^2(W)$ and

$$\text{(1.1)} \qquad \int_W h\,\Delta\varphi\,d\lambda^d = 0 \qquad \text{for every } \varphi \in \mathcal{C}_c^\infty(W)$$

(where λ^d denotes Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{C}_c^\infty(W)$ the space of all real functions on W with compact support which are differentiable infinitely many times). In fact, it suffices to know that $h \in \mathcal{C}(W)$ and h satisfies (1.1) to conclude that h is harmonic on W.

A relatively compact open set U in $\mathbb{R}^d$ is *regular* if, for every continuous real function φ on the boundary ∂U, there exists a unique extension to a continuous real function h on the closure $\overline{U}$, the *solution to the Dirichlet problem* for U and φ, which is harmonic on U. It is positive provided $\varphi \geq 0$. For example, every open ball is regular and the solution to the corresponding Dirichlet problem is given by the Poisson kernel.

We shall study solutions (and supersolutions) to equations

$$\Delta u = u\mu, \tag{1.2}$$

where μ is a signed Radon measure on $\mathbb{R}^d$ belonging to a certain class (we shall assume that μ is a (local) *Kato measure*; for the definition see Section 2). If μ has a density, say V, with respect to λ^d, then $V \in \mathcal{L}^p_{\text{loc}}(\mathbb{R}^d)$, $p < d/2$, is a sufficient property; see Corollary 2.7).

DEFINITION 1.1. *A continuous real function h on an open set W in $\mathbb{R}^d$ is called* $(\Delta - \mu)$-harmonic*, if $\Delta u - u\mu = 0$ in the sense of distributions, that is, if*

$$\int h\,\Delta\varphi\,d\lambda^d - \int u\varphi\,d\mu = 0 \qquad \text{for every } \varphi \in \mathcal{C}_c^\infty(W).$$

The set of all $(\Delta - \mu)$-harmonic functions on W will be denoted by $\mathcal{H}^{\Delta-\mu}(W)$.

A relatively compact open set U in $\mathbb{R}^d$ will be called $(\Delta - \mu)$-regular if, for every $\varphi \in C(\partial U)$, there exists a unique function $h \in \mathcal{H}^{\Delta-\mu}(U)$ such that $h \to \varphi$ at ∂U, and $h \geq 0$ provided $\varphi \geq 0$.

It will turn out every regular set is $(\Delta - \mu)$-regular, if we have a *positive perturbation*, that is, if $\mu \geq 0$ (see Theorem 4.2). For *negative perturbations*, that is, if $\mu \leq 0$, this may fail.

To get some feeling for these possibilities let us first look at the case $d = 1$ and $\mu = \alpha\lambda^1$, that is, at the equation

$$u'' = \alpha u, \tag{1.3}$$

where $\alpha \in \mathbb{R}$. Let $U = (a, b)$, $-\infty < a < b < \infty$, and φ a real function on $\partial U = \{a, b\}$.

a) If $\alpha = 0$, then the solutions to (1.3) are the functions which are (locally) affinely linear and, of course,

$$h := \frac{b - x}{b - a}\,\varphi(a) + \frac{x - a}{b - a}\,\varphi(b)$$

is the solution to the (classical) Dirichlet problem. A slightly different approach (which will be useful when dealing with $\alpha \neq 0$) is the following. For $x \in [a, b]$, let

$$h_1(x) := b - x \quad \text{ and } \quad h_2(x) := x - a.$$

Then h_1, h_2 are positive affinely linear functions on $[a, b]$,

$$h_1(a) > 0, \quad h_1(b) = 0, \quad h_2(a) = 0, \quad h_2(b) > 0. \tag{1.4}$$

Hence

$$h := \frac{\varphi(a)}{h_1(a)}\, h_1 + \frac{\varphi(b)}{h_2(b)}\, h_2 \tag{1.5}$$

is the solution to the Dirichlet problem.

b) Let us now assume that $\alpha > 0$ (positive perturbation) and let $\gamma := \sqrt{\alpha}$. Then the solutions to $u'' = \alpha u$ are linear combinations of $e^{\gamma x}$ and $e^{-\gamma x}$. So the functions

$$h_1 := e^{\gamma(b-x)} - e^{\gamma(x-b)} \quad \text{and} \quad h_2 := e^{\gamma(x-a)} - e^{\gamma(a-x)}$$

are positive solutions to $u'' = \alpha u$. Since they satisfy (1.4), (1.5) yields a solution to the Dirichlet problem for $u'' = \alpha u$, a solution which is positive provide $\varphi \geq 0$ (and even strictly positive on (a, b), if at least one of the values of φ is stricly positive); the uniqueness is easily verified. So U is $(\Delta - \alpha\lambda^1)$-regular. Let us note that now the functions h_1, h_2 are strictly convex, and hence the solution to the Dirichlet problem is strictly smaller than the solution in the case $\alpha = 0$.

c) Finally, let us assume that $\alpha < 0$ (negative perturbation), and let $\gamma := \sqrt{-\alpha}$. Then the solutions to $u'' = \alpha u$ are linear combinations of $\sin \gamma x$ and $\cos \gamma x$ and, up to multiples, the functions

$$h_1 := \sin \gamma(b - x) \quad \text{and} \quad h_2 := \sin \gamma(x - a)$$

are the only solutions which vanish at b and a, respectively. We observe that, for $j = 1, 2$,

$$h_j \geq 0 \text{ on } U \quad \text{if and only if} \quad \gamma(b - a) \leq \pi.$$

Therefore $U = (a, b)$ is $(\Delta - \alpha\lambda^1)$-regular if and only if

$$\gamma(b - a) < \pi,$$

and then, again, the solution to the Dirichlet problem for the given function φ is obtained by (1.5). Now the solution is strictly concave, and hence strictly larger than the solution in the case $\alpha = 0$. Moreover, we see that, assuming $\varphi \geq 0$ and $\varphi \neq 0$, the solution h is increasing to ∞ as $\gamma \uparrow \pi/(b - a)$.

If $\gamma(b - a) = \pi$, then $h_0 := h_1 = h_2 = \sin \gamma(x - a)$ is a stricty positive function in $\mathcal{H}^{\Delta-\mu}(U) \cap \mathcal{C}_0(U)$ (where $\mathcal{C}_0(U) := \{f \in \mathcal{C}(U)\colon f \to 0 \text{ at } \partial U\}$). If $\gamma(b - a) = k\pi$, $k \in \mathbb{N}$, $k \geq 2$, then still $\sin \gamma(b - a) \in \mathcal{H}^{\Delta-\mu}(U) \cap \mathcal{C}_0(U)$, but $\sin \gamma(b - a)$, as well as every solution to $u'' = \alpha u$ on U, changes its sign.

If $\gamma(b - a) > \pi$ and $\gamma(b - a)$ is not a multiple of π, a function h, defined by (1.5), still satisfies $h(a) = \varphi(a)$, $h(b) = \varphi(b)$, and $h'' = \alpha h$, but h changes its sign (even if $\varphi(a)$ and $\varphi(b)$ are both positive).

2 Kato measures, potential kernels, and $(\Delta - \mu)$-harmonic functions

For $x \in \mathbb{R}^d$, $d \geq 1$, and $r > 0$, let $B(x, r)$ denote the open ball with center x and radius r, that is,

$$B(x, r) := \{y \in \mathbb{R}^d\colon |y - x| < r\}.$$

We define a (Green) function $G\colon \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, \infty]$ by

$$G(x,y) := \begin{cases} \frac{1}{2}(1-|x-y|), & d=1,\\ \frac{1}{2\pi}\ln\frac{1}{|x-y|}, & d=2,\\ \frac{1}{(d-2)\omega_d}|x-y|^{2-d}, & d\ge 3. \end{cases}$$

Let us observe that $G>0$, if $d \ge 3$, whereas, for $d \le 2$, $G(x,y)>0$ if and only if $|x-y|<1$.

Given a Radon measure ρ on $\mathbb{R}^d$ with compact support, we may define a function $G\rho\colon \mathbb{R}^d \to (-\infty,\infty]$ by

$$G\rho(x) := \int G(x,y)\, d\rho(y), \qquad x \in \mathbb{R}^d, \tag{2.1}$$

and note that $G\rho$ is superharmonic on $\mathbb{R}^d$. It is not difficult to verify that

$$\Delta G(\cdot, y) = -\varepsilon_y, \qquad y \in \mathbb{R}^d,$$

(in the sense of distributions), and hence, by Fubini's theorem,

$$\Delta G\rho = -\rho. \tag{2.2}$$

In particular, $G\rho$ is harmonic outside the support of ρ.

DEFINITION 2.1. *A signed Radon measure on $\mathbb{R}^d$ is called a (local)* Kato measure, *if, for every open ball B in $\mathbb{R}^d$,*

$$G(1_B|\mu|) \in \mathcal{C}(\mathbb{R}^d). \tag{2.3}$$

REMARKS 2.2. *It is easily verified that, for every signed Radon measure μ on $\mathbb{R}^d$ the following holds.*

1. *μ is a Kato measure if and only if μ^+ and μ^- are Kato measures.*
2. *If (B_n) is a sequence of open balls covering $\mathbb{R}^d$, then μ is a Kato measure provided* (2.3) *holds for every ball B_n, $n \in \mathbb{N}$.*
3. *If μ is a Kato measure and f is a locally bounded Borel measurable function on $\mathbb{R}^d$, then $f\mu$ is a Kato measure.*

For every regular set U in $\mathbb{R}^d$, let H_U denote the harmonic kernel for U, that is, $H_U(x,\cdot)$ is the harmonic measure μ_x^U, if $x \in U$, and $H_U(x,\cdot) = \varepsilon_x$, if $x \in U^c$. If V,U are regular sets such that $V \subset U$, then

$$H_V H_U = H_U. \tag{2.4}$$

DEFINITION 2.3. *Let μ be a signed Radon measure on $\mathbb{R}^d$. For every regular set U and every $f \in \mathcal{B}_b(U)$, let*

$$K_U^\mu f := G(f\mu) - H_U G(f\mu) \tag{2.5}$$

(*where* 2.1 *is extended in an obvious way to signed measures*).

LEMMA 2.4. *Let A_n, $n \in \mathbb{N} \cup \{\infty\}$, be bounded Borel sets such that $A_n \downarrow A_\infty$. Then $G(1_{A_n}\mu) \to G(1_{A_\infty}\mu)$ locally uniformly as $n \to \infty$.*

Proof. We may assume that $\mu \geq 0$. By bounded convergence, $G(1_{A_n}\mu) \to G(1_{A_\infty}\mu)$ pointwise. If $d \geq 3$, then the proof is immediately finished by Dini's lemma. If $d \leq 2$, we may consider a ball B such that $A_1 \subset B$. Then, by Dini's lemma, $K_B^\mu 1_{A_n} \downarrow K_B^\mu 1_{A_\infty}$ locally uniformly on B. Moreover, $H_B G(1_{A_n}\mu) \to H_B G(1_{A_\infty}\mu)$ locally uniformly on B. Since $K_B^\mu 1_{A_n} + H_B G(1_{A_n}\mu) = G(1_{A_n}\mu)$, $n \in \mathbb{N} \cup \{\infty\}$, the claim follows. □

The following statements are easily verified (the last property follows immediately from (2.4) and (2.5)).

PROPOSITION 2.5. *If μ is a Kato measure and U is a regular set, then the following holds:*

1. $K_U^{\mu^\pm}(\mathcal{B}_b^+(U)) \subset \mathcal{S}_b^+(U)$.
2. $K_U^\mu(\mathcal{B}_b(U)) \subset \mathcal{C}_0(U)$.
3. *If (f_n) is a bounded sequence in $B_b(U)$ such that $f_n \to f$ pointwise, then $K_U^\mu f_n \to K_U^\mu f$ (even uniformly).*
4. *If V is a regular set and $V \subset U$, then*

$$K_V^\mu = K_U^\mu - H_V K_U^\mu. \tag{2.6}$$

If $d \geq 2$, then Kato measures do not charge points. Indeed, for every $x \in \mathbb{R}^d$, $0 \leq |\mu|(\{x\})G(\cdot, x) \leq G(1_{B(x,1)}|\mu|) < \infty$, Knowing that $G(x,x) = \infty$ we see that $\mu(\{x\}) = 0$.

PROPOSITION 2.6. *For every signed Radon measure μ on $\mathbb{R}^d$, which does not charge points, the following statements are equivalent:*

(i) *μ is a Kato measure.*

(ii) *For every $R > 0$, $\lim_{\varepsilon\to 0} \sup_{|x|\leq R} \|G(1_{B(x,\varepsilon)}|\mu|)\|_\infty = 0$.*

(iii) *For every $R > 0$, $\lim_{\varepsilon\to 0} \sup_{|x|\leq R} \|K_{B(x,\varepsilon)}^{|\mu|} 1\|_\infty = 0$.*

(iv) *For every $x \in \mathbb{R}^d$, $\lim_{\varepsilon\to 0} \|K_{B(x,\varepsilon)}^{|\mu|} 1\|_\infty = 0$.*

Proof. We may assume without loss of generality that $\mu \geq 0$.

(i) ⇒ (ii): Let $R > 0$, $A := \overline{B(0,R)}$, $x \in A$, and $\eta \in (0, 1/2)$. Knowing that $G > 0$ on $B(x,\eta) \times B(x,\eta)$, the functions $G(1_{B(x,\varepsilon)}\mu)$, $0 < \varepsilon \leq \eta$, are continuous, and $\mu(\{x\}) = 0$, we obtain, by Dini's theorem, that there exists $\varepsilon(x) \in (0,\eta)$ such that $G(1_{B(x,\varepsilon(x))}\mu) < \eta$. We may choose points $x_1, \ldots, x_m \in A$ such that A is covered by the balls $B(x_j, \varepsilon(x_j))$, $1 \leq j \leq m$. Now let $\varepsilon := (1/2)\min\{\varepsilon(x_1), \ldots, \varepsilon(x_m)\}$. If $x \in A$, then $B(x,\varepsilon) \subset B(x_j, \varepsilon(x_j))$ for some $1 \leq j \leq m$, and hence $0 \leq G(1_{B(x,\varepsilon)}\mu) \leq G(1_{B(x_j,\varepsilon(x_j))}\mu) < \eta$.

(ii) ⇒ (iii): Obvious, since $0 \leq K_{B(x,\varepsilon)}^\mu 1 \leq G(1_B(x,\varepsilon)\mu)$, if $\varepsilon \in (0, 1/2)$.

(iii) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (i): Let U be regular, $x \in U$, $\eta > 0$, and $\varepsilon \in (0, 1/2)$ such that $B := B(x, \varepsilon) \subset U$ and $K_B^\mu 1 \leq \eta$. Then

$$p := G(1_U \mu) = G(1_{U \setminus B}\mu) + G(1_B \mu) = G(1_{U\setminus B}\mu) + H_B G(1_B \mu) + K_B^\mu 1,$$

where the first two terms on the right side are harmonic on B, and hence continuous on B. So there exists $\delta > 0$ such that $|p(y) - p(x)| < 3\eta$ provided $|x - y| < \delta$. □

COROLLARY 2.7. *Let $p > d/2$ and $V \in \mathcal{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$. Then $V\lambda^d$ is a Kato measure.*

Proof. We may suppose that $V \geq 0$ and shall prove the result only for $d \geq 3$ (leaving the cases $d = 1$ and $d = 2$ to the reader). Let $1/p + 1/q = 1$. Then

$$\frac{1}{q} = 1 - \frac{1}{p} > 1 - \frac{2}{d} = \frac{d-2}{d}. \tag{2.7}$$

Let $R > 0$, $|x| \leq R$, $\varepsilon \in (0, 1/2)$, $y \in B(0, \varepsilon)$, and $B := B(y, 2\varepsilon)$. Then $B(x, \varepsilon) \subset B \subset B(0, R+1)$. Hence, by Hölder's inequality,

$$\begin{aligned} G(1_{B(x,\varepsilon)} V\lambda^d)(y) &\leq G(1_B V \lambda^d)(y) = \frac{1}{(d-2)\omega_d} \int_B |y - z|^{2-d} V(z)\, d\lambda^d(z) \\ &\leq \frac{I_\varepsilon}{(d-2)\omega_d} \left(\int_{B(0,R+1)} V^p(z)\, d\lambda^d(z) \right)^{1/p}, \end{aligned}$$

where the integral is finite and

$$I_\varepsilon := \left(\int_B |y - z|^{(2-d)q}\, d\lambda^d(z) \right)^{1/q} = \omega_d \int_0^{2\varepsilon} t^{d-1-(d-2)q}\, dt.$$

By (2.7), $d - (d-2)q > 0$. Thus $\lim_{\varepsilon \to 0} I_\varepsilon = 0$. □

In the following let μ be a signed Radon measure on $\mathbb{R}^d$ which is a Kato measure. If $d = 1$, we assume that μ does not charge points.

PROPOSITION 2.8. *1. For every real function h on an open set W in $\mathbb{R}^d$, the following statements are equivalent:*

(i) $h \in \mathcal{H}^{\Delta-\mu}(W)$.

(ii) *For every regular U with $\overline{U} \subset W$,*

$$h \in \mathcal{B}_b(U) \quad \text{and} \quad h + K_U^\mu h \quad \text{is harmonic on } U. \tag{2.8}$$

(ii′) *There exists a covering of W by regular sets such that $\overline{U} \subset W$ and (2.8) holds.*

2. For every real function h on a regular set U the following statements are equivalent:

(i) $h \in \mathcal{H}_b^{\Delta-\mu}(U)$.

(ii) $h \in \mathcal{B}_b(U)$ *and* $h + K_U^\mu h \in \mathcal{H}(U)$.

Proof. Straightforward using $K_U^\mu(B_b(U)) \subset \mathcal{C}_0(U)$ and $\Delta(h + K_U^\mu h) = \Delta h - h\mu$ on U. □

Consequence: If U is regular and $\varphi \in \mathcal{C}(\partial U)$, then $h \in \mathcal{H}^{\Delta-\mu}(U)$ and h tends to φ at ∂U if and only if

$$h + K_U h = H_U \varphi. \tag{2.9}$$

So we are led to the following questions:

- Is $I + K_U^\mu$ invertible on the space of all $H_U\varphi$, $\varphi \in \mathcal{C}(\partial U)$?
- If yes, is $(I + K_U^\mu)^{-1} H_U \varphi \geq 0$ provided $\varphi \geq 0$?

Before starting to answer these questions let us note a convergence property of $(\Delta - \mu)$-harmonic functions.

COROLLARY 2.9. *Let W be an open set and (h_n) an increasing sequence of $(\Delta - \mu)$-harmonic functions such that $h := \sup h_n$ is locally bounded. Then h is $(\Delta - \mu)$-harmonic.*

Proof. Let B be a ball, $\overline{B} \subset W$. Then $H_B h_n = h_n + K_B^\mu h_n$, for every $n \in \mathbb{N}$. By dominated convergence, $\lim H_B h_n = H_B h$ and $\lim_{n\to\infty} K_B^\mu h_n = K_B^\mu h$. Thus $H_B h = h + K_B^\mu h$. Since $H_B h$ is harmonic on B, we obtain that h is $(\Delta-\mu)$-harmonic on B. □

PROPOSITION 2.10. *Let U be a regular set, $s \in S^+(U)$, and suppose that $\mu \geq 0$. Let $f \in \mathcal{B}_b(U)$ such that $s \geq K_U^\mu f$ on $\{f > 0\}$. Then $s \geq K_U^\mu f$ on U.*

Proof. By assumption,

$$s + K_U^\mu f^- \geq K_U^\mu f^+ \quad \text{on } \{f^+ > 0\}. \tag{2.10}$$

Then $t := s + K_U^\mu f^- \in \mathcal{S}^+(U)$. Let A be a compact set in $\{f > 0\}$, $W := U \setminus A$, and $g := K_U^\mu(1_A f)$. Clearly, g is harmonic on W. By (2.10), $t \geq g$ on A. Moreover, for every $z \in \partial U$, $\liminf_{x\to z}(t(x) - g(x)) \geq 0$, since $t \geq 0$ and $g \in \mathcal{C}_0(U)$. By the minimum principle, $t - g \geq 0$ on W. Hence $t \geq g$ on U. This clearly implies that $t \geq K_U^{\mu^+} f$ on U. □

3 Dominant functions and resolvents

In this section, we shall provide general results which in view of Proposition 2.10 will establish that the answers to the questions raised at the end of Section 2 are always YES, if μ is positive.

Let L be a bounded kernel on a measurable space $(E, \mathcal{E})$.

DEFINITION 3.1. *A function $u \in \mathcal{E}^+$ is called* L-dominant *if, for every $f \in \mathcal{E}_b$,*

$$u \geq Lf \quad \textit{provided} \quad u \geq Lf \quad \text{on } \{f > 0\}.$$

REMARKS 3.2. 1. If u is L-dominant and $\alpha, \beta > 0$, then αu is (βL)-dominant (trivial).

2. If u is a finite L-dominant function, then all functions $\alpha u + Lg$, $\alpha \geq 0$, $g \in \mathcal{E}_b^+$ are L-dominant.

Indeed, if $f \in \mathcal{E}_b$ and $\alpha u + Lg \geq Lf$ on $\{f > 0\}$, then, for every $\alpha' > \alpha$, $\alpha' u \geq L(f - g)$ on the set $\{f > 0\}$ which contains the set $\{f - g > 0\}$, and hence $\alpha' u \geq L(f - g)$ on E. Letting $\alpha' \downarrow \alpha$ we obtain that $\alpha u + Lg \geq Lf$.

REMARK 3.3. Let us note the following identities which will be useful. If S, T are bounded operators on $(E, \mathcal{E}_b)$ such that $I + S$ and $I + T$ are invertible, then

$$(I + S)^{-1} - (I + T)^{-1} = (I + T)^{-1}(T - S)(I + S)^{-1} \tag{3.1}$$

(to see this, it suffices to apply $I + S$ from the right, and $I + T$ from the left), in particular (taking $T = 0$),

$$(I + S)^{-1} = I - (I + S)^{-1}S. \tag{3.2}$$

The following lemma is very simple, but powerful.

LEMMA 3.4. *Let u be L-dominant and $f, g \in \mathcal{E}_b$ such that $f \leq u$ and $Lf = g + Lg$. Then $g \leq u$.*

Proof. Since $g = L(f - g)$, we know that $u \geq L(f - g)$ on the set $\{u \geq g\}$ which contains the set $\{f - g > 0\}$. Thus $u \geq L(f - g)$, that is, $u \geq g$. □

PROPOSITION 3.5. *Suppose that $I + L$ is invertible and u is a bounded L-dominant function. Then $0 \leq (I + L)^{-1}L \leq L$, the operator $(I + L)^{-1}L$ is σ-continuous, and*

$$0 \leq (I + L)^{-1}Lu \leq u \quad \text{and} \quad 0 \leq (I + L)^{-1}u \leq u. \tag{3.3}$$

Proof. Let v be any bounded L-dominant function and $g := (I + L)^{-1}Lv$ so that $g + Lg = Lv$. By Remark 3.2.2, 0 is L-dominant (as well as every Lf, $f \in \mathcal{E}_b^+$). By Lemma 3.4, $g \leq v$ and $-g \leq 0$ (using $-v \leq 0$). So

$$0 \leq (I + L)^{-1}Lv \leq v, \qquad 0 \leq (I + L)^{-1}v \leq v, \tag{3.4}$$

where the second inequalities follow by (3.2). Thus (3.3) holds and, applying (3.4) to the functions $v = Lf$, $f \in \mathcal{E}_b^+$, we see that $0 \leq (I + L)^{-1}L \leq L$.

Finally, let $f_n \in \mathcal{E}_b^+$, $f_n \downarrow 0$. Since $0 \leq (I + L)^{-1}Lf_n \leq Lf_n$ and $Lf_n \downarrow 0$, we obtain that $\lim_{n\to\infty}(I + L)^{-1}Lf_n = 0$. □

For the remainder of this section, let us assume that there exists a bounded L-dominant function $u \geq 1$.

PROPOSITION 3.6. *$I + L$ is invertible.*

Proof. Clearly, $I + \alpha L$ is invertible, if $0 < \alpha < \|L\|^{-1} = \|L1\|_\infty^{-1}$. Let us fix a real $c > \|u\|_\infty$ and suppose that, for some $\alpha > 0$, the operator $I + \alpha L$ is invertible. By Proposition 3.5,

$$\|(I + \alpha L)^{-1}L\| \leq \alpha^{-1}\|(I + \alpha L)^{-1}(\alpha L)u\|_\infty \leq \alpha^{-1}\|u\|_\infty < c/\alpha.$$

So, for every $\beta \in [\alpha, (1+1/c)\alpha]$,

$$\|(\beta-\alpha)(I+\alpha L)^{-1}L\| < \frac{(\beta-\alpha)c}{\alpha} \le 1,$$

and hence

$$I+\beta L = (I+\alpha L)(I+(\beta-\alpha)(I+\alpha L)^{-1}L)$$

is invertible. A straightforward induction shows that $I+\alpha L$ is invertible for every $\alpha > 0$. □

For every $\alpha \ge 0$, let

$$L_\alpha := (I+\alpha L)^{-1}L$$

so that, by (3.2),

$$(I+\alpha L)^{-1} = I - \alpha L_\alpha. \tag{3.5}$$

PROPOSITION 3.7. *$(L_\alpha)_{\alpha\ge 0}$ is a resolvent, that is, for all $\alpha, \beta > 0$,*

$$L_\beta - L_\alpha = (\alpha-\beta)L_\alpha L_\beta \qquad \text{(resolvent equation)}. \tag{3.6}$$

If $v \in \mathcal{E}_b^+$ is L-dominant, then $\alpha L_\alpha v \le v$ and $\alpha \mapsto \alpha L_\alpha v$ is increasing.

Moreover, $\alpha \mapsto L_\alpha$ is decreasing and $L = L_0 = \sup_{\alpha>0} L_\alpha$.

Proof. By Proposition 3.2, the resolvent equation follows from (3.1), taking $S = \beta L$, $T = \alpha L$, and then applying L from the right. Moreover, we see that, for every L-dominant $v \in \mathcal{E}_b^+$, the mapping $\alpha \mapsto (I+\alpha L)^{-1}v$ is decreasing and positive, and therefore, by (3.5), the mapping $\alpha \mapsto \alpha L_\alpha v$ is increasing and bounded by v.

The resolvent equation shows that $\alpha \mapsto L_\alpha$ is decreasing and $L = L_\alpha + \alpha L_\alpha L$, where $\lim_{\alpha\to 0} \alpha L_\alpha L = 0$, since $\alpha L_\alpha L1 \le \alpha L^2 1$. Thus $L = L_0 = \sup_{\alpha>0} L_\alpha$. □

PROPOSITION 3.8. *Let $D := d/d\alpha$ and $f \in \mathcal{E}_b$. Then, for all $n \in \mathbb{N}$ and $\alpha > 0$,*

$$D^n L_\alpha f = (-1)^n\, n!\, L_\alpha^{n+1} f \qquad \textit{and} \qquad D^n(\alpha L_\alpha) f = (-1)^{n+1}\, n!\, L_\alpha^n (I-\alpha L_\alpha) f. \tag{3.7}$$

Proof. Let $g \in \mathcal{E}_b$. Then, for all $i, j \in \mathbb{N} \cup \{0\}$ and $\alpha, \beta \in (0, \infty)$, by the resolvent equation

$$(L_\beta - L_\alpha) L_\beta^i L_\alpha^j = (\beta-\alpha) L_\beta^{i+1} L_\alpha^{j+1} g,$$

where

$$|L_\beta^{i+1} L_\alpha^{j+1} g| \le \|g\|_\infty L_\beta^{i+1} L_\alpha^{j+1} u \le \alpha^{-(i+1)} \beta^{-(j+1)} \|g\|_\infty u.$$

Let $n \in \mathbb{N}$. Recalling that $L_\beta^n - L_\alpha^n = (L_\beta - L_\alpha)(L_\beta^{n-1} + L_\beta^{n-2} L_\alpha + \dots L_\alpha^{n-1})$, we obtain that $\lim_{\beta\to\alpha} L_\beta^n g = L_\alpha^n g$ and

$$D(L_\alpha^n g) = -n L_\alpha^{n+1} g. \tag{3.8}$$

By induction, the first equality in (3.7) follows. Moreover,

$$D(\alpha L_\alpha g) = L_\alpha g - \alpha L_\alpha^2 g = L_\alpha (I - \alpha L_\alpha) g,$$

and, for every every $n \in \mathbb{N}$, by (3.8),

$$\begin{aligned} D(L_\alpha^{n-1}(I-\alpha L_\alpha)g) &= D(L_\alpha^{n-1} g - \alpha L_\alpha^n g) \\ &= -(n-1)L_\alpha^n g - L_\alpha^n g + n\alpha L_\alpha^{n+1} g = -n L_\alpha^n (I - \alpha L_\alpha) g. \end{aligned}$$

By induction, the proof of (3.7) is finished. □

COROLLARY 3.9. *For every bounded L-dominant $v > 0$,*

$$(I+L)^{-1}v \geq v\,\exp(-Lv/v) > 0.$$

Proof. Let us fix $x \in E$, $\varepsilon \in (0,1)$, and define

$$\varphi(\alpha) := \varepsilon + (I+\alpha L)^{-1}v(x) = \varepsilon + v(x) - \alpha L_\alpha v(x), \qquad \alpha > 0.$$

Then $\varphi > 0$ and, by Proposition 3.8, $(-1)^n\varphi^{(n)} \geq 0$, for all $n \in \mathbb{N}$. So, by Bernstein's theorem, $\psi := \ln\varphi$ is convex. In particular, $\psi(1) \geq \psi(\varepsilon) + (1-\varepsilon)\psi'(\varepsilon)$.[1] Since $\psi' = \varphi'/\varphi$, we obtain that

$$\varphi(1) \geq \varphi(\varepsilon)\exp((1-\varepsilon)\varphi'(\varepsilon)/\varphi(\varepsilon)).$$

We have $\varphi(1) = \varepsilon + (I+L)^{-1}v(x)$, $\varphi(\varepsilon) = \varepsilon + (I+\varepsilon L)^{-1}v(x)$, and

$$\varphi'(\varepsilon) = -L_\varepsilon(I - \varepsilon L_\varepsilon)v(x)$$

whence $\lim_{\varepsilon\to 0}\varphi'(\varepsilon) = -Lv(x)$. Thus the proof is finished letting $\varepsilon \to 0$. □

REMARK 3.10. *If we only want to know that $(I+L)^{-1}v > 0$ (and this will be sufficient for our applications of Corollary 3.9), we may proceed as follows. Let $x \in E$ and $\varphi(\alpha) := (I+\alpha L)^{-1}v(x)$, $\alpha \geq 0$. Then $\varphi \geq 0$, $\varphi(0) > 0$, and φ is decreasing. Assuming that φ is not strictly positive, there hence exists $\alpha_0 > 0$such that $\varphi > 0$ on $[0,\alpha_0)$ and $\varphi = 0$ on (α_0,∞). Since φ can be represented by a Taylor series around α_0, we obtain a contradiction.*

4 $(\Delta-\mu)$-regular sets and $(\Delta-\mu)$-superharmonic functions

PROPOSITION 4.1. *If $\mu \geq 0$, then, for every regular set U, $I + K_U^\mu$ is invertible and, for every $s \in \mathcal{S}_b^+(U)$, $0 \leq (I+K_U^\mu)^{-1}s \leq s$.*

Proof. By Proposition 2.10, every $u \in \mathcal{S}_b^+(U)$ is K_U^μ-dominant. Hence the result follows immediately from Proposition 3.5. □

THEOREM 4.2. *If $\mu \geq 0$, then every regular U is $(\Delta-\mu)$-regular, the corresponding harmonic kernel is*

$$H_U^{\Delta-\mu} = (I+K_U^\mu)^{-1}H_U, \tag{4.1}$$

and, for every $\varphi \in \mathcal{C}^+(\partial U)$ with $H_U\varphi > 0$,

$$H_U\varphi \geq H_U^{\Delta-\mu}\varphi \geq e^{-K_U^\mu H_U\varphi/H_U\varphi}\,H_U\varphi.$$

In particular, $1 \geq H_U^{\Delta-\mu}1 \geq e^{-K_U^\mu 1}$.

Proof. If $\varphi \in \mathcal{C}^+(\partial U)$, then $s := H_U\varphi \in \mathcal{H}_b^+(U) \subset \mathcal{S}_b^+(U)$. Hence the statements follow from Proposition 4.1 and Corollary 3.9. □

[1]For the moment, we do not want to worry about limits at 0.

To treat the general case let us note that, for every regular set U,

$$I + K_U^\mu = (I + K_U^{\mu^+})(I - (I + K_U^{\mu^+})^{-1} K_U^{\mu^-}), \tag{4.2}$$

where

$$L_U^\mu := (I + K_U^{\mu^+})^{-1} K_U^{\mu^-}$$

is a bounded kernel and $L_U^\mu \le K_U^{\mu^-}$.

LEMMA 4.3. *Let L be a bounded kernel on a measurable space $(E, \mathcal{E})$. Then the following statements are equivalent.*

(i) *The function $\sum_{n=0}^\infty L^n 1$ is bounded.*

(ii) *The operator $I - L$ on $(\mathcal{E}_b, \|\cdot\|_\infty)$ is invertible and $(I - L)^{-1} = \sum_{n=0}^\infty L^n$.*

(iii) *The operator $I - L$ on $(\mathcal{E}_b, \|\cdot\|_\infty)$ is invertible and its inverse is positive.*

(iv) *There exists $g \in \mathcal{E}_b^+$ such that $1 + Lg \le g$.*

(v) *There exist $c > 0$ and $\gamma \in (0,1)$ such that, for every $n \in \mathbb{N}$, $L^n 1 \le c\gamma^n$.*

(vi) *The spectral radius $\rho(L) := \lim_{n\to\infty} \|L^n\|^{1/n}$ is strictly less than 1.*

Proof. (i) ⇒ (ii) ⇒ (iii): Trivial.

(iii) ⇒ (iv): The function $g := (I - L)^{-1} 1 \in \mathcal{E}_b^+$ satisfies $1 + Lg = g$.

(iv) ⇒ (v): Of course, $c := \|g\|_\infty \ge 1$. Since $c^{-1} g + Lg \le 1 + Lg \le g$, and hence $Lg \le (1 - c^{-1}) g$, we obtain that, for every $n \in \mathbb{N}$, $L^n g \le (1 - c^{-1})^n g \le c(1 - c^{-1})^n$.

(v) ⇒ (vi): Trivial, since $\|L^n\| = \|L^n 1\|_\infty$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} c^{1/n} = 1$.

(vi) ⇒ (i): Let $\gamma \in (\rho(L), 1)$. Then there exists $m \in \mathbb{N}$ such that $L^m 1 \le \gamma$, and hence $\sum_{n=0}^\infty L^n 1 \le (1 - \gamma)^{-1} \sum_{j=0}^{m-1} L^j 1$ is bounded. □

DEFINITION 4.4. *A regular set U is called μ-*admissible*, if $I - L_U^\mu$ is invertible and its inverse is positive.*

REMARK 4.5. In particular, every regular set U satisfying $\|K_U^{\mu^-} 1\|_\infty < 1$, is μ-admissible. So, by Proposition 2.6, there exists a base of μ-admissible balls.

PROPOSITION 4.6. *For every regular set U the following properties are equivalent:*

1. *U is μ-admissible.*

2. *$I + K_U^\mu$ is invertible and*

$$(I + K_U^\mu)^{-1} = \sum_{n=0}^\infty (L_U^\mu)^n (I + K_U^{\mu^+})^{-1}. \tag{4.3}$$

3. *$I + K_U^\mu$ is invertible and, for every $s \in \mathcal{S}_b^+(U)$, $(I + K_U^\mu)^{-1} s \ge 0$.*

Proof. (1) $\Rightarrow$ (2): Immediate consequence of Lemma 4.3 and (4.2).

(2) $\Rightarrow$ (3): Recall that, by Proposition 4.1, $(I + K_U^{\mu^+})^{-1}s \geq 0$, for every $s \in \mathcal{S}_b^+(U)$.

(3) $\Rightarrow$ (1): By (4.2), $I - L_U^\mu$ is invertible and

$$(I - L_U^\mu)^{-1} = (I + K_U^\mu)^{-1}(I + K_U^{\mu^+}).$$

Let $s := 1 + K_U^{\mu^+}1$. Then $s \in \mathcal{S}_b^+(U)$, hence

$$g := (I - L_U^\mu)^{-1}1 = (I + K_U^\mu)^{-1}s \geq 0.$$

Thus U is μ-admissible, by Proposition 4.3. □

By the considerations at the end of Section 2, we now obtain the following.

COROLLARY 4.7. *If U is regular and μ-admissible, then U is $(\Delta - \mu)$-regular and*

$$H_U^{\Delta-\mu} = (I + K_U^\mu)^{-1}H_U = (I - L_U^\mu)^{-1}H_U^{\Delta-\mu^+} = \sum_{n=0}^{\infty}(L_U^\mu)^n(I + K_U^{\mu^+})^{-1}H_U.$$

In particular, $H_U^{\Delta-\mu} \geq H_U^{\Delta-\mu^+}$ (whence $H_U^{\Delta-\mu} \geq H_U$, if $\mu \leq 0$) and $H_U^{\Delta-\mu}$ maps $\mathcal{B}_b(\partial U)$ into $\mathcal{H}_b^{\Delta-\mu}(U)$.

REMARK 4.8. For all μ-admissible balls B and $x \in B$,

$$\operatorname{supp}(H_B^{\Delta-\mu}(x,\cdot)) = \partial B. \tag{4.4}$$

Indeed, if $\varphi \in \mathcal{C}^+(\partial B)$ and $\varphi \neq 0$, then $h := H_B\varphi > 0$ on B and hence

$$H_B^{\Delta-\mu}(\varphi) \geq H_B^{\Delta-\mu^+}(\varphi) \geq h \exp(-K_B^{\mu^+}h/h) > 0.$$

PROPOSITION 4.9. *Let ν be a Kato measure on $\mathbb{R}^d$ such that $\mu \leq \nu$, and let U be a regular set. Then the following holds:*

1. *For every $s \in \mathcal{S}_b^+(U)$, $(I + K_U^{\nu^+})^{-1}s \leq (I + K_U^{\mu^+})^{-1}s$.*
2. *$L_U^\nu \leq L_U^\mu$.*
3. *If U is μ-admissible, then U is ν-admissible and, for every $s \in \mathcal{S}_b^+(U)$,*

$$(I + K_U^\nu)^{-1}s \leq (I + K_U^\mu)^{-1}s.$$

In particular,

$$H_U^{\Delta-\nu} \leq H_U^{\Delta-\mu}.$$

Proof. Of course, $\mu^+ \leq \nu^+$, $\nu^- \leq \mu^-$. For every $s \in \mathcal{S}_b^+(U)$, by (3.1),

$$(I + K_U^{\mu^+})^{-1}s - (I + K_U^{\nu^+})^{-1}s = (I + K_U^{\nu^+})^{-1}K^{\nu^+-\mu^+}(I + K_U^{\mu^+})^{-1}s \geq 0.$$

Moreover, for every $f \in \mathcal{B}_b^+(U)$,

$$L_U^\nu f = (I + K_U^{\nu^+})^{-1}K_U^{\nu^-}f \leq (I + K_U^{\mu^+})^{-1}K_U^{\nu^-}f \leq (I + K_U^{\mu^+})^{-1}K_U^{\mu^-}f = L_U^\mu f.$$

By Lemma 4.3 and Corollary 4.7, the proof is finished. □

The following converse generalizes what we know in the case, where ν is the measure μ^+.

COROLLARY 4.10. *Let ν be a Kato measure on $\mathbb{R}^d$ such that $\mu \le \nu$. Let U be a regular set which is ν-admissible, and let L be the positive operator $(I+K_U^\nu)^{-1}K_U^{\nu-\mu}$. Then the following statements are equivalent:*

1. *U is μ-admissible.*
2. *$\sum_{n=0}^\infty L^n 1$ is bounded.*
3. *$I + K_U^\mu$ is invertible and $(I+K_U^\mu)^{-1} = \sum_{n=0}^\infty L^n (I+K^\nu)^{-1}$.*

Proof. Obviously,

$$I + K_U^\mu = (I + K_U^\nu)(I - L), \tag{4.5}$$

Let us suppose that U is μ-admissible. Then the operator $I - L$ is invertible and

$$(I - L)^{-1} = (I + K_U^\mu)^{-1}(I + K_U^\nu).$$

There exists $a > 0$ such that $h := aH_U^{\Delta-\mu}1 \ge 1$ on U. Obviously, $s := h + K_U^\nu h = a + K_U^{\nu-\mu}h \in \mathcal{S}_b^+(U)$. Therefore $g := (I-L)^{-1}h = (I+K_U^\mu)^{-1}s \ge 0$. Thus (2) holds by Lemma 4.3.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) follow immediately from (4.5), Lemma 4.3, and Proposition 4.6. $\square$

DEFINITION 4.11. *Given an open set W in $\mathbb{R}^d$, a lower semicontinuous function $s\colon W \to (-\infty, \infty]$, which is finite on a dense subset of W, is called $(\Delta - \mu)$-*superharmonic*, if, for every μ-admissible regular set V with $\overline{V} \subset W$,*

$$H_V^{\Delta-\mu}s \le s.$$

Let $\mathcal{S}^{\Delta-\mu}(W)$ denote the set of all $(\Delta-\mu)$-superharmonic functions on W.

The following is easily verified.

PROPOSITION 4.12. *For every open set W the following holds.*

1. *$\mathcal{S}^{\Delta-\mu}(W)$ is a convex cone.*
2. *If $s, t \in \mathcal{S}^{\Delta-\mu}(W)$, then $s \wedge t \in \mathcal{S}^{\Delta-\mu}(W)$.*
3. *If (s_n) is an increasing sequence in $\mathcal{S}^{\Delta-\mu}(W)$, and $s := \sup s_n < \infty$ on a dense subset of W, then $s \in \mathcal{S}^{\Delta-\mu}(W)$.*
4. *$\mathcal{S}^{\Delta-\mu}(W) \cap (-\mathcal{S}^{\Delta-\mu}(W)) = \mathcal{H}^{\Delta-\mu}(W)$.*
5. *If $s \in \mathcal{S}^{\Delta-\mu}(W)$ and $\tilde{W}$ is an open set in W, then $s|_{\tilde{W}} \in \mathcal{S}^{\Delta-\mu}(\tilde{W})$.*
6. *If ν is a Kato measure such that $\mu \le \nu$, then*

$$\mathcal{S}_+^{\Delta-\mu}(W) \subset \mathcal{S}_+^{\Delta-\nu}(W) \quad \text{and} \quad -\mathcal{S}_+^{\Delta-\nu}(W) \subset -\mathcal{S}_+^{\Delta-\mu}(W). \tag{4.6}$$

PROPOSITION 4.13. *Let U be a regular set and $f \in \mathcal{B}_b(U)$.*

1. *If $f + K_U^\mu f \in \mathcal{S}(U)$, then $f \in \mathcal{S}^{\Delta-\mu}(U)$.*
2. *If $f \in \mathcal{C}(U)$ and $\Delta f - f\mu \le 0$, then $f \in \mathcal{S}^{\Delta-\mu}(U)$.*

Proof. Let V be regular and μ-admissible, $\overline{V} \subset U$. Then, by (2.6),

$$s := (I + K_V^\mu)(f - H_V^{\Delta-\mu} f) = (f + K_U^\mu f) - H_V(f + K_U^\mu f).$$

If $f + K_U^\mu f \in \mathcal{S}(U)$, then f is lower semicontinuous and $s \in \mathcal{S}_b^+(U)$, hence

$$f - H_V^{\Delta-\mu} f = (I + K_V^\mu)^{-1} s \ge 0.$$

This proves (1), and (2) is a consequence of (1). □

Later on, we shall be able to prove that the converse directions hold as well.

PROPOSITION 4.14. *Let W be an open connected set and $s \in \mathcal{S}^{\Delta-\mu}(W)$, $s \ge 0$, $s \ne 0$. Then $s > 0$.*

Proof. Let $A := \{s = 0\}$. Then A is closed in W. Suppose that there is a point $x \in W \cap \partial A$. Let $r > 0$ such that $B := B(x, r)$ is μ-admissible (which is true, if r is sufficiently small). Since $H_B^{\Delta-\mu} s(x) \le s(x) = 0$ and $\operatorname{supp}(H_B^{\Delta-\mu}(x, \cdot)) = \partial B$, we see that $\partial B \subset A$. So A contains a neighborhood of x, A is open. Since $A \ne W$, we finally conclude that $A = \emptyset$, that is, $s > 0$. □

5 A general minimum principle and first consequences

Let us first establish a general minimum principle (which is of independent interest).

PROPOSITION 5.1. *Let W be a relatively compact open set, $s_0 \in \mathcal{C}(W)$ such that $\inf s_0(W) > 0$, and $s_1 \colon W \to (-\infty, \infty]$ such that s_1 is lower semicontinuous and $\liminf_{x \to z} s_1(x) \ge 0$, for every $z \in \partial W$.*

Assume that there exists a closed half-space H in $\mathbb{R}^d$ such that $0 \in \partial H$ and, for every $x \in W$, there exists a measure σ on W satisfying $\sigma(W \setminus (x + H)) > 0$ and

$$\sigma(s_j) \le s_j(x), \qquad j = 0, 1.$$

Then $s_1 \ge 0$.

Proof. Suppose that *not* $s_1 \ge 0$, and let $u := s_1/s_0$. Then $u \colon W \to (-\infty, \infty]$, u is lower semicontinuous, and $\liminf_{x \to z} u(x) \ge 0$, for every $z \in \partial W$. Hence

$$-\infty < \alpha := \inf u(W) < 0 \quad \text{and} \quad A := \{u = \alpha\}$$

is a non-empty compact set in W. Shifting H we see that there exists $x \in A$ with

$$A \subset x + H.$$

We choose a corresponding measure σ and define

$$\tau := \frac{s_0}{s_0(x)}\,\sigma.$$

Then

$$\tau(1) = \frac{\sigma(s_0)}{s_0(x)} \le 1 \quad \text{and} \quad \tau(u) = \frac{1}{s_0(x)}\sigma(s_1) \le u(x),$$

hence

$$\alpha = u(x) \ge \tau(u) \ge \alpha\tau(1) \ge \alpha.$$

Therefore $\tau(1) = 1$ and τ is supported by A. However, $\tau(W \setminus (x+H)) > 0$, by assumption. Since $A \subset x+H$, we have a contradiction, and the proof is finished. □

Let W be an open set in $\mathbb{R}^d$. For technical reasons we shall now introduce a class of functions which is seemingly larger than $\mathcal{S}^{\Delta-\mu}(W)$, but will turn out to be equal to $\mathcal{S}^{\Delta-\mu}(W)$.

DEFINITION 5.2. *Let $\tilde{\mathcal{S}}^{\Delta-\mu}(W)$ denote the set of all lower semicontinuous function $s\colon W \to (-\infty, \infty]$, which are finite on a dense subset of W, such that, for every $x \in W$, there exist μ-admissible balls B_n satisfying $x \in B_n$, $\overline{B}_n \subset W$,* $\operatorname{diam} B_n < 1/n$, *and*

$$H_{B_n}^{\Delta-\mu} s(x) \le s(x), \qquad n \in \mathbb{N}.$$

These functions satisfy the following minimum principle.

PROPOSITION 5.3. *Let W be relatively compact open such that there exists a function $s_0 \in \mathcal{S}^{\Delta-\mu}(W) \cap \mathcal{C}(W)$ satisfying* $\inf s_0(W) > 0$, *and let $s \in \tilde{\mathcal{S}}^{\Delta-\mu}(W)$ such that* $\liminf_{x\to z} s(x) \ge 0$, *for every $z \in \partial U$. Then $s \ge 0$.*

Proof. Let H be any half-space in $\mathbb{R}^d$ such that $0 \in \partial H$. Given $x \in W$, we may take $\sigma := H_B^{\Delta-\mu}(x,\cdot)$, where B is any μ-admissible ball such that $x \in B$, $\overline{B} \subset W$, and $H_B^{\Delta-\mu} s(x) \le s(x)$. Since $\operatorname{supp}(\sigma) = \partial B$, we know that $\sigma(W \setminus (x+H)) > 0$. So the minimum principle follows from Proposition 5.1. □

The following result is preliminary. For a characterization of $(\Delta-\mu)$-regular sets see Theorem 6.4 and Theorem 6.5.

COROLLARY 5.4. *Let U be a relatively compact open set and let us consider the following properties.*

(i) *U is $(\Delta-\mu)$-regular.*

(ii) *There exists $h \in \mathcal{H}^{\Delta-\mu}(U)$ such that* $\inf h(U) > 0$.

(iii) *There exists $s \in \mathcal{S}^{\Delta-\mu}(U) \cap \mathcal{C}(U)$ such that* $\inf s(U) > 0$.

(iv) *If $s \in \tilde{\mathcal{S}}^{\Delta-\mu}(U)$ and* $\liminf_{x\to z} s(x) \ge 0$, *for every $z \in \partial U$, then $s \ge 0$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. (i) ⇒ (ii). Let $h := H_U^{\Delta-\mu}1$. Then $h > 0$ on U, by Proposition 4.14. Moreover, $\lim_{x\to z} h(x) = 1$, for every $z \in \partial U$. Hence $\inf h(U) > 0$.

(ii) ⇒ (iii). Trivial.

(iii) ⇒ (iv). Proposition 5.3. □

PROPOSITION 5.5. *Let W be an open set in $\mathbb{R}^d$. Then $\tilde{\mathcal{S}}^{\Delta-\mu}(W) = \mathcal{S}^{\Delta-\mu}(W)$. Moreover, for every $s \in \mathcal{S}^{\Delta-\mu}(W)$ and every $(\Delta-\mu)$-regular set U with $\overline{U} \subset W$, $H_U^{\Delta-\mu} s \le s$.*

Proof. Let $v \in \tilde{\mathcal{S}}^{\Delta-\mu}(W)$ and let U be a $(\Delta-\mu)$-regular set, $\overline{U} \subset W$. We consider $f \in \mathcal{C}(W)$, $f \le v$, and define $s := v - H_U^{\Delta-\mu} f$. Then $s \in \tilde{\mathcal{S}}^{\Delta-\mu}(U)$, s is lower semicontinuous on W, and $s = v - f \ge 0$ on ∂U. Hence, by Corollary 5.4, $s \ge 0$, $H_U^{\Delta-\mu} f \le v$. The proof is finished choosing $f_n \in \mathcal{C}(W)$ such that $f_n \uparrow v$. □

COROLLARY 5.6. *The mapping $W \mapsto \mathcal{S}^{\Delta-\mu}(W)$ is a sheaf: If W is the union of open sets W_i, $i \in I$, and $s\colon W \to \overline{\mathbb{R}}$, then $s \in \mathcal{S}^{\Delta-\mu}(W)$ if and only if, for all $i \in I$, $s|_{W_i} \in \mathcal{S}^{\Delta-\mu}(W_i)$.*

Proof. Clearly, the statement holds for $W \mapsto \tilde{\mathcal{S}}^{\Delta-\mu}(W)$. So the result follows immediately from Proposition 5.5. □

COROLLARY 5.7. *Let W be an open set, $s \in \mathcal{S}^{\Delta-\mu}(W)$, and let U be $(\Delta-\mu)$-regular, $\overline{U} \subset W$. Then $H_U^{\Delta-\mu} s \in \mathcal{S}^{\Delta-\mu}(W)$.*

Proof. Let $\tilde{s} := H_U^{\Delta-\mu} s$, and $f_n \in \mathcal{C}(W)$, $f_n \uparrow s$. Then $H_U^{\Delta-\mu} f_n \in \mathcal{C}(W)$ and $H_U^{\Delta-\mu} f_n \uparrow \tilde{s} \le s$. Therefore $\tilde{s}$ is lower semicontinuous.

Let $x \in W$ and let B be a μ-admissible ball such that $x \in B$ and $\overline{B} \subset W$. Then

$$H_B^{\Delta-\mu} \tilde{s}(x) \le H_B^{\Delta-\mu} s(x) \le s(x),$$

where $s(x) = \tilde{s}(x)$, if $x \in W \setminus U$. If $x \in U$ and $\overline{B} \subset U$, then $H_B^{\Delta-\mu} H_U^{\Delta-\mu}(x,\cdot) = H_U^{\Delta-\mu}(x,\cdot)$, and hence

$$H_B^{\Delta-\mu} \tilde{s}(x) = \tilde{s}(x).$$

Thus $\tilde{s} \in \tilde{\mathcal{S}}^{\Delta-\mu}(W)$, by Proposition 5.5. □

PROPOSITION 5.8. *Let W be an open set, $s \in \mathcal{S}^{\Delta-\mu}(W)$ and $t \in -\mathcal{S}^{\Delta-\mu}(W)$ such that $t \le s$ and both s and t are locally bounded.*[2] *Then there exists a function $h \in \mathcal{H}^{\Delta-\mu}(W)$ such that $t \le h \le s$.*

Proof. The proof is the same as in classical potential theory. Let $\mathcal{V} := \{V_m \colon m \in \mathbb{N}\}$ be a base of $(\Delta-\mu)$-regular sets which are relatively compact in W and cover W. Let (U_n) be a sequence in $\mathcal{V}$ containing every V_m, $m \in \mathbb{N}$, infinitely many times. We define

$$s_n := H_{U_n}^{\Delta-\mu} H_{U_{n-1}}^{\Delta-\mu} \cdots H_{U_2}^{\Delta-\mu} H_{U_1}^{\Delta-\mu} s, \qquad n \in \mathbb{N}.$$

By Corollary 5.7, the sequence (s_n) is contained in $\mathcal{S}^{\Delta-\mu}(W)$ and, of course, it is decreasing. Moreover, it follows immediately by induction that $s_n \ge t$, $n \in \mathbb{N}$. Let

$$h := \lim_{n\to\infty} s_n.$$

Then $s \ge h \ge t$. Given $m \in \mathbb{N}$, we know that $s_n = H_{V_m}^{\Delta-\mu} s_{n-1}$ for infinitely many $n \in \mathbb{N}$, and hence $H_{V_m}^{\Delta-\mu} h = h$. Since h is bounded on $\overline{V}_m$, we obtain that h is $(\Delta-\mu)$-harmonic on each V_m, $m \in \mathbb{N}$. Thus $h \in \mathcal{H}^{\Delta-\mu}(W)$. □

[2] It follows from the proof and Corollary 8.4 that the last assumption can be omitted.

6 Characterization of $(\boldsymbol{\Delta}-\boldsymbol{\mu})$-superharmonic functions and $(\boldsymbol{\Delta}-\boldsymbol{\mu})$-regular sets

LEMMA 6.1. *Let U be a regular μ-admissible set and $s \in \mathcal{S}^{\Delta-\mu}(U)$, $s \geq 0$. Then s is $(I+K_U^\mu)^{-1}K_U^{\mu^-}$-dominant.*

Proof (See the proof of Proposition 2.10). Let $L := (I+K_U^\mu)^{-1}K_U^{\mu^-}$ and $f \in \mathcal{B}_b(U)$ such that $s \geq Lf$ on $\{f>0\}$, that is,

$$s + Lf^- \geq Lf^+ \quad \text{on } \{f^+>0\}. \tag{6.1}$$

Then $(I+K_U^\mu)Lf^- = K_U^{\mu^-}f^- \in \mathcal{S}_b^+(U)$, and hence $Lf^- \in \mathcal{S}^{\Delta-\mu}(U)$, by Proposition 4.13. So $t := s + Lf^- \in \mathcal{S}^{\Delta-\mu}(U)$. Let A be a compact set in $\{f>0\}$, $W := U \setminus A$, and $g := L(1_A f)$. If V is any regular set such that $\overline{V} \subset W$, then

$$(I+K_V^\mu)g = (I+K_U^\mu)g - H_V K_U^\mu g,$$

and hence g is $(\Delta-\mu)$-harmonic on V. So g is $(\Delta-\mu)$-harmonic on W. By (6.1), $t \geq g$ on A. Moreover, for every $z \in \partial U$, $\liminf_{x\to z}(t(x)-g(x)) \geq 0$, since $s \geq 0$ and $g \in \mathcal{C}_0(U)$. By Corollary 5.4 and Proposition 5.1, $t-g \geq 0$ on W. Hence $t \geq g$ on U. This clearly implies that $t \geq Lf$ on U. □

THEOREM 6.2. *Let U be a regular set and $f \in \mathcal{B}_b(U)$.*

1. *$f \in \mathcal{S}^{\Delta-\mu}(U)$ if and only if $f + K_U^\mu f \in \mathcal{S}(U)$.*
2. *If f is continuous, then $f \in \mathcal{S}^{\Delta-\mu}(U)$ if and only if $\Delta f - f\mu \leq 0$.*

Proof. Let V be a regular set such that $\overline{V} \subset U$ and $\|K_V^{|\mu|}1\|_\infty < 1/3$. Then V is μ-admissible and $\|(I+K_U^\mu)^{-1}\| < (1-(1/3))^{-1} = 3/2$. Hence

$$S := (I+K_V^\mu)^{-1}K_V^{\mu^+} \quad \text{and} \quad T := (I+K_V^\mu)^{-1}K_V^{\mu^-}$$

are bounded operators on $\mathcal{B}_b(V)$ having a norm which is strictly smaller than $1/2$. By Proposition 4.6, $S \geq 0$ and $T \geq 0$. So we may define

$$L := (I+T)^{-1}S$$

and know that $\|L\| < 1$.

Let us fix $f \in \mathcal{S}^{\Delta-\mu}(U)$ and let

$$v := (I+K_V^\mu)(f - H_V^{\Delta-\mu}f), \quad w := f - H_V^{\Delta-\mu}f.$$

Then $w \in \mathcal{S}^{\Delta-\mu}(V)$, $w \geq 0$. If $g \in \mathcal{B}_b^+(V)$, then, by Proposition 4.13, $Sg \in \mathcal{S}^{\Delta-\mu}(V)$, since $(I+K_V^\mu)Sg = K_V^{\mu^+}g$ is superharmonic on V. If t is any positive function in $\mathcal{S}^{\Delta-\mu}(V)$, then t is T-dominant, by Lemma 6.1, and hence

$$(I+T)^{-1}t \geq 0,$$

by Proposition 3.5. In particular, $(I+T)^{-1}w \geq 0$ and $Lg \geq 0$, for every $g \in \mathcal{B}_b^+(V)$. We have

$$\begin{aligned} I + K_V^\mu &= (I - (I+K_V^\mu)^{-1}K_V^\mu)^{-1} = (I - S + T)^{-1} \\ &= (I-L)^{-1}(I+T)^{-1} = \sum_{n=0}^\infty L^n (I+T)^{-1}. \end{aligned}$$

Thus $v = (I + K_V^\mu)w \geq 0$. Since, by (2.6) (cf. also the proof of Proposition 4.13),

$$v = (f + K_U^\mu f) - H_V(f + K_U^\mu f),$$

we see that $f + K_U^\mu f \in \mathcal{S}(U)$.

If, conversely, $f + K_U^\mu f \in \mathcal{S}(U)$, then $f \in \mathcal{S}^{\Delta-\mu}(U)$, by Proposition 4.13. Again, (2) is a consequence of (1). □

For the characterization of $(\Delta-\mu)$-regular sets we shall need the following result which will also be useful in connection with a discussion of eigenvalues.

PROPOSITION 6.3. *Let U be a regular set. Then K_U^μ is a compact operator on $\mathcal{B}_b(U)$.*

Proof. It clearly suffices to consider the case $\mu \geq 0$. Let $\varepsilon > 0$ and $x \in U$. By Proposition 2.6, there exists an open ball A in U, centered at x, such that $K_U^\mu 1_A \leq \varepsilon$. There exists an open ball $\tilde{A} \subset A$, centered at x, such that, for all harmonic functions g on A satisfying $|g| \leq K_U^\mu 1$ and all $y \in \tilde{A}$,

$$|g(y) - g(x)| < \varepsilon \tag{6.2}$$

(see, for instance, [**?**, Lemma 1.5.6]). Now let $f \in \mathcal{B}_b(U)$, $|f| \leq 1$. Then

$$K_U^\mu f = K_U^\mu(1_{U\setminus A} f) + K_U^\mu(1_A f),$$

where $g := K_U^\mu(1_{U\setminus A} f)$ is harmonic on A, $|g| \leq K_U^\mu 1$, and $|K_U^\mu(1_A f)| \leq K_U^\mu 1_A \leq \varepsilon$. Hence, by (6.2), for all $y \in \tilde{A}$,

$$|K_U^\mu f(y) - K_U^\mu f(x)| \leq |g(y) - g(x)| + 2\varepsilon < 3\varepsilon.$$

Therefore the set $\{K_U^\mu f \colon f \in \mathcal{B}_b(U), |f| \leq 1\}$ is equicontinuous at x. Since $K_U^\mu(\mathcal{B}_b(U))$ is contained in $\mathcal{C}_0(U)$, the claim follows by Arzéla-Ascoli's theorem. □

THEOREM 6.4. *For every regular set U, the following statements are equivalent.*

(i) *U is $(\Delta-\mu)$-regular.*

(ii) *There exists $h \in \mathcal{H}^{\Delta-\mu}(U)$ such that* $\inf h(U) > 0$.

(iii) *There exists $s \in \mathcal{S}^{\Delta-\mu}(U) \cap \mathcal{C}(U)$ such that* $\inf s(U) > 0$.

(iv) *If $s \in \mathcal{S}^{\Delta-\mu}(U)$ and* $\liminf_{x\to z} s(x) \geq 0$*, for every $z \in \partial U$, then $s \geq 0$.*

(v) *U is μ-admissible.*

Proof. By Proposition 5.4, (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).

(iv) ⇒ (v): Let $f \in \mathcal{B}_b(U)$ and $f + K_U^\mu f = 0$ so that $f \in \mathcal{H}^{\Delta-\mu}(U) \cap \mathcal{C}_0(U)$. Hence, by (iv), $\pm f \geq 0$, $f = 0$. So $I + K_U^\mu$ is injective. By Proposition 6.3, K_U^μ is compact. Therefore $I + K_U^\mu$ is bijective.

Now let $s \in \mathcal{S}_b^+(U)$ and $t := (I + K_U^\mu)^{-1} s$. Then $t \in \mathcal{S}^{\Delta-\mu}(U)$, by Proposition 4.13. Since $K_U^\mu t \in \mathcal{C}_0(U)$, we know that, for every $z \in \partial U$,

$$\liminf_{x \to z} t(x) = \liminf_{x \to z} s(x) \geq 0.$$

By (iv), $t \geq 0$. Therefore U is μ-admissible, by Proposition 4.6.

(v) ⇒ (i): Proposition 4.7. □

THEOREM 6.5. *Let U be a relatively compact open set in $\mathbb{R}^d$. Then U is $(\Delta - \mu)$-regular if and only if U is regular and μ-admissible.*

Proof. By Proposition 4.7 and Theorem 6.4, it suffices to show that U is regular if it is $(\Delta - \mu)$-regular. So let us suppose that U is $(\Delta - \mu)$-regular, and let $\varphi \in \mathcal{C}(\partial U)$, $0 \leq \varphi \leq 1$.

a) Let us first consider the case, where $\mu \geq 0$, and let

$$t := H_U^{\Delta-\mu}\varphi, \quad t_0 := H_U^{\Delta-\mu}(1-\varphi), \quad s := 1 - t_0.$$

By Theorem 4.2 (applied to regular sets V with $\overline{V} \subset U$), t, t_0 are subharmonic on U. Therefore $s - t$ is superharmonic on U. Since both s and t tend to φ at ∂U, the (classical) minimum principle implies that $s - t \geq 0$, that is, $t \leq s$. By Proposition 5.8, there exists a harmonic function g on U such that $t \leq g \leq s$. Of course, g tends to φ at ∂U. So U is regular.

b) Let us now consider the general case. Let B be a ball such that $\overline{U} \subset B$. If $a > 0$ is sufficiently large, then $h_0 := a(H_B^{\Delta-\mu^+}1)|_U \geq 1$ on U. Let

$$u := H_U^{\Delta-\mu}\varphi, \quad u_0 = H_U^{\Delta-\mu}(h_0 - \varphi), \quad v := h_0 - u_0.$$

By Corollary 4.7, u, u_0 are $(\Delta - \mu^+)$-superharmonic on U. Therefore $u - v$ is $(\Delta - \mu^+)$-superharmonic on U. Since both u and v tend to φ at ∂U, we obtain, by Proposition 5.3, that $u - v \geq 0$, that is, $v \leq u$. By Proposition 5.8, there exists a $(\Delta - \mu^+)$-harmonic function h such that $u \leq h \leq v$. Of course, h tends to φ at ∂U. Hence U is $(\Delta - \mu^+)$-regular. By (a), U is regular. □

7 ν-eigenvalues of $\Delta - \mu$

Throughout this section let μ be a signed Kato measure, ν a positive Kato measure on $\mathbb{R}^d$, and U a connected regular set.

PROPOSITION 7.1. *Suppose that ν charges every non-empty finely open[3] set in U[4]. Then there exists $\alpha > 0$ such that U is $(\mu + \alpha\nu)$-admissible.*

[3] The fine topology is the coarsest topology τ on $\mathbb{R}^d$ such that every superharmonic function on $\mathbb{R}^d$ is τ-continuous.

[4] This holds, for example, if $\nu \geq V\lambda^d$, $V > 0$ on U.

Proof. By Lemma 2.4, there exists a compact set A in U such that $K_U^{\mu^-} 1_{U\setminus A} \le 1/3$. Let $s := K_U^{\mu^-} 1_A$ and define functions $v_\alpha \in \mathcal{C}_b^+(U)$, $\alpha > 0$, by

$$v_\alpha := (I + K_U^{\mu^+ + \alpha\nu})^{-1} s.$$

We know that the mapping $\alpha \mapsto v_\alpha$ is decreasing (e. g., by Proposition 4.9). Let $v := \inf_{\alpha>0} v_\alpha$. Since

$$v_\alpha + K_U^{\mu^+ + \alpha\nu} v_\alpha = s, \tag{7.1}$$

we obtain that $w := \sup_{\alpha>0} K_U^{\mu^+ + \alpha\nu} v_\alpha \in \mathcal{S}_b^+(U)$ and $v + w = s$. Therefore v is finely continuous. Moreover $\alpha K_U^\nu v \le \alpha K_U^\nu v_\alpha \le s$, for every $\alpha > 0$, and hence $K_U^\nu v = 0$. Thus $v = 0$.

Since the functions v_α are continuous and $\alpha \mapsto v_\alpha$ is decreasing, we conclude that v_α tends to zero locally uniformly on U as $\alpha \to \infty$. So there exists $\alpha > 0$ such that $v_\alpha \le 1/3$ on A. Then, by (7.1), $K_U^{\mu^+ + \alpha\nu} v_\alpha + 1/3 \ge s$ on A, and hence on U, by Proposition 2.10. Therefore $v_\alpha \le 1/3$, and hence

$$(I + K_U^{\mu^+ + \alpha\nu})^{-1} K_U^{\mu^-} 1 \le K_U^{\mu^-} 1_{U\setminus A} + v_\alpha \le 2/3,$$

Thus U is $(\mu + \alpha\nu)$-admissible, by Proposition 4.10. □

Unless stated otherwise we shall assume from now on that U is $\mu+\alpha\nu$-admissible for some $\alpha \in \mathbb{R}$ and that $\nu(U) > 0$.

DEFINITION 7.2. *Let $\Gamma_{\mu,\nu}(U)$ denote the set of all $\alpha \in \mathbb{R}$ such that U is $(\mu+\alpha\nu)$-admissible and*

$$\alpha_0 := \alpha_0(\mu, \nu, U) := \inf \Gamma_{\mu,\nu}(U).$$

For every $\alpha \in \mathbb{R}$, let $\mathcal{E}^\alpha_{\mu,\nu}(U)$ denote the set of all $h \in \mathcal{C}_0(U)$ such that

$$\Delta h - h\mu = \alpha h \nu,$$

that is

$$\mathcal{E}^\alpha_{\mu,\nu}(U) = \mathcal{H}^{\Delta-\mu-\alpha\nu}(U) \cap \mathcal{C}_0(U) = \{h \in \mathcal{B}_b(U) \colon h + K_U^\mu + \alpha K_U^\nu h = 0\}.$$

*α is called a ν-*eigenvalue of $\Delta - \mu$, *if $\mathcal{E}^\alpha_{\mu,\nu}(U) \ne \{0\}$, and then the functions in $\mathcal{E}^\alpha_{\mu,\nu}(U) \setminus \{0\}$ are called ν-eigenfunctions of $\Delta - \mu$.*

Let $E_{\mu,\nu}(U)$ denote the set of ν-eigenvalues of $\Delta - \mu$.

THEOREM 7.3. *There exists $\alpha_0 \in \mathbb{R}$ such that $\Gamma_{\mu,\nu}(U) = (\alpha_0, \infty)$. Moreover, α_0 is the maximal ν-eigenvalue of $\Delta - \mu$.*

Proof. By assumption, $\Gamma_{\mu,\nu}(U) \ne \emptyset$. Let $\alpha \in \Gamma_{\mu,\nu}(U)$. Then $[\alpha, \infty) \subset \Gamma_{\mu,\nu}(U)$ by Proposition 4.9. For every $0 < \varepsilon < \|(I + K_U^\mu + \alpha K_U^\nu)^{-1} K_U^\nu\|^{-1}$,

$$(I + K_U^\mu + (\alpha - \varepsilon) K_U^\mu)^{-1} = \sum_{n=0}^\infty [\varepsilon (I + K_U^\mu + \alpha K_U^\nu)^{-1}]^n (I + K_U^\mu + \alpha K_U^\nu)^{-1},$$

and hence $\alpha - \varepsilon \in \Gamma_{\mu,\nu}(U)$. This proves that $\Gamma_{\mu,\nu}(U) = (\alpha_0, \infty)$, where $\alpha_0 < \infty$.

We show next that $\alpha_0 > -\infty$. Let $\alpha \in \Gamma_{\mu,\nu}(U)$, $L := (I + K_U^\mu + \alpha\nu)^{-1} K_U^\nu$, and let C be a compact set in U such that $\nu(C) > 0$. Then $K_U^\nu 1_C > 0$ on U and hence the $(\Delta - \mu)$-superharmonic function $L1_C$ is strictly positive on U. So there exists $\beta \in (0, \infty)$ such that $\beta L 1_C \geq 1_C$. By induction, $(\beta L)^n 1_C \geq 1_C$ for every $n \in \mathbb{N}$, and hence $\sum_{n=0}^\infty (\beta L)^n 1 = \infty$ on C. Therefore, by Corollary 4.10 (with $\mu + (\alpha - \beta)\nu, \mu$ instead of μ, ν, respectively), $\alpha - \beta \notin \Gamma_{\mu,\nu}(U)$, $\alpha_0 \geq \alpha - \beta$.

Finally, assume that the operator $I + K_U^\mu + \alpha_0 K_U^\nu$ is injective. Then, by Proposition 6.3, $I + K_U^\mu + \alpha_0 K_U^\nu$ is invertible and, for every $s \in \mathcal{S}_b^+(U)$,

$$(I + K_U^\mu + \alpha_0 K_U^\nu)^{-1} s = \lim_{n\to\infty} (I + K_U^\mu + (\alpha_0 + (1/n)) K_U^\nu)^{-1} s \geq 0.$$

So $\alpha_0 \in \Gamma_{\mu,\nu}(U)$, by Proposition 4.6, contradicting the fact that $\Gamma_{\mu,\nu}(U) = (\alpha_0, \infty)$. This finishes the proof. □

COROLLARY 7.4. *Let U be a regular set. Then U is μ-admissible if and only if $I + K_U^\mu + \alpha K_U^\nu$ is injective for every $\alpha \geq 0$.*

THEOREM 7.5. *The set $\mathcal{E}_{\mu,\nu}^\alpha(U)$ is at most countable, it is contained in $(-\infty, \alpha_0]$ and has no finite accumulation point. For each $\alpha \in \mathcal{E}_{\mu,\nu}^\alpha(U)$, the eigenspace $\mathcal{E}_{\mu,\nu}^\alpha(U)$ is finite-dimensional, and eigenfunctions for different eigenvalues are linearly independent.*

Proof. Let $\beta \in \Gamma_{\mu,\nu}(U)$ and let L be the positive operator $(I + K_U^\mu + \beta\nu)^{-1} K_U^\nu$ on $B_b(U)$. It is compact, since K_U^ν is compact, by Proposition 6.3. We note that

$$I + K_U^\mu + \alpha K_U^\nu = (I + K_U^\mu + \beta K_U^\nu)(I + (\alpha - \beta)L).$$

So $I + K_U^\mu + \alpha K_U^\nu$ is invertible (injective) if and only if $I + (\alpha - \beta)L$ is invertible (injective) and, for every $\alpha \neq 0$, α is a ν-eigenvalue of $\Delta - \mu$ if and only if $-1/(\alpha - \beta)$ is an eigenvalue of L. So the result follows from the general theory of Fredholm operators. □

REMARK 7.6. *In fact, it can be shown that the eigenspaces $\mathcal{E}_{\mu,\nu}^\alpha(U)$, $\alpha \in E_{\mu,\nu}(U)$, are pairwise orthogonal in $\mathcal{L}^2(U,\nu)$ and that their (direct) sum is dense in $\mathcal{L}^2(U,\nu)$. In particular, $E_{\mu,\nu}(U)$ is countable (see* [**?**]*).*

By our means, we are able to prove the following (where (1) and (4) could as well be proved using $\mathcal{L}^2$-theory).

THEOREM 7.7. *The following holds.*

1. *For the greatest ν-eigenvalue α_0 of $\Delta - \mu$, the space of all ν-eigenfunctions of $\Delta - \mu$ is one-dimensional and consists of multiples of a strictly positive function h_0.*
2. $\mathcal{S}_+^{\Delta - (\mu + \alpha_0\nu)}(U) = \mathbb{R}^+ h_0$.
3. *If $\alpha < \alpha_0$, then $\mathcal{S}_+^{\Delta - (\mu + \alpha\nu)}(U) = \{0\}$.*
4. *For every $\alpha \in \mathcal{E}_{\mu,\nu}^\alpha(U) \setminus \{\alpha_0\}$, there is no positive ν-eigenfunction of $\Delta - \mu$.*

Proof. Let (γ_n) be a sequence in $\Gamma_{\mu,\nu}(U)$ which is decreasing to α_0. For every $n \in \mathbb{N}$, let

$$g_n := H_U^{\Delta-(\mu+\gamma_n\nu)}1 \quad \text{and} \quad c_n := \|g_n\|_\infty.$$

By Proposition 4.9, for every $n \in \mathbb{N}$, $1 \le g_n \le g_{n+1}$ and

$$g_n + (K_U^\mu + \gamma_n K_U^\nu) g_n = 1. \tag{7.2}$$

If $\sup c_n < \infty$, then $g := \lim_{n\to\infty} g_n$ is bounded and $g + (K_U^\mu + \alpha_0 K_U^\nu) g = 1$. So $g \in \mathcal{H}^{\Delta+\mu+\alpha_0\nu}(U)$ and $\inf g(U) > 0$, by Proposition 4.14. Therefore $\alpha_0 \in \Gamma_{\mu,\nu}(U)$, by Theorem 6.4, a contradiction. Thus $\sup c_n = \infty$.

Since K_U^μ, K_U^ν are compact operators on $(\mathcal{B}_b(U), \|\cdot\|_\infty)$ mapping $\mathcal{B}_b(U)$ into $C_0(U)$, there is a subsequence (h_n) of $(c_n^{-1} g_n)$ such that $(K_U^\mu h_n + \gamma_n K_U^\nu h_n)$ converges uniformly to a function $h_0 \in \mathcal{C}_0^+(U)$. By (7.2), the sequence (h_n) itself converges uniformly to h_0 and $h_0 + K_U^\mu h_0 + \alpha_0 K_U h_0 = 0$, that is, $h_0 \in E_{\mu,\nu}^{\alpha_0}(U)$. Of course, $\|h_0\|_\infty = 1$, since $\|h_n\|_\infty = 1$ for every $n \in \mathbb{N}$. Since $h_0 \ge 0$, we hence see, by Proposition 4.14, that $h_0 > 0$ on U.

Let $\mu_0 := \mu + \alpha_0\nu$, $\alpha \le \alpha_0$, and $s \in \mathcal{S}^{\Delta-\mu-\alpha\nu}(U) \setminus \{0\}$, $s \ge 0$. Then s is $(\Delta-\mu_0)$-superharmonic, by (4.6). Using Lemma 2.4, we may find a compact set $A \ne \emptyset$ in U such that $W := U \setminus A$ is regular and $\|K_U^{|\mu_0|} 1_W\|_\infty < 1$. Then $\|K_W^{|\mu_0|} 1\|_\infty < 1$, and hence W is μ_0-admissible. Let

$$a := \sup\{\alpha \ge 0 \colon \alpha h_0 \le s \text{ on } A\} \quad \text{and} \quad t := s - a h_0.$$

Then $t \in \mathcal{S}^{\Delta-\mu_0}(U)$, $t \ge 0$ on A, and there exists a point $x_0 \in A$ such that $t(x_0) = 0$. Clearly, $\liminf_{y\to z} t(y) \ge 0$ for every $z \in \partial W$ (recall that $h_0 \to 0$ at ∂U). Therefore, by Proposition 5.3, $t \ge 0$ on W. So $t \ge 0$ on U. Since U is connected and $t(x_0) = 0$, we conclude, by Proposition 4.14, that $t = 0$, that is, $s = a h_0$.

Since $-h_0 \in \mathcal{S}^{\Delta-\mu-\alpha\nu}(U)$, by (4.6), we obtain that $s \in \mathcal{H}^{\Delta-\mu-\alpha\nu}(U)$, and hence $s \in E_{\mu,\nu}^\alpha(U) \cap E_{\mu,\nu}^{\alpha_0}(U)$. Thus $s = 0$, if $\alpha < \alpha_0$.

Finally, let $h \in E_{\mu,\nu}^{\alpha_0}$. There exists $b > 0$ such that $g := b h_0 - h \ge 0$ on A. Since $h \in \mathcal{C}_0(U)$ and hence $\tilde{g} \in \mathcal{C}_0(U)$, we obtain, by Proposition 5.3, that $g \ge 0$ on W as well. By the preceding considerations, there exists $c > 0$ such that $b h_0 - h = c h_0$ and therefore $h \in \mathbb{R} h_0$. □

8 Harnack's inequalities

For the moment, let us suppose that $d \ge 3$ and let $x, y, z \in \mathbb{R}^d$. Then, by the triangle inequality, $|x-y| \le |x-z| + |z-y|$, hence $2|x-z| \ge |x-y|$ or $2|z-y| \ge |x-y|$. So $|x-z|^{2-d} \wedge |z-y|^{2-d} \le 2^{d-2}|x-y|^{2-d}$ and therefore

$$G(x,z)G(z,y) \le 2^{d-2} G(x,y)\big(G(x,z) + G(z,y)\big).$$

For every regular set V, we have a corresponding Green function G_V, defined by

$$G_V(\cdot, y) := G(\cdot, y) - H_V G(\cdot, y), \qquad y \in V,$$

and then, for all $f \in \mathcal{B}_b(V)$ and $x \in V$,

$$K_V^\mu f(x) = \int G_V(x,y) f(y)\, d\mu(y).$$

If V is an open ball $B(x_0, r)$, then

$$G_V(x,y) = \begin{cases} \dfrac{1}{(d-2)\omega_d}\left\{1-(1+\Phi(x,y))^{1-d/2}\right\}|x-y|^{2-d}, & \text{if } d \ge 3,\\ \dfrac{1}{4\pi}\ln\bigl\{1+\Phi(x,y)\bigr\}, & \text{if } d = 2,\\ \dfrac{1}{2}\Bigl\{|x-y| + \dfrac{r^2-(x-x_0)(y-x_0)}{r}\Bigr\}, & \text{if } d = 1, \end{cases}$$

where

$$\Phi(x,y) := \frac{(r^2-\|x-x_0\|^2)(r^2-|y-y_0|^2)}{r^2\|x-y\|^2}$$

(see [**?**]). Note that G_V is symmetric, that is, $G_V(x,y) = G_V(y,x)$, for all $x, y \in V$ (this is true as well if V is not a ball).

Given $d \ge 1$, there exists $C_\Delta > 1$, such that, for all open balls V in $\mathbb{R}^d$ and $x, y, z \in V$, the following similar inequality holds

$$G_V(x,z)G_V(z,y) \le C_\Delta G_V(x,y)(G_{V'}(x,z) + G_{V'}(y,z)), \tag{8.1}$$

where V' denotes the open ball which has the same center as V, but double radius. (It is easily seen that V' cannot be replaced by V, but it requires considerable efforts to obtain (8.1).)

PROPOSITION 8.1. *Let ν be a positive Radon measure on $\mathbb{R}^d$ and let V be an open ball in $\mathbb{R}^d$. Then, for every $s \in \mathcal{S}^+(V)$,*

$$K_V^\nu s \le 2C_\Delta \|K_{V'}^\nu 1\|_\infty\, s.$$

Proof. For all $x, y \in V$,

$$\begin{aligned} K_V^\nu G_V(\cdot, y)(x) &= \int G_V(x,z)G_V(z,y)\, d\nu(z) \\ &\le C_\Delta G_V(x,y) \int (G_{V'}(x,z) + G_{V'}(y,z))\, d\nu(z) \\ &= C_\Delta G_V(x,y)(K_{V'}^\nu 1(x) + K_{V'}^\nu 1(y)) \le 2C_\Delta \|K_{V'}^\nu 1\|_\infty G_V(x,y). \end{aligned}$$

Integrating with respect to a Radon measure ρ on V, we obtain that $K_V^\nu(G_V\rho) \le 2C_\Delta\|K_{V'}^\nu 1\|_\infty G_V\rho$. The proof is finished using the fact that every $s \in \mathcal{S}^+(V)$ is the increasing limit of a sequence $(G_V\rho_n)$. □

COROLLARY 8.2. *Let μ be a signed Radon measure on $\mathbb{R}^d$ and V an open ball in $\mathbb{R}^d$ such that $K_{V'}^{|\mu|}1 \le 1/(6C_\Delta)$. Then, for every $s \in \mathcal{S}_b^+(V)$,*

$$\frac{1}{2}\, s \le (I + K_V^\mu)^{-1} s \le \frac{3}{2}\, s. \tag{8.2}$$

In particular,

$$\frac{1}{2}\, H_V \le H_V^{\Delta-\mu} \le \frac{3}{2}\, H_V. \tag{8.3}$$

Proof. By Proposition 8.1, $\|K_V^{|\mu|}\| \le 1/3$ and, for every $s \in \mathcal{S}_b^+(V)$, $K_V^{|\mu|}s \le \frac{1}{3}s$. Therefore

$$(I + K_V^\mu)^{-1}s = s + \sum_{n=0}^\infty (-K_V^\mu)^{n+1}s,$$

where

$$|\sum_{n=0}^\infty (-K_V^\mu)^{n+1}s| \le \sum_{n=1}^\infty (K_V^{|\mu|})^n s \le \sum_{n=1}^\infty \bigl(\frac{1}{3}\bigr)^n s = \frac{1}{2}\, s.$$

This proves (8.2), and (8.3) is an immediate consequence. □

Looking at the Poisson kernel, it is easily verified that there exists $c_0 > 0$ such that, for every ball $V = B(x_0, r)$ in $\mathbb{R}^d$ and all $x, y \in B(x_0, r/2)$,

$$H_V(x, \cdot) \le c_0 H_V(y, \cdot). \tag{8.4}$$

THEOREM 8.3. *For every connected open set W in X and every compact set A in W, there exists $c > 0$ such that, for every positive $h \in \mathcal{H}^{\Delta-\mu}(W)$,*

$$\sup h(A) \le c \inf h(A). \tag{8.5}$$

Proof. Let $h \in \mathcal{H}^{\Delta-\mu}(W)$, $h \ge 0$. Let $x_0 \in W$ and $r > 0$ such that the closure of $V := B(x_0, r)$ is contained in W and $K_{V'}^{|\mu|}1 \le 1/(6C_\Delta)$. By Corollary 8.2 and (8.5), for all $x, y \in B(x_0, r/2)$,

$$h(x) = H_V^{\Delta-\mu}h(x) \le \frac{3}{2}\, H_V h(x) \le \frac{3c_0}{2}\, H_V h(y) \le 3c_0 H_V^{\Delta-\mu}h(y) = 3c_0 h(y).$$

The proof is finished by a standard (and straightforward) covering argument. □

The following strong convergence property is now an immediate consequence of Theorem 8.3 and Corollary 2.9.

COROLLARY 8.4. *Let (h_n) be an increasing sequence of $(\Delta - \mu)$-harmonic functions on a connected open set W in X such that $h := \sup h_n$ is finite at some point. Then h is $(\Delta - \mu)$-harmonic on W.*