

Brownian motion and potential theory

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1 Brownian motion (Wiener measure)

Let Ω denote the set of all (continuous) *paths* (or *trajectories*) $\omega: [0, \infty) \rightarrow \mathbb{R}^d$, $d \geq 1$. The (random) *position at time* $t \geq 0$ will also be denoted by X_t :

$$X_t(\omega) := \omega(t) \quad (t \geq 0, \omega \in \Omega).$$

Let \mathcal{F} denote the σ -Algebra on Ω which is generated by the events

$$\{X_t \in U\} := \{\omega \in \Omega: X_t(\omega) \in U\} \quad (U \text{ open in } \mathbb{R}^d).$$

The *Wiener measure* P is the (unique) measure on (Ω, \mathcal{F}) having the following properties:

- (i) $X_0 = 0$ P -almost surely.
- (ii) For all $0 \leq s < t$, $X_t - X_s$ is Gaussian (normally distributed) with mean 0 and variance $t - s$, i.e.,

$$P(X_t - X_s \in U) = \left(\frac{1}{2\pi(t-s)} \right)^{d/2} \int_U e^{-\frac{|x|^2}{2(t-s)}} dx.$$

- (iii) For all $0 \leq t_0 < t_1 < \dots < t_n$, the increments

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$$

are (probabilistically) independent.

REMARKS 1.1. 1. P is rotationally invariant.

2. For every real $a > 0$, P is invariant under multiplication of vectors in \mathbb{R}^d by a and division of time by a^2 (that is, under the mapping $\omega \mapsto (t \mapsto a\omega(t/a^2))$).

3. For P -almost every $\omega \in \Omega$, the (continuous) path $t \mapsto X_t(\omega)$ is nowhere differentiable (has unbounded variation)!

4. Brownian motion on \mathbb{R}^d (starting at the origin) "is" the sum of d independent (one-dimensional) Brownian motions on the coordinate axes.

Without going into all details, let us briefly discuss *one* way to construct the Wiener measure (there are various entirely different possibilities). For all $t > 0$, $x \in \mathbb{R}^d$, for every Borel measurable subset B of \mathbb{R}^d and Borel measurable function $f \geq 0$ on \mathbb{R}^d , we define

$$g_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right),$$

$$P_t f(x) = f * g_t(x) = \int f(y) g_t(x-y) dy, \quad P_t(x, B) = P_t 1_B(x) = \int_B g_t(x-y) dy.$$

It is easily seen that $P_t f$ is continuous if f is bounded (e.g. using Lebesgue's convergence theorem). Therefore $P_t f = \sup_n P_t(f \wedge n)$ is lower semicontinuous for every Borel measurable function $f \geq 0$.

A straightforward computation shows that $g_s * g_t = g_{s+t}$ and therefore

$$(1.1) \quad P_s P_t = P_{s+t},$$

that is, $(P_t)_{t>0}$ is a *semigroup*. We complete it to a semigroup $(P_t)_{t \geq 0}$ by the identity operator I , that is, we define

$$P_0 f := f.$$

In terms of the measures $P_t(x, \cdot)$, we may write

$$P_t f(x) = \int f(y) P_t(x, dy).$$

Having to deal with iterated integrations of this type, we shall mostly write

$$P_{t_1, \dots, t_n}(x, d(x_1, \dots, x_n))$$

instead of

$$P_{t_1}(x, dx_1) P_{t_2-t_1}(x_1, dx_2) \dots P_{t_n-t_{n-1}}(x_{n-1}, dx_n)$$

to get shorter formulas.

Given a finite set $I = \{t_1, \dots, t_n\}$, where $0 \leq t_1 < \dots < t_n < \infty$, we define a probability measure P_I on $(\mathbb{R}^d)^I$ by

$$P_I(B) := \int_B P_{t_1, \dots, t_n}(0, d(x_1, \dots, x_n)) \quad (B \text{ Borel measurable subset of } (\mathbb{R}^d)^I),$$

that is, for all Borel measurable subsets B_1, \dots, B_n of \mathbb{R}^d ,

$$P_I(B_1 \times \dots \times B_n) = \int_{B_1} \dots \int_{B_n} P_{t_1, \dots, t_n}(0, d(x_1, \dots, x_n))$$

(where of course the first integration is with respect to x_n , the next with respect to x_{n-1} etc.). The semigroup property (1.1) implies that the family (P_I) is *projective*, that is, if I and J are finite subsets of $[0, \infty)$, $I \subset J$, and π_I^J denotes the canonical projection from $(\mathbb{R}^d)^J$ on $(\mathbb{R}^d)^I$, then

$$(1.2) \quad \pi_I^J(P_J) = P_I$$

(it suffices to consider the case where J and I differ by one element, and then (1.2) follows easily from (1.1)). Having (1.2), a general theorem of Kolomogorov implies the existence of a probability measure \tilde{P} on $\tilde{\Omega} := (\mathbb{R}^d)^{[0, \infty)}$ such that

$$\pi_I(\tilde{P}) = P_I$$

for every finite set $I \subset [0, \infty)$, where π_I denotes the canonical projection from $(\mathbb{R}^d)^{[0, \infty)}$ on $(\mathbb{R}^d)^I$.

Let \tilde{E} denote the expectation with respect to \tilde{P} . For every $0 \leq t < \infty$, let $\pi_t := \pi_{\{t\}}$, that is, $\pi_t((x_s)_{s \geq 0}) = x_t$. Then, for all $0 \leq t_1 < \dots < t_n < \infty$ and every Borel measurable function $f \geq 0$ on \mathbb{R}^d ,

$$\tilde{E}(f(\pi_{t_1}, \dots, \pi_{t_n})) = \int \dots \int f(x_1, \dots, x_n) P_{t_1, \dots, t_n}(0, d(x_1, \dots, x_n)).$$

In particular, $\tilde{E}(f(\pi_0)) = P_0 f(0) = f(0)$, that is, $\pi_0 = 0$ P -almost surely. Further, for all $0 < s < t < \infty$ and every Borel measurable function $f \geq 0$ on \mathbb{R}^d ,

$$\begin{aligned} \tilde{E}(f(\pi_t - \pi_s)) &= \int f(x_2 - x_1) P_s(0, dx_1) P_{t-s}(x_1, dx_2) \\ &= P_{t-s} f(0) \int P_s(0, dx_1) = P_{t-s} f(0) = \int f(y) g_{t-s}(y) dy. \end{aligned}$$

More generally, for all $0 \leq t_0 < t_1 < \dots < t_n < \infty$ and Borel measurable positive functions f_1, \dots, f_n on \mathbb{R}^d ,

$$\begin{aligned} \tilde{E}\left(\prod_{j=1}^n f_j(\pi_{t_j} - \pi_{t_{j-1}})\right) &= \int \dots \int \left(\prod_{j=1}^n f_j(x_j - x_{j-1})\right) P_{t_0, \dots, t_n}(0, d(x_0, \dots, x_n)) \\ &= \prod_{j=1}^n P_{t_j - t_{j-1}} f_j(0) = \prod_{j=1}^n \tilde{E}(f_j(\pi_{t_j} - \pi_{t_{j-1}})), \end{aligned}$$

that is, the increments $\pi_{t_1} - \pi_{t_0}, \pi_{t_2} - \pi_{t_1}, \dots, \pi_{t_n} - \pi_{t_{n-1}}$ are independent and all differences $\pi_t - \pi_s$ are normally distributed with mean 0 and variance $t - s$.

The following estimate will be very useful. For all $\eta > 0, t \geq 0$, and $\delta > 0$,

$$\begin{aligned} \tilde{P}(|\pi_{t+\delta} - \pi_t| \geq \eta) &= \tilde{P}(|\pi_\delta| \geq \eta) \leq \frac{2d}{\sqrt{2\pi\delta}} \int_{\eta/\sqrt{d}}^{\infty} \exp\left(-\frac{\xi^2}{2\delta}\right) d\xi \\ &\leq \frac{2d}{\sqrt{2\pi\delta}} \frac{\delta\sqrt{d}}{\eta} \int_{\eta/\sqrt{d}}^{\infty} \frac{\xi}{\delta} \exp\left(-\frac{\xi^2}{2\delta}\right) d\xi = \frac{d}{\sqrt{\pi}} \sqrt{\frac{2d\delta}{\eta^2}} \exp\left(-\frac{\eta^2}{2d\delta}\right). \end{aligned}$$

In particular,

$$(1.3) \quad \lim_{\delta \rightarrow 0} \tilde{P}(|\pi_{t+\delta} - \pi_t| \geq \eta) = 0$$

and, defining $\varepsilon_n := 2^{-n/4}$, we see that, for all $n \in \mathbb{N}$ and $i = 0, 1, 2, \dots$,

$$(1.4) \quad \tilde{P}(|\pi_{(i+1)2^{-n}} - \pi_{i2^{-n}}| \geq \varepsilon_n) \leq \frac{d}{\sqrt{\pi}} \sqrt{\frac{2d}{2^{n/2}}} \exp\left(-\frac{2^{n/2}}{2d}\right) =: \alpha_n.$$

Of course, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, and it is easily seen (root criterion) that

$$(1.5) \quad \sum_{n=1}^{\infty} n2^n \alpha_n < \infty.$$

Let D denote the set of all positive dyadic numbers, that is,

$$D := \bigcup_{n=1}^{\infty} D_n, \quad D_n := \{i2^{-n} : i = 0, 1, 2, \dots\}.$$

The key to a modification of the process which has continuous paths is the following result.

LEMMA 1.2. *There exists an event $\tilde{\Omega}_0$ in $\tilde{\Omega}$ such that $\tilde{P}(\tilde{\Omega}_0) = 1$ and, for every $\tilde{\omega} \in \tilde{\Omega}_0$, the mapping $t \mapsto \pi_t(\tilde{\omega})$ is uniformly continuous on each set $D \cap [0, K]$, $K > 0$.*

Proof. We define a (decreasing) sequence (A_m) of events by

$$A_m := \bigcup_{n=m}^{\infty} \bigcup_{i=0}^{n2^n-1} \{|\pi_{(i+1)2^{-n}} - \pi_{i2^{-n}}| \geq \varepsilon_n\} \quad (m \in \mathbb{N}).$$

By definition of α_n , we obtain the inequalities

$$\tilde{P}(A_m) \leq \sum_{n=m}^{\infty} n2^n \alpha_n.$$

Therefore (1.5) implies that the intersection A of the sequence (A_m) has probability zero.

We define $\tilde{\Omega}_0 := A^c$ and consider $\tilde{\omega} \in \tilde{\Omega}_0$. Then there exists $m \in \mathbb{N}$ such that

$$(1.6) \quad \tilde{\omega} \in \bigcap_{n=m}^{\infty} \bigcap_{i=0}^{n2^n-1} \{|\pi_{(i+1)2^n} - \pi_{i2^n}| < \varepsilon_n\}.$$

We fix $K > 0$, $\varepsilon > 0$, and choose $k \geq K \vee m$ such that

$$\sum_{l=k}^{\infty} \varepsilon_l < \frac{\varepsilon}{2}.$$

We claim that the mapping $\varphi: t \mapsto \pi_t(\tilde{\omega})$ satisfies

$$|\varphi(s) - \varphi(t)| < \varepsilon, \quad \text{whenever } s, t \in D \cap [0, K], \quad |s - t| < 2^{-k}.$$

Indeed, let $s, t \in D \cap [0, K]$ such that $0 < t - s < 2^{-k}$. There exists $\tilde{k} \geq k$ and $r \in [s, t] \cap D_{\tilde{k}}$ such that $r - 2^{-\tilde{k}} < s$ and $r + 2^{-\tilde{k}} > t$. The dyadic representation of $t - r$ yields $a_\nu \in \{0, 1\}$ such that $t - r = \sum_{\nu=0}^{\infty} a_\nu 2^{-\nu}$. Of course, $a_\nu = 0$ for all $\nu \leq \tilde{k}$. We define

$$t_n := r + \sum_{\tilde{k} < \nu \leq \tilde{k}+n} a_\nu 2^{-\nu} \quad (n = 0, 1, 2, \dots).$$

Then $t_0 = r$. Moreover, for every $n \geq 1$, $t_n \in D_{\tilde{k}+n}$, $t_n = t_{n-1}$ or $t_n - t_{n-1} = 2^{-(\tilde{k}+n)}$. Therefore (1.6) implies that

$$|\varphi(t_n) - \varphi(t_{n-1})| < \varepsilon_{\tilde{k}+n} \quad (n \in \mathbb{N}).$$

Since $t \in D$, it is clear that $t_n = t$ for some $n > 0$. So we see that

$$|\varphi(t) - \varphi(r)| \leq \sum_{n=1}^{\infty} |\varphi(t_n) - \varphi(t_{n-1})| < \sum_{l=k}^{\infty} \varepsilon_l < \frac{\varepsilon}{2}.$$

Using the dyadic representation of $r - s$, we obtain the inequality $|\varphi(r) - \varphi(s)| < \varepsilon/2$. Thus $|\varphi(t) - \varphi(s)| < \varepsilon$. \square

Next we define random variables $\tilde{X}_t: \tilde{\Omega} \rightarrow \mathbb{R}^d$, $t \geq 0$, by

$$\tilde{X}_t(\tilde{\omega}) = \begin{cases} \lim_{s \in D, s \rightarrow t} \pi_s(\tilde{\omega}), & \text{if } \tilde{\omega} \in \tilde{\Omega}_0, \\ 0, & \text{if } \tilde{\omega} \in \tilde{\Omega} \setminus \tilde{\Omega}_0. \end{cases}$$

By Lemma 1.2, we know that the limits exists, that the mappings $t \mapsto \tilde{X}_t(\tilde{\omega})$, $\tilde{\omega} \in \tilde{\Omega}$, are continuous, and that

$$\tilde{X}_t = \pi_t \quad \tilde{P}\text{-almost surely,} \quad \text{whenever } t \in D.$$

Let us now consider an arbitrary $t \geq 0$. We choose a sequence (s_n) in D such that $\lim_{n \rightarrow \infty} s_n = t$. Since $\lim_{n \rightarrow \infty} \tilde{X}_{s_n} = \tilde{X}_t$ and $\tilde{X}_{s_n} = \pi_{s_n}$ \tilde{P} -almost surely for every $n \in \mathbb{N}$, we see that, for every $\eta > 0$,

$$\lim_{n \rightarrow \infty} \tilde{P}(|\pi_{s_n} - \tilde{X}_t| \geq \eta) = 0.$$

By (1.3), this implies that

$$(1.7) \quad \tilde{X}_t = \pi_t \quad \tilde{P}\text{-almost surely.}$$

We finally define a (measurable!) mapping $\Phi: \tilde{\Omega} \rightarrow \Omega$ by

$$(\Phi(\tilde{\omega}))(t) = \tilde{X}_t(\tilde{\omega})$$

and take

$$P := \Phi(\tilde{P}).$$

Let E denote the expectation with respect to the probability measure P on (Ω, \mathcal{F}) . By (1.7), we see that, for all $0 \leq t_1 < \dots < t_n < \infty$ and every Borel measurable function $f \geq 0$ on \mathbb{R}^d ,

$$\begin{aligned} E(f(X_{t_1}, \dots, X_{t_n})) &= \tilde{E}(f(\pi_{t_1}, \dots, \pi_{t_n})) \\ &= \int \dots \int f(x_1, \dots, x_n) P_{t_1, \dots, t_n}(0, d(x_1, \dots, x_n)). \end{aligned}$$

Thus P is the Wiener measure (the uniqueness is easily established).

Shifting by $x \in \mathbb{R}^d$ we get a probability measure P^x on (Ω, \mathcal{F}) such that

$$(1.8) \quad E^x(f(X_{t_1}, \dots, X_{t_n})) = \int \dots \int f(x_1, \dots, x_n) P_{t_1, \dots, t_n}(x, d(x_1, \dots, x_n)),$$

where of course E^x denotes the expectation with respect to P^x . In particular, $X_0 = x$ P^x -almost surely.

We shall frequently write $E^x(Y; A)$ instead of $E^x(1_A Y)$ and we shall say that a statement holds “*almost surely*” if it holds P^x -almost surely for every $x \in \mathbb{R}^d$.

2 Weak Markov property

Given $s \geq 0$, we define the *shift operator* $\theta_s: \Omega \rightarrow \Omega$ by $(\theta_s(\omega))(t) = \omega(s+t)$, that is,

$$(2.1) \quad X_t \circ \theta_s = X_{s+t}.$$

For every $s \geq 0$, let

$$\mathcal{F}_s := \sigma(X_t: 0 \leq t \leq s)$$

be the σ -algebra generated by all X_t , $0 \leq t \leq s$. Intuitively, \mathcal{F}_s is the history of Brownian motion up to time s . Of course, $\mathcal{F}_s \subset \mathcal{F}_t$, whenever $0 \leq s \leq t < \infty$.

It follows immediately from (2.1) that every shift θ_s , $0 \leq s < \infty$, is measurable, that is, $\theta_s^{-1}(A) \in \mathcal{F}$ for every $A \in \mathcal{F}$. In fact,

$$(2.2) \quad \theta_s^{-1}(A) \in \mathcal{F}_{s+t} \quad \text{for all } s, t \geq 0 \text{ and } A \in \mathcal{F}_t.$$

The *weak Markov property* expresses the fact that Brownian motion after time s depends only on the position X_s and not on further details of the history up to time s :

PROPOSITION 2.1 (Weak Markov property). *Let $Y: \Omega \rightarrow [0, \infty]$ be \mathcal{F} -measurable and $s \geq 0$. Then the function $x \mapsto E^x(Y)$ is Borel measurable and, for every $x \in \mathbb{R}^d$ and all $A \in \mathcal{F}_s$,*

$$(2.3) \quad E^x(Y \circ \theta_s; A) = E^x(E^{X_s}(Y); A).^1$$

Proof. By standard monotone class theorems, it suffices to consider the case

$$A = \{X_{s_i} \in B_i \text{ for every } 1 \leq i \leq m\}, \quad Y = 1_{\{X_{t_j} \in C_j \text{ for every } 0 \leq j \leq n\}},$$

where $B_1, \dots, B_m, C_0, \dots, C_n$ are Borel subsets of \mathbb{R}^d and

$$0 \leq s_1 < \dots < s_m = s, \quad 0 = t_0 < t_1 < \dots < t_n < \infty.$$

Then, for every $x \in \mathbb{R}^d$,

$$E^x(Y) = \varepsilon_x(C_0) \int_{C_1} \dots \int_{C_n} P_{t_1, \dots, t_n}(x, d(y_1, \dots, y_n))$$

showing that $x \mapsto E^x(Y)$ is Borel measurable. Therefore the function $\omega \mapsto E^{X_s(\omega)}(Y)$ is \mathcal{F} -measurable (even \mathcal{F}_s -measurable). Moreover,

$$\begin{aligned} & E^x(E^{X_s}(Y); A) \\ &= \int_{B_1} \dots \int_{B_m} E^{x_m}(Y) P_{s_1, \dots, s_m}(x, d(x_1, \dots, x_m)) \\ &= \int_{B_1} \dots \int_{B_{m-1}} \int_{B_m \cap C_0} \int_{C_1} \dots \int_{C_n} P_{s_1, \dots, s_m, s+t_1, \dots, s+t_n}(x, d(x_1, \dots, x_m, y_1, \dots, y_n)) \\ &= E^x(Y \circ \theta_s; A), \end{aligned}$$

where the last inequality holds, since $Y \circ \theta_s$ is the characteristic function of the set $\{X_{s+t_j} \in C_j \text{ for every } 0 \leq j \leq n\}$. \square

COROLLARY 2.2. *Let $Y: \Omega \rightarrow [0, \infty]$ be \mathcal{F} -measurable. Then, for every $x \in \mathbb{R}^d$ and for every \mathcal{F}_s -measurable function $Z: \Omega \rightarrow [0, \infty]$,*

$$(2.4) \quad E^x(Z \cdot Y \circ \theta_s) = E^x(Z \cdot E^{X_s}(Y)).$$

Proof. It suffices to recall that every such function Z is the limit of an increasing sequence of functions of the form $\sum_{j=1}^n \alpha_j 1_{A_j}$ with $\alpha_j \geq 0$ and $A_j \in \mathcal{F}_s$. \square

¹Using conditional expectations this can, equivalently, be written as $E^x(Y \circ \theta_s | \mathcal{F}_s) = E^{X_s}(Y)$.

3 Stopping times

A function $S: \Omega \rightarrow [0, \infty]$ is called a *stopping time* if, for every $t > 0$,

$$\{S < t\} \in \mathcal{F}_t.$$

Intuitively, it is a random time such that the event $\{S < t\}$ is determined by the history of Brownian motion before time t . Clearly, every constant time is a stopping time. Moreover, for many subsets A of \mathbb{R}^d , the first hitting time of A , defined by

$$T_A(\omega) := \inf\{t > 0: X_t(\omega) \in A\},$$

is a stopping time (see the next section).

LEMMA 3.1. *For all stopping times S, T, T_n and $t \geq 0$ the following holds:*

1. $S \wedge T$, $S \vee T$, and $S + T$ are stopping times.
2. $\inf_n T_n$, $\sup_n T_n$, $\liminf_{n \rightarrow \infty} T_n$, and $\limsup_{n \rightarrow \infty} T_n$ are stopping times.
3. $t + S \circ \theta_t$ is a stopping time.

Proof. 1. For every $t > 0$,

$$\{S \wedge T < t\} = \{S < t\} \cup \{T < t\} \in \mathcal{F}_t, \quad \{S \vee T < t\} = \{S < t\} \cap \{T < t\} \in \mathcal{F}_t,$$

and

$$\{S + T < t\} = \bigcup_{0 < r < t, r \in \mathbb{Q}} \{S < r\} \cap \{T < t - r\} \in \mathcal{F}_t.$$

2. For every $t > 0$,

$$\{\inf_n T_n < t\} = \bigcup_{n=1}^{\infty} \{T_n < t\} \in \mathcal{F}_t, \quad \{\sup_n T_n < t\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \{T_n < t - \frac{1}{m}\} \in \mathcal{F}_t.$$

So we see that $\inf_n T_n$ and $\sup_n T_n$ are stopping times. This implies that

$$\liminf_{n \rightarrow \infty} T_n = \sup_n \inf_{m \geq n} T_m \quad \text{and} \quad \limsup_{n \rightarrow \infty} T_n = \inf_n \sup_{m \geq n} T_m$$

are also stopping times.

3. If $r \leq t$, then $\{t + S \circ \theta_t < r\} = \emptyset$. If $r > t$, then, by (2.2),

$$\{t + S \circ \theta_t < r\} = \theta_t^{-1}(\{S < r - t\}) \in \mathcal{F}_{r-t+t} = \mathcal{F}_t.$$

□

In the following we shall need the “history” associated with a stopping time S . We define

$$\mathcal{F}_S^+ := \{A \in \mathcal{F}: A \cap \{S < t\} \in \mathcal{F}_t \text{ for every } t > 0\}.$$

It is immediately seen that \mathcal{F}_S^+ is a σ -algebra. Indeed, $\Omega \in \mathcal{F}_S^+$ by definition of a stopping time, and

$$A^c \cap \{S < t\} = (A \cap \{S < t\})^c \cap \{S < t\} \in \mathcal{F}_t,$$

whenever $A \in \mathcal{F}_S^+$ and $t > 0$. Therefore $A \in \mathcal{F}_S^+$ implies that $A^c \in \mathcal{F}_S^+$. Moreover, it follows immediately from the definition that any countable union of events in \mathcal{F}_S^+ is contained in \mathcal{F}_S^+ .

If S is a constant stopping time, that is, if $S \equiv s$, $s \in \mathbb{R}^+$, then, for every $A \in \mathcal{F}$,

$$A \cap \{S < t\} = \begin{cases} \emptyset, & \text{if } t \leq s, \\ A, & \text{if } t > s. \end{cases}$$

Hence, for every $s \geq 0$,

$$\mathcal{F}_s^+ = \bigcap_{t>s} \mathcal{F}_t \supset \mathcal{F}_s,$$

which justifies the superscript “+”. We note that, for every stopping time T and for every $s \geq 0$,

$$(3.1) \quad \{T \leq s\} = \bigcap_{t>s} \{T < t\} \in \mathcal{F}_s^+.$$

PROPOSITION 3.2. *For all stopping times S, T, S_n the following holds:*

1. S is \mathcal{F}_S^+ -measurable.
2. If $S \leq T$, then $\mathcal{F}_S^+ \subset \mathcal{F}_T^+$.
3. $\mathcal{F}_{\inf S_n}^+ = \bigcap_{n=1}^{\infty} \mathcal{F}_{S_n}^+$.
4. $\{S < T\}, \{S = T\}, \{S > T\} \in \mathcal{F}_S^+ \cap \mathcal{F}_T^+$.

Proof. 1. Let $s > 0$. Then, for every $t > 0$,

$$\{S < s\} \cap \{S < t\} = \{S < s \wedge t\} \in \mathcal{F}_{s \wedge t} \subset \mathcal{F}_t,$$

whence $\{S < s\} \in \mathcal{F}_S^+$.

2. Assume $S \leq T$ and let $A \in \mathcal{F}_S^+$. Then, for every $t > 0$,

$$A \cap \{T < t\} = (A \cap \{S < t\}) \cap \{T < t\} \in \mathcal{F}_t.$$

3. Let $S := \inf S_n$. By Proposition 3.1, S is a stopping time. By (2), $\mathcal{F}_S^+ \subset \bigcap_{n=1}^{\infty} \mathcal{F}_{S_n}^+$. Consider now $A \in \bigcap_{n=1}^{\infty} \mathcal{F}_{S_n}^+$. Then $A \in \mathcal{F}_S^+$ since, for every $t > 0$,

$$A \cap \{S < t\} = \bigcup_{n=1}^{\infty} (A \cap \{S_n < t\}) \in \mathcal{F}_t.$$

4. Let $t > 0$ and $t_n := (1 - \frac{1}{n})t$, $n \in \mathbb{N}$. By (1) and (2), all $S \wedge t_n$ and $T \wedge t_n$ are $\mathcal{F}_{t_n}^+$ -measurable and hence \mathcal{F}_t -measurable. Therefore

$$\{T \leq S\} \cap \{S < t\} = \bigcup_{n=1}^{\infty} \{T < t\} \cap \{S < t\} \cap \{T \wedge t_n \leq S \wedge t_n\} \in \mathcal{F}_t,$$

whence $\{T \leq S\} \in \mathcal{F}_S^+$ and $\{S < T\} = \{T \leq S\}^c \in \mathcal{F}_S^+$. Moreover,

$$\{S \leq T\} = \bigcap_{n=1}^{\infty} \{S < T + (1/n)\} \in \mathcal{F}_S^+$$

$\{S > T\} = \{S \leq T\}^c \in \mathcal{F}_S^+$, and $\{S = T\} = \{S \leq T\} \cap \{T \leq S\} \in \mathcal{F}_S^+$. Because of the symmetry of (4), the proof is complete. \square

It will be useful to approximate arbitrary stopping times by stopping times admitting only countably many values. For every $n \in \mathbb{N}$, let

$$D_n := \{i2^{-n} : i = 0, 1, 2, \dots\} \cup \{\infty\}.$$

Given a stopping time S , we define functions $S_n : \Omega \rightarrow D_n$, $n \in \mathbb{N}$, by

$$(3.2) \quad S_n(\omega) := \begin{cases} i2^{-n} & \text{on } \{(i-1)2^{-n} \leq S < i2^{-n}\}, i \in \mathbb{N}, \\ \infty & \text{on } \{S = \infty\}. \end{cases}$$

It is easily seen that

$$S_n \downarrow S \quad \text{as } n \uparrow \infty.$$

If $0 < t \leq 2^{-n}$, then $\{S_n < t\} = \emptyset$. If $2^{-n} < t < \infty$ and i is the largest natural number such that $i2^{-n} < t$, then

$$\{S_n < t\} = \{S < i2^{-n}\} \in \mathcal{F}_{i2^{-n}} \subset \mathcal{F}_t.$$

So every S_n is a stopping time. The sequence (S_n) will be called *canonical approximating sequence* for S .

Given a stopping time S , we define the position X_S and the shift operator θ_S on the subset $\{S < \infty\}$ by

$$X_S(\omega) := X_{S(\omega)}(\omega), \quad (\theta_S(\omega))(t) = \omega(S(\omega) + t).$$

Then, for every $t \geq 0$,

$$X_t \circ \theta_S = X_{S+t} \quad \text{on } \{S < \infty\}.$$

PROPOSITION 3.3. *For every stopping time S , the mapping X_S from $\{S < \infty\}$ to \mathbb{R}^d is \mathcal{F}_S^+ -measurable.*

Proof. Let (S_n) be the approximating sequence for S . Fix $n \in \mathbb{N}$, $t > 0$, and a Borel measurable subset B of \mathbb{R}^d . Let $m \in \mathbb{N}$ be maximal such that $m2^{-n} \leq t$. Then

$$\{X_{S_n} \in B\} \cap \{S_n < t\} = \bigcup_{i=1}^m \{(i-1)2^{-n} \leq S < i2^{-n}, X_{i2^{-n}} \in B\} \in \mathcal{F}_t.$$

Therefore every X_{S_n} , $n \in \mathbb{N}$, is $\mathcal{F}_{S_n}^+$ -measurable on the set $\{S < \infty\}$. By (2) and (3) of Proposition 3.2, we conclude that, for every $n \in \mathbb{N}$, $X_S = \lim_{m \rightarrow \infty} X_{S_m}$ is \mathcal{F}_{S_n} -measurable and therefore the limit X_S is \mathcal{F}_S -measurable on $\{S < \infty\}$. \square

LEMMA 3.4. *Let S, T be stopping times, and $r \geq 0$.*

1. $\theta_S^{-1}(A) \in \mathcal{F}_{S+r}^+$ for all $A \in \mathcal{F}_r$.
2. $S + T \circ \theta_S$ (having the value ∞ on $\{S = \infty\}$) is a stopping time and

$$(3.3) \quad X_T \circ \theta_S = X_{S+T \circ \theta_S} \quad \text{on } \{S + T \circ \theta_S < \infty\}.$$

Proof. 1. For every $0 \leq t \leq r$ and every Borel measurable subset B of \mathbb{R}^d ,

$$\theta_S^{-1}(\{X_t \in B\}) = \{X_{S+t} \in B\} \cap \{S < \infty\} \in \mathcal{F}_{S+t}^+ \subset \mathcal{F}_{S+r}^+.$$

2. For every $t > 0$,

$$\{S + T \circ \theta_S < t\} = \bigcup_{0 < r < t, r \in \mathbb{Q}} \theta_S^{-1}(\{T < r\}) \cap \{S + r < t\} \in \mathcal{F}_t,$$

since $\theta_S^{-1}(\{T < r\}) \in \mathcal{F}_{S+r}^+$ by (1). Therefore $S + T \circ \theta_S$ is a stopping time. The identity (3.3) follows immediately from the definitions. \square

4 Hitting times

Let us now look more closely at hitting times. The *first hitting time* T_A (*first entry time* D_A resp.) for a subset A of \mathbb{R}^d is defined by

$$T_A(\omega) := \inf\{t > 0: X_t(\omega) \in A\}, \quad D_A(\omega) := \inf\{t \geq 0: X_t(\omega) \in A\}.$$

Obviously, $D_A = 0$ P^x -almost surely, if $x \in A$. Moreover,

$$T_A = 0 \quad P^x\text{-almost surely, if } x \in \overset{\circ}{A}.$$

Clearly, $D_A \leq T_A$ and $D_A(\omega) = T_A(\omega)$ if $X_0(\omega) \notin A$. It is easily verified that

$$(4.1) \quad (s + D_A \circ \theta_s)(\omega) = \inf\{t \geq s: X_t(\omega) \in A\} \downarrow T_A(\omega) \quad \text{as } s > 0, s \downarrow 0.$$

Moreover, trivially

$$(4.2) \quad D_{A_n} \downarrow D_A, \quad T_{A_n} \downarrow T_A, \quad \text{whenever } A_n \uparrow A,$$

and

$$(4.3) \quad T_{A_1 \cup A_2} = T_{A_1} \wedge T_{A_2} \quad \text{and} \quad T_{\partial A} \leq T_A \vee T_{A^c},$$

where the inequality follows from the continuity of the paths.

We note that $A_n \downarrow A$ does not necessarily imply that $D_{A_n} \uparrow D_A$ or $T_{A_n} \uparrow T_A$. Indeed, consider the case $d = 1$. Then $A_n := (0, 1/n)$, $n \in \mathbb{N}$, decreases to the empty set (having first entry time ∞), whereas $\lim D_{A_n} = \lim T_{A_n} = T_0 < \infty$ almost surely (see (8.1)).

Even for a Borel measurable B set it is not clear, if T_B and D_B are stopping times. If T_B is a stopping time, then D_B is a stopping time, since $\{D_B < t\} = \{X_0 \in B\} \cup \{T_B < t\}$. Of course, $\{T_B < t\}$ is the union of the sets $\{X_s \in B\}$, $0 < s < t$, each of which is contained in \mathcal{F}_t , but we are dealing with an uncountable union! We shall see that this problem can be overcome for K_σ -sets.

A subset B of \mathbb{R}^d is called K_σ -set, if it is a countable union of compact sets. We note that, of course, all open and all closed subsets of \mathbb{R}^d are K_σ -sets.

PROPOSITION 4.1. *For every K_σ -set B , the first hitting time T_B and the first entry time D_B are stopping times.*

Proof. If U is an open subset of \mathbb{R}^d , then

$$\{T_U < t\} = \bigcup_{0 < r < t, r \in \mathbb{Q}} \{X_r \in U\} \in \mathcal{F}_t,$$

and therefore T_U and D_U are stopping times.

Suppose now that K is a compact subset of \mathbb{R}^d . Let (U_m) be the sequence of (open) $(1/m)$ -neighborhoods of K and fix $s > 0$. Then, for every $m \in \mathbb{N}$,

$$s + D_{U_m} \circ \theta_s \leq s + D_{U_{m+1}} \circ \theta_s \leq s + D_K \circ \theta_s.$$

If $\omega \in \Omega$ such that $t := \sup_m (s + D_{U_m} \circ \theta_s)(\omega) < \infty$, then

$$X_t(\omega) = \lim_{m \rightarrow \infty} X_{s + D_{U_m} \circ \theta_s}(\omega) \in \bigcap_{m=1}^{\infty} \bar{U}_m = K$$

and therefore $t \geq (s + D_K \circ \theta_s)(\omega)$. Thus

$$(4.4) \quad s + D_{U_m} \circ \theta_s \uparrow s + D_K \circ \theta_s \quad \text{as } m \uparrow \infty.$$

Applying Lemma 3.1 several times, we conclude first that $s + D_K \circ \theta_s$ is a stopping time for every $s > 0$ and then that $T_K = \inf((1/n) + D_K \circ \theta_{1/n})$ (and D_K) are stopping times. Recalling (4.2), we finally see that, for every K_σ -set B , the times T_B and D_B are stopping times. \square

LEMMA 4.2. *For every bounded subset A of \mathbb{R}^d , $T_{A^c} < \infty$ almost surely.*

Proof. Fix $x \in \mathbb{R}^d$. For every $\varepsilon > 0$, there exists $t > 0$ such that $P^x(X_t \in A) < \varepsilon$ and hence $P^x(T_{A^c} = \infty) \leq P^x(X_t \in A) < \varepsilon$. \square

Definition 4.3. Given $x \in \mathbb{R}^d$ and $r > 0$, we define

$$B(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}, \quad S(x, r) := \{y \in \mathbb{R}^d : |y - x| = r\}.$$

We denote the uniform distribution of mass 1 on $B(x, r)$ by $\lambda_{x,r}$ and the uniform distribution of mass 1 on $S(x, r)$ by $\sigma_{x,r}$, that is, $\lambda_{x,r}$ is normed volume measure on $B(x, r)$ and $\sigma_{x,r}$ is normed surface measure on $S(x, r)$.

LEMMA 4.4. For all $x \in \mathbb{R}^d$ and $r > 0$, the distribution of $X_{T_{B(x,r)^c}}$ with respect to P^x is $\sigma_{x,r}$.

Proof. We may assume without loss of generality that $x = 0$. By Lemma 4.2, $T_{B(0,r)^c} < \infty$ almost surely. So the statement follows immediately from the invariance of P under rotations. \square

5 Strong Markov property

In the following we shall prove the *strong Markov property* which, essentially, is obtained from the weak Markov property (2.3) replacing the constant time s by a stopping time S . It is a precise formulation for the idea that Brownian motion after the stopping time S does only depend on the position at the (random) time S and not on further details from the past of Brownian motion up to time S (and infinitesimally past S).

A first version of the strong Markov property is gained by the following result.

PROPOSITION 5.1. Let S be a stopping time and $t \geq 0$. Then, for every Borel measurable subset B of \mathbb{R}^d ,

$$(5.1) \quad P^x(X_{S+t} \in B; S < \infty) = E^x(P^{X_S}(X_t \in B); S < \infty).$$

For every Borel measurable function $f \geq 0$ on \mathbb{R}^d ,

$$(5.2) \quad E^x(f(X_{S+t}); S < \infty) = E^x(E^{X_S}(f(X_t)); S < \infty).$$

Proof. Clearly, (5.1) is a special case of (5.2). To prove (5.2) we may assume without loss of generality that f is continuous and bounded. Let (S_n) be the sequence defined by (3.2), fix $n \in \mathbb{N}$, and define

$$A_i := \{S_n = i2^{-n}\}, \quad s_i := i2^{-n} \quad (i \in \mathbb{N}).$$

We know that $A_i \in \mathcal{F}_{s_i}$ for every $i \in \mathbb{N}$. Thus, by the weak Markov property (Proposition 2.1),

$$\begin{aligned} E^x(f(X_{S_n+t}); S < \infty) &= \sum_{i=1}^{\infty} E^x(f(X_{s_i+t}); A_i) \\ &= \sum_{i=1}^{\infty} E^x(E^{X_{s_i}}(f(X_t)); A_i) = E^x(E^{X_{S_n}}(f(X_t)); S < \infty) = E^x(P_t f(X_{S_n}); S < \infty), \end{aligned}$$

where $P_t f$ is continuous. Letting n tend to infinity and using the continuity of the paths, we see that (5.2) holds. \square

COROLLARY 5.2. *Let S be a stopping time and $t \geq 0$. Then, for every Borel measurable function $f \geq 0$ on \mathbb{R}^d and for every function $Z: \{S < \infty\} \rightarrow [0, \infty]$ which is \mathcal{F}_S^+ -measurable,*

$$(5.3) \quad E^x(Z \cdot f(X_{S+t}); S < \infty) = E^x(Z \cdot E^{X_S}(f(X_t))); S < \infty).$$

Proof. It suffices to consider the case $Z = 1_A$, $A \in \mathcal{F}_S^+$. The little trick is to define

$$\tilde{S} := \begin{cases} S & \text{on } A, \\ \infty & \text{on } A^c. \end{cases}$$

Then \tilde{S} is a stopping time since, for every $t > 0$,

$$\{\tilde{S} < t\} = A \cap \{S < t\} \in \mathcal{F}_t.$$

By (5.2),

$$\begin{aligned} E^x(Z \cdot f(X_{S+t}); S < \infty) &= E^x(f(X_{\tilde{S}+t}); \tilde{S} < \infty) \\ &= E^x(E^{X_{\tilde{S}}}(f(X_t)); \tilde{S} < \infty) = E^x(Z \cdot E^{X_S}(f(X_t))); S < \infty. \end{aligned}$$

□

PROPOSITION 5.3. *For every stopping time S and for every \mathcal{F} -measurable $Y: \{S < \infty\} \rightarrow [0, \infty]$,*

$$(5.4) \quad E^x(Y \circ \theta_S; S < \infty) = E^x(E^{X_S}(Y); S < \infty).$$

Proof. Again by monotone class theorems, it suffices to consider the case

$$Y = \prod_{j=1}^n f_j \circ X_{t_j}$$

where f_1, \dots, f_n are positive bounded Borel measurable functions on \mathbb{R}^d and $0 \leq t_1 < \dots < t_n < \infty$. We argue by induction on n . For $n = 1$ (5.4) holds by (5.2). Suppose that the statement holds for some $n \in \mathbb{N}$. Then $S + t_n$ is a stopping time and, by Proposition 3.2,2 and Proposition 3.3, every function $f_j \circ X_{t_j} \circ \theta_S = f_j \circ X_{S+t_j}$, $1 \leq j \leq n$, is $\mathcal{F}_{S+t_n}^+$ -measurable. By Corollary 5.2,

$$\begin{aligned} E^x\left(\prod_{j=1}^{n+1} f_j \circ X_{t_j} \circ \theta_S\right) &= E^x\left(\prod_{j=1}^n f_j \circ X_{S+t_j} \cdot f_{n+1} \circ X_{t_{n+1}-t_n} \circ \theta_{S+t_n}\right) \\ &= E^x\left(\prod_{j=1}^n f_j \circ X_{S+t_j} \cdot E^{X_{S+t_n}}(f_{n+1} \circ X_{t_{n+1}-t_n})\right). \end{aligned}$$

The last term has the form $E^x(\prod_{j=1}^n g_j \circ X_{t_j} \circ \theta_S)$ with $g_j = f_j$ for every $1 \leq j \leq n-1$ and $g_n(y) = f_n(y)E^y(f_{n+1} \circ X_{t_{n+1}-t_n})$. Therefore, by induction hypothesis and by

Corollary 2.2,

$$\begin{aligned} E^x \left(\prod_{j=1}^{n+1} f_j \circ X_{t_j} \circ \theta_S \right) &= E^x \left(E^{X_S} \left(\prod_{j=1}^n g_j \circ X_{t_j} \right) \right) \\ &= E^x \left(E^{X_S} \left[\prod_{j=1}^n f_j \circ X_{t_j} \cdot E^{X_{t_n}} (f_{n+1} \circ X_{t_{n+1}-t_n}) \right] \right) = E^x \left(E^{X_S} \left(\prod_{j=1}^{n+1} f_j \circ X_{t_j} \right) \right). \end{aligned}$$

□

Our most general form of the strong Markov property is the following.

THEOREM 5.4. *Let S be a stopping time and let $Y: \{S < \infty\} \rightarrow [0, \infty]$ be \mathcal{F} -measurable. Then, for every \mathcal{F}_S^+ -measurable function $Z: \{S < \infty\} \rightarrow [0, \infty]$,*

$$(5.5) \quad E^x(Z \cdot Y \circ \theta_S; S < \infty) = E^x(Z \cdot E^{X_S}(Y); S < \infty).^2$$

Proof. See the proof of Corollary 5.2. □

For hitting times we have the following consequence.

COROLLARY 5.5. *Let S be a stopping time and let B be a K_σ -set in \mathbb{R}^d . Then, for every $x \in \mathbb{R}^d$ and for every Borel measurable function $f \geq 0$ on \mathbb{R}^d ,*

$$(5.6) \quad E^x(f \circ X_{T_B}; S < T_B < \infty) = E^x(E^{X_S}(f \circ X_{T_B}; T_B < \infty); S < T_B)$$

Proof. The definition of T_B implies that

$$S + T_B \circ \theta_S = T_B \quad \text{on } \{S < T_B\}.$$

By (3.3), $X_{S+T_B \circ \theta_S} = X_{T_B} \circ \theta_S$ on $\{S + T_B \circ \theta_S < \infty\}$. Therefore

$$f \circ X_{T_B} 1_{\{S < T_B < \infty\}} = (f \circ X_{T_B} 1_{\{T_B < \infty\}}) \circ \theta_S 1_{\{S < T_B\}} \quad \text{on } \{S < \infty\}.$$

By Proposition 3.3, the function $Y := f \circ X_{T_B} 1_{\{T_B < \infty\}}$ is \mathcal{F} -measurable. By Proposition 3.2, the function $Z := 1_{\{S < T_B\}}$ is \mathcal{F}_S^+ -measurable. So the statement follows from the preceding theorem. □

Moreover, we note the following zero–one–law.

PROPOSITION 5.6 (Blumenthal's 0–1–law). *Let $x \in \mathbb{R}^d$. Then $P^x(A) \in \{0, 1\}$ for every $A \in \mathcal{F}_0^+$. In particular, $P^x(T = 0) \in \{0, 1\}$ for every stopping time T .*

Proof. Fix $x \in \mathbb{R}^d$ and $A \in \mathcal{F}_0^+$. Applying (5.5) to $S = 0$ and $Y = Z = 1_A$ we see that

$$P^x(A) = E^x(1_A \cdot 1_A \circ \theta_0) = E^x(1_A \cdot P^{X_0}(A)) = (P^x(A))^2.$$

□

²Using conditional expectations this can be written as $E^x(Y \circ \theta_S | \mathcal{F}_S^+) = E^{X_S}(Y)$.

A first consequence of Blumenthal's 0–1–law is the following result.

COROLLARY 5.7. *Let C be a convex cone and H be a hyperplane in \mathbb{R}^d such that $H \cap \overset{\circ}{C} \neq \emptyset$. Then the convex cone $C' := H \cap C$ satisfies*

$$T_{C'} = 0 \quad P^0\text{-almost surely.}$$

Proof. We may assume without loss of generality that $H = \{x \in \mathbb{R}^d : x_1 = 0\}$. Let $A := \{x \in \mathbb{R}^d : x_1 \geq 0\}$. By Proposition 5.6, $T_A = 0$ P^0 -almost surely or $T_A > 0$ P^0 -almost surely. If $T_A > 0$ P^0 -almost surely, then, by the symmetry of Brownian motion, $T_{A^c} > 0$ P^0 -almost surely. Thus $0 < T_A \wedge T_{A^c} = T_{\mathbb{R}^d} = 0$ P^0 -almost surely, a contradiction. Therefore $T_A = 0$ P^0 -almost surely and, by the symmetry of Brownian motion, $T_{A^c} = 0$ P^0 -almost surely. By (4.3), this implies that $T_H = 0$ P^0 -almost surely.

We may cover H by finitely many cones obtained from C' by rotations within H . If $T_{C'} > 0$ P^0 -almost surely, this would lead to the contradiction $T_H > 0$ P^0 -almost surely. Thus $T_{C'} = 0$ P^0 -almost surely. \square

6 Some basic facts from potential theory

Throughout this section let U denote an open subset of \mathbb{R}^d , $d \geq 1$. A function $h \in \mathcal{C}^2(U)$ is called *harmonic*, if

$$\Delta h := \sum_{j=1}^d \frac{\partial^2 h}{\partial x_j^2} = 0 \quad \text{on } U.$$

The set of all harmonic functions on U is denoted by $\mathcal{H}(U)$. Clearly, every affinely linear function is harmonic. If $d = 2$, the real and imaginary parts of holomorphic functions on U are harmonic. Let us now determine which functions on $\mathbb{R}^d \setminus \{0\}$ that are invariant under rotations are harmonic. To that end we define

$$r(x) := |x| = \sqrt{x_1^2 + \cdots + x_d^2} \quad (x \in \mathbb{R}^d)$$

and consider functions of the form $\varphi \circ r$, where $\varphi \in \mathcal{C}^2((0, \infty))$. For the moment, we fix $1 \leq j \leq d$. Then

$$(6.1) \quad \frac{\partial(\varphi \circ r)}{\partial x_j} = \frac{x_j}{r} \varphi' \circ r = x_j \psi \circ r,$$

where $\psi(t) = \varphi'(t)/t$, $\psi'(t) = \varphi''(t)/t - \varphi'(t)/t^2$. Hence

$$\begin{aligned} \frac{\partial^2(\varphi \circ r)}{\partial x_j^2} &= \psi \circ r + x_j \frac{\partial(\psi \circ r)}{\partial x_j} \\ &= \psi \circ r + x_j \frac{x_j}{r} \psi' \circ r = \frac{\varphi' \circ r}{r} + \frac{x_j^2}{r^2} \varphi'' \circ r - x_j^2 \frac{\varphi' \circ r}{r^3}, \end{aligned}$$

$$\Delta(\varphi \circ r) = \varphi'' \circ r + (d-1) \frac{\varphi' \circ r}{r}.$$

If $d \geq 3$, we conclude that $\varphi \circ r$ is harmonic on $\mathbb{R}^d \setminus \{0\}$ if and only if $\varphi(t) = a + bt^{2-d}$, $a, b \in \mathbb{R}$. Moreover, we get $\varphi(t) = a + b \ln t$ on \mathbb{R}^2 and $\varphi(t) = a + bt$ on \mathbb{R} .

Defining the *Green function* G for \mathbb{R}^d by

$$G(x, y) = \begin{cases} -|x - y|, & \text{if } d = 1, \\ -\ln |x - y|, & \text{if } d = 2, \\ |x - y|^{2-d}, & \text{if } d \geq 3, \end{cases}$$

we see that, for every $y \in \mathbb{R}^d$,

$$(6.2) \quad \Delta G(\cdot, y) = 0 \quad \text{on } \mathbb{R}^d \setminus \{y\}.$$

Moreover, since $G(\cdot, y) \in C^\infty(\mathbb{R}^d \setminus \{y\})$, all partial derivatives

$$\frac{\partial}{\partial x_{j_1}} \frac{\partial}{\partial x_{j_2}} \cdots \frac{\partial}{\partial x_{j_m}} G(\cdot, y)$$

are harmonic on $\mathbb{R}^d \setminus \{y\}$. In particular, using (6.1), we see that

$$(6.3) \quad \frac{x_j}{r^d} \in \mathcal{H}(\mathbb{R}^d \setminus \{0\}) \quad (1 \leq j \leq d, d \geq 1).$$

PROPOSITION 6.1 (Minimum principle). *Let $f: U \rightarrow (-\infty, \infty]$ be lower semicontinuous such that $\liminf_{x \rightarrow z} f(x) \geq 0$ for every $z \in \partial U$ and (if U is unbounded) $\liminf_{|x| \rightarrow \infty} f(x) \geq 0$. Then $f \geq 0$ provided f has one of the following properties:*

- (a) *For every $x \in U$, there exists a radius $r_x > 0$ such that $\overline{B(x, r_x)} \subset U$ and $\int f d\sigma_{x, r_x} \leq f(x)$.*
- (b) *$f \in C^2(U)$ and $\Delta f \leq 0$.*

Proof. Assume the contrary, that is, that $f(x_0) < 0$ for some $x_0 \in U$. Choose $0 < \varepsilon < |f(x_0)|$ and define

$$g := f + \varepsilon.$$

Then $g \geq 0$ outside a compact subset of U and $g(x_0) < 0$. Therefore

$$-\infty < \alpha := \inf g(U) < 0$$

and $K := \{g = \alpha\}$ is a non-empty compact subset of U .

1. Suppose that (a) holds. Consider $x \in K$ with minimal distance to U^c (maximal $|x|$ if $U = \mathbb{R}^d$). Then $\sigma_{x, r_x}(\{g > \alpha\}) > 0$, $\int g d\sigma_{x, r_x} > \alpha = g(x)$, a contradiction.

2. Now suppose (b). Choose an open set V such that \overline{V} is a compact subset of U and $g \geq 0$ on $U \setminus V$. There exists a real $a > 0$ such that $\overline{V} \subset B(0, a)$. Choose $\delta > 0$ such that \tilde{g} defined by

$$(6.4) \quad \tilde{g}(x) := g(x) + \delta \left(a - \sum_{j=1}^d x_j^2 \right) \quad (x \in V)$$

satisfies $\tilde{g}(x_0) < 0$. Clearly,

$$(6.5) \quad \Delta \tilde{g} = \Delta f - 2\delta d < 0.$$

Since $\tilde{g} > g \geq 0$ on ∂V and $\tilde{g}(x_0) < 0$, there exists a point $x \in V$ where \tilde{g} attains its minimum. For every $1 \leq j \leq d$, the function

$$\varphi_j: t \mapsto \tilde{g}(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)$$

has a minimum at x_j and hence $\varphi_j''(x_j) \geq 0$. Therefore

$$\Delta \tilde{g}(x) = \sum_{j=1}^d \varphi_j''(x_j) \geq 0,$$

contradicting (6.5). □

We note that the condition at infinity cannot be dropped: Consider the function $x \mapsto -x_1$ on $\{x \in \mathbb{R}^d: x_1 > 0\}$ or the function $x \mapsto |x|^{2-d} - 1$ on $\{x \in \mathbb{R}^d: |x| > 1\}$, $d \geq 3$.

COROLLARY 6.2. *Let h_1, h_2 be harmonic functions on U such that*

$$\lim_{x \rightarrow z} (h_1 - h_2)(x) = 0 \quad \text{for every } x \in \partial U \cup \{\infty\}.$$

Then $h_1 = h_2$.

Proof. It suffices to apply Proposition 6.1 to the functions $f = \pm(h_1 - h_2)$. □

Given $a \in \mathbb{R}^d$ and $r > 0$, the *Poisson kernel* K for $B(a, r)$ is defined by

$$K(x, z) := r^{d-2} \frac{r^2 - |x - a|^2}{|x - z|^d} \quad (x \in B(a, r), z \in S(a, r)).$$

PROPOSITION 6.3. *Let $a \in \mathbb{R}^d$, $r > 0$, define $B = B(a, r)$, $S = S(a, r)$, $\sigma = \sigma_{a, r}$, and let K be the Poisson kernel for B . Then the following holds:*

1. *For every $z \in S$, $K(\cdot, z) \in \mathcal{H}(B)$ and, for every $\varepsilon > 0$, $\lim_{x \rightarrow z} K(x, \tilde{z}) = 0$ uniformly for $\tilde{z} \in S \setminus B(z, \varepsilon)$.*
2. *For every bounded Borel measurable function f on S , the Poisson integral*

$$I_K f: x \mapsto \int K(x, z) f(z) d\sigma(z)$$

is harmonic on B and $I_K f(a) = \int f d\sigma$.

3. *$I_K 1 = 1$.*
4. *If $z \in S$ and f is a bounded Borel measurable function on S which is continuous at z , then $\lim_{x \rightarrow z} I_K f(x) = f(z)$.*

5. If $f \in \mathcal{C}(S)$, then $I_K f$ is the function $h \in \mathcal{H}(U)$ such that $\lim_{x \rightarrow z} h(x) = f(z)$ for every $z \in S$.

Proof. 1. We may assume without loss of generality that $z = 0$ (we may translate B). Then $|a|^2 = r^2$ and

$$K(x, 0) = r^{d-2}(-|x|^{2-d} + 2 \sum_{j=1}^d a_j \frac{x_j}{|x|^d}) \in \mathcal{H}(\mathbb{R}^d \setminus \{0\})$$

by (6.2) and (6.3). The uniform convergence is easily verified.

2. The continuity of the partial derivatives leads to “ $\Delta f = f \Delta$ ”. This shows that $I_K f$ is harmonic. Moreover, $I_K f(a) = \sigma(f)$ since $K(a, \cdot) = 1$.

3. Fix $0 < \rho < r$. By rotational invariance, $I_K 1$ is constant on $S(a, \rho)$. By minimum principle (Proposition 6.1), it follows that $I_K 1$ is constant on $B(a, \rho)$. Since $I_K 1(a) = 1$, we see that $I_K 1 = 1$ on $B(a, \rho)$.

4. We may assume without loss of generality that $f(z) = 0$ (since we can replace f by $f - f(z)$). Let us fix $\varepsilon > 0$. There exists $\delta > 0$ such that

$$|f(\tilde{z}) - f(z)| < \varepsilon, \quad \text{whenever } \tilde{z} \in S, |\tilde{z} - z| < \delta.$$

Taking $f_1 := 1_{S \cap B(z, \delta)}$ and $f_2 := f - f_1$ we obtain the inequality $|I_K f_1| \leq \varepsilon I_K 1 = \varepsilon$. Moreover, by (1)

$$I_K f_2(x) = \int_{|\tilde{z}-z| \geq \delta} K(x, \tilde{z}) f(\tilde{z}) d\sigma(\tilde{z}) \rightarrow 0 \quad \text{as } x \rightarrow z.$$

5. By (4) and Corollary 6.2. □

COROLLARY 6.4. *For every locally bounded Borel measurable function on U the following statements are equivalent:*

1. f is harmonic.
2. f has the mean value property, that is, $\sigma_{x,r}(f) = f(x)$ for all $x \in U$ and $r > 0$ such that $\overline{B(x, r)} \subset U$.

Proof. (1) \Rightarrow (2): We fix $x \in U$ and $r > 0$ such that $\overline{B(x, r)} \subset U$, and define $B := B(x, r)$, $S := S(x, r)$. As before, let I_K be the Poisson integral for B . By Proposition 6.2, $f|_B = I_K(f|_S)$. In particular, $f(x) = \int f d\sigma_{x,r}$.

(2) \Rightarrow (1): The mean value property implies that $f(x) = \int f d\lambda_{x,r}$, whenever $x \in U$ and $r > 0$ such that $\overline{B(x, r)} \subset U$. Therefore f is continuous by Lebesgue’s convergence theorem. Using the previous notation, we see that $g := f|_B - I_K(f|_S)$ has the mean value property on B . Since $\lim_{x \rightarrow z} g(x) = 0$ for every $z \in S$, our minimum principle implies that $f|_B = I_K(f|_S)$ whence $f|_B \in \mathcal{H}(B)$. □

A Borel measurable function $s: U \rightarrow (-\infty, \infty]$ is called *supermedian* on U , if it is locally lower bounded and

$$\int s d\sigma_{x,r} \leq s(x),$$

whenever $x \in U$ and $r > 0$ such that $\overline{B(x, r)} \subset U$. It is called *hyperharmonic* if it is supermedian and lower semicontinuous. If, in addition, always $\int s d\sigma_{x,r} < \infty$ it is called *superharmonic*.

Obviously, a function h on U is harmonic if and only if h and $-h$ are superharmonic on U . Clearly, every minimum of finitely many superharmonic functions on U is superharmonic. We shall see later on that the functions $G(\cdot, y)$, $y \in \mathbb{R}^d$, are superharmonic on \mathbb{R}^d (see Section 10).

7 Dirichlet problem

As before let U denote an open subset of \mathbb{R}^d . Given a continuous bounded function f on the boundary ∂U of U , the Dirichlet problem consists in finding a continuous extension to the closure \overline{U} of U which is harmonic on U . If U is a ball, the Dirichlet problem is solved by the Poisson integral (see Proposition 6.3). We shall see that, in the general case, the Dirichlet problem can be solved using Brownian motion.

Given a Borel measurable function f on ∂U which is bounded from below, let $H_U f(x)$, $x \in U$, denote the expected value of f at the points where Brownian motion starting at x exits U , that is

$$H_U f(x) := E^x(f(X_{T_{U^c}}); T_{U^c} < \infty).$$

We note that

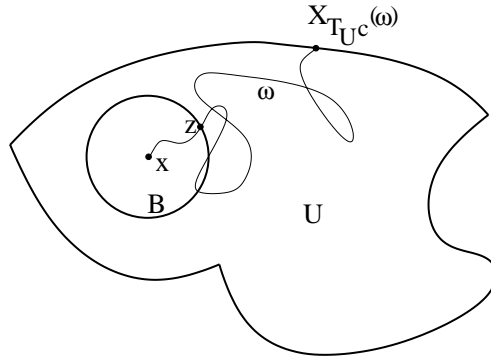
$$H_U 1(x) = P^x(T_{U^c} < \infty)$$

is the probability that Brownian motion, starting at x , ever leaves U .

PROPOSITION 7.1. *For every $f \in \mathcal{C}_b(\partial U)$, the function $H_U f$ is bounded and harmonic on U .*

Proof. Obviously, $|H_U f| \leq \sup_{z \in \partial U} |f(z)| < \infty$. So $H_U f$ is bounded.

We fix $x \in U$ and $r > 0$ such that the closure of the ball $B := B(x, r)$ is contained in U , and let $S := T_{B^c}$, $T := T_{U^c}$. By Lemma 4.2 and Lemma 4.4, $S < \infty$ almost surely and the distribution of X_S with respect to P^x is $\sigma_{x,r}$. If a Brownian path starts at x and hits U^c in finite time, then it has to hit ∂B before. Therefore $S < T$ on $\{T < \infty\}$.



Applying Corollary 5.5, we obtain that

$$\begin{aligned} H_U f(x) &= E^x(f \circ X_T; T < \infty) = E^x(E^{X_s}(f \circ X_T; T < \infty)) \\ &= \int E^z(f \circ X_T; T < \infty) d\sigma_{x,r}(z) = \int H_U f(z) d\sigma_{x,r}(z). \end{aligned}$$

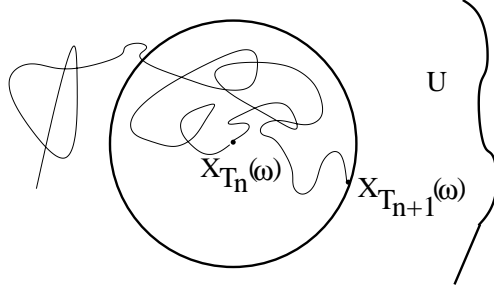
By Proposition 6.4, we conclude that $H_U f$ is harmonic. \square

PROPOSITION 7.2. *Let $s \geq 0$ be lower semicontinuous function on \bar{U} which is hyperharmonic on U . Then $H_U s \leq s$ on U .*

Proof. If $U = \mathbb{R}^d$, then $H_U s = 0 \leq s$. So we may suppose that $U \neq \mathbb{R}^d$. We define $T_n: \Omega \rightarrow [0, \infty]$, $n = 0, 1, 2, \dots$, recursively by $T_0 = 0$,

$$T_{n+1}(\omega) = \inf\{t > T_n(\omega) : |X_t(\omega) - X_{T_n}(\omega)| \geq \frac{1}{2} \text{dist}(X_{T_n}(\omega), \partial U)\},$$

if $T_n(\omega) < \infty$, and $T_{n+1}(\omega) = \infty$ if $T_n(\omega) = \infty$.



Equivalently, $T_{n+1} = T_n + S \circ \theta_{T_n}$, where

$$S(\omega) = \inf\{t > 0 : |X_t(\omega) - X_0(\omega)| \geq \frac{1}{2} \text{dist}(X_0(\omega), \partial U)\}.$$

It is easily checked that S is a stopping time and that, by Lemma 4.2, $T_n < \infty$ almost surely for every $n \in \mathbb{N}$. By Lemma 3.4, (T_n) is an increasing sequence of stopping times. Moreover,

$$(7.1) \quad \lim_{n \rightarrow \infty} T_n = T_{U^c} \quad P^x\text{-almost surely.}$$

Indeed, the definition of the sequence (T_n) trivially implies that $T_n \leq T_{U^c}$ for every $n \in \mathbb{N}$. Fix $\omega \in \Omega$ and assume that $t := \lim_{n \rightarrow \infty} T_n(\omega) < T_{U^c}(\omega)$. Then

$$K := \{X_s(\omega) : 0 \leq s \leq t\}$$

is a compact subset of U and therefore $\varepsilon := \text{dist}(K, U^c) > 0$. There exists $\delta > 0$ such that

$$|X_s(\omega) - X_{s'}(\omega)| < \frac{\varepsilon}{2}, \quad \text{whenever } 0 \leq s \leq s' \leq t, \quad s' - s < \delta.$$

This implies that $T_{n+1}(\omega) - T_n(\omega) \geq \delta$ for every $n \in \mathbb{N}$ and consequently

$$t \geq T_{N+1}(\omega) \geq N\delta$$

for every $N \in \mathbb{N}$, which is impossible. Thus (7.1) holds and, fixing $x \in U$,

$$\lim_{n \rightarrow \infty} X_{T_n} = X_{T_{U^c}} \quad P^x\text{-almost surely on } \{T_{U^c} < \infty\}.$$

Since s is lower semicontinuous on \overline{U} , we therefore conclude that

$$s(X_{T_{U^c}}) \leq \liminf_{n \rightarrow \infty} s(X_{T_n}) \quad P^x\text{-almost surely on } \{T_{U^c} < \infty\}.$$

Using Fatou's lemma and the positivity of s on ∂U we see that

$$H_U s(x) = E^x(s(X_{T_{U^c}}); T_{U^c} < \infty) \leq E^x(\liminf_{n \rightarrow \infty} s(X_{T_n})) \leq \liminf_{n \rightarrow \infty} E^x(s(X_{T_n})).$$

For every $y \in U$, let $r(y) := \text{dist}(y, U^c)/2$. Lemma 4.4 shows that

$$E^y(s(X_S)) = \int s \, d\sigma_{y, r(y)} \leq s(y) \quad \text{for every } y \in U.$$

By the strong Markov property (Proposition 5.3), the preceding inequalities imply that

$$\begin{aligned} E^x(s(X_{T_n})) &= E^x(s(X_S) \circ \theta_{T_{n-1}}) \\ &= E^x(E^{X_{T_{n-1}}}(s(X_S))) \leq E^x(s(X_{T_{n-1}})) \end{aligned}$$

for every $n \in \mathbb{N}$. Combining these estimates, we finally get the estimate

$$H_U s(x) \leq \lim_{n \rightarrow \infty} E^x(s \circ X_{T_n}) \leq E^x(s \circ X_{T_0}) = s(x).$$

□

The following result shows that, at least if U is bounded and the Dirichlet problem admits a solution, the solution can be found using Brownian motion.

COROLLARY 7.3. *Suppose that T_{U^c} is finite almost surely, that $f \in \mathcal{C}_b(\partial U)$, and that the Dirichlet problem for f admits a bounded solution h . Then $H_U f = h$.*

Proof. We may assume without loss of generality that $|h| \leq 1$ and therefore $|f| \leq 1$. By Proposition 7.2,

$$H_U(1 - f) \leq 1 - h \quad \text{and} \quad H_U(f + 1) \leq h + 1,$$

where $H_U 1 = 1$ since, by assumption, $T_{U^c} < \infty$ almost surely. Thus $H_U f = h$. □

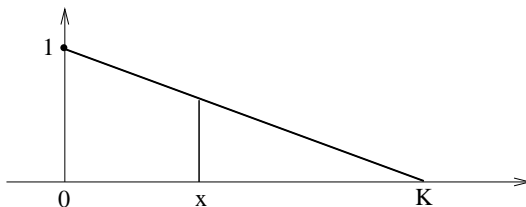
REMARKS 7.4. 1. If U is bounded, then $T_{U^c} < \infty$ almost surely by Lemma 4.2 and the Dirichlet problem for a given function f admits at most one solution.

2. If $d = 3$, $U = \overline{B(0, 1)^c}$, and $f = 1$, then the constant 1 as well as the function $x \mapsto 1/|x|$ are bounded solutions to the Dirichlet problem!

8 A first application: Recurrence/transience

Let us first consider the simple case of the real line, that is, $d = 1$. Obviously, for all $0 < x < K < \infty$,

$$P^x(T_{\{0\}} < T_{\{K\}}) = H_{(0,K)}1_{\{0\}}(x) = \frac{K-x}{K}$$



Letting K tend to ∞ , we see that

$$P^x(T_{\{0\}} < \infty) = 1 \quad (0 < x < \infty).$$

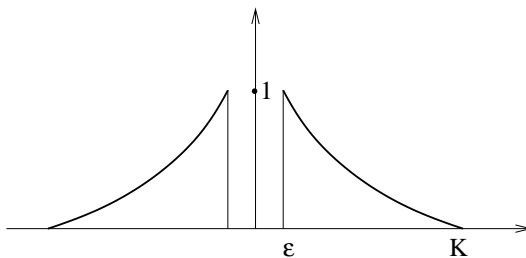
Similarly for $x < 0$, and we already know from Corollary 5.7 that $P^0(T_{\{0\}} = 0) = 1$ (of course, this can be shown in a much simpler way). Thus, for all $y \in \mathbb{R}$,

$$(8.1) \quad T_{\{y\}} < \infty \quad \text{almost surely,}$$

Brownian motion on \mathbb{R} is *recurrent*.

Next let us look at the plane, that is, the case $d = 2$. For all $0 < \varepsilon < |x| < K < \infty$,

$$(8.2) \quad P^x(T_{\overline{B}(0,\varepsilon)} < T_{B(0,K)^c}) = \frac{\ln K - \ln |x|}{\ln K - \ln \varepsilon}.$$



Letting K tend to infinity, we see that, for every $|x| > \varepsilon$,

$$T_{\overline{B}(0,\varepsilon)} < \infty \quad P^x\text{-almost surely,}$$

Brownian motion on \mathbb{R}^2 is *recurrent*.

However, letting first ε tend to zero and then K tend to ∞ , we see that

$$(8.3) \quad T_{\{0\}} = \infty \quad P^x\text{-almost surely.}$$

This does hold not only for every $x \neq 0$, but also for $x = 0$. Indeed, let $T := T_{\{0\}}$, $s > 0$. The weak Markov property (Proposition 2.1) implies that

$$P^0(T \circ \theta_s < \infty) = E^0(P^{X_s}(T < \infty)) = 0,$$

where the last equality follows from (8.3) and the fact that $X_s \neq 0$ P^0 -almost surely. Thus $T \circ \theta_s = \infty$ P^0 -almost surely. To finish the argument it suffices to note that $s + T \circ \theta_s \downarrow T$ as $s \downarrow 0$.

Finally, we assume that $d \geq 3$. Then, for all $0 < \varepsilon < |x| < K < \infty$,

$$(8.4) \quad P^x(T_{\overline{B}(0,\varepsilon)} < T_{B(0,K)^c}) = \frac{|x|^{2-d} - K^{2-d}}{\varepsilon^{2-d} - K^{2-d}}$$

and therefore

$$(8.5) \quad P^x(T_{\overline{B}(0,\varepsilon)} < \infty) = \left(\frac{\varepsilon}{|x|}\right)^{d-2}$$

which tends to zero as $|x|$ tends to ∞ ! Therefore Brownian motion on \mathbb{R}^3 is *transient*.

Moreover, using (8.3) from the 2-dimensional case, we see that for every linear subspace L of dimension at most $d-2$,

$$(8.6) \quad T_L = \infty \quad \text{almost surely.}$$

In particular, for every $x \in \mathbb{R}^d$,

$$(8.7) \quad T_x = \infty \quad \text{almost surely.}$$

(which can also be deduced from (8.5)).

9 Dirichlet problem – continued

Again, let U be an arbitrary open subset of \mathbb{R}^d .

THEOREM 9.1. *For every $z \in \partial U$, the following statements are equivalent:*

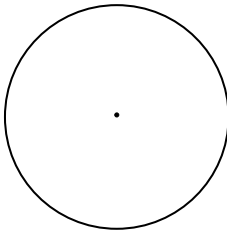
1. z is regular, that is, for every continuous bounded function f on ∂U ,

$$\lim_{x \rightarrow z} H_U f(x) = f(z).$$

2. There exists a barrier for z , that is, a superharmonic function $s > 0$ on some $U \cap B(z, r)$, $r > 0$, which tends to 0 at z .

3. $T_{U^c} = 0$ P^z -almost surely (that is, U^c is not thin at z).

EXAMPLES 1. 1. Zaremba (1911): U punctured unit ball, $U = B(0, 1) \setminus \{0\}$, $d \geq 2$, $z = 0$:

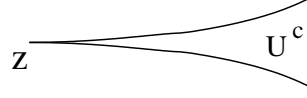


$$T_{\{0\}} = \infty \quad P^0\text{-almost surely}$$

implies that

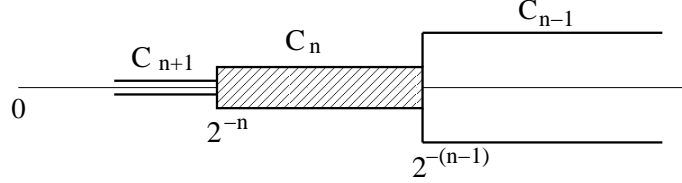
$$T_{U^c} = T_{B(0,1)^c} > 0 \quad P^0\text{-a.s.}$$

2. *Lebesgue spine* (Lebesgue 1913):



Rough construction: Since $T_{\mathbb{R} \times \{0\} \times \{0\}} = \infty$ P^0 -almost surely by (8.6), there exist $r_n > 0$ such that the cylinders

$$C_n := \{(x, y, z) \in \mathbb{R}^3 : 2^{-n} \leq x \leq 2^{-(n-1)}, y^2 + z^2 \leq r_n^2\}$$



satisfy

$$P^0(T_{C_n} < 1) < 2^{-n}.$$

Defining spines

$$S_m := \{0\} \cup \bigcup_{n>m} C_n, \quad m = 0, 1, 2, \dots,$$

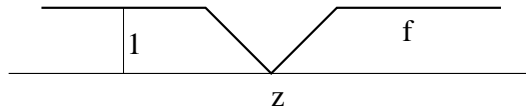
we have $T_{S_0} > 0$ P^0 -almost surely since, for every $m \in \mathbb{N}$,

$$P^0(T_{S_0} = 0) = P^0(T_{S_m} = 0) \leq P^0(T_{S_m} < 1) \leq \sum_{n=m+1}^{\infty} P^0(T_{C_n} < 1) < 2^{-m}.$$

Proof of Theorem 9.1. It suffices to treat the case $d > 1$.

(1) \Rightarrow (2): Define

$$f(\tilde{z}) := 1 \wedge |\tilde{z} - z| \quad (\tilde{z} \in \partial U).$$



Then $H_U f$ is harmonic on U (whence superharmonic) and

$$\lim_{x \rightarrow z} H_U f(x) = f(z) = 0.$$

Since $T_{\{z\}} = \infty$ almost surely, we know that

$$X_{T_{U^c}} \neq z \quad \text{almost surely on } \{T_{U^c} < \infty\}.$$

Therefore

$$\{H_U f > 0\} = \{H_U 1 > 0\}.$$

Since $\lim_{x \rightarrow z} H_U 1(x) = 1$ (again by regularity of z), it is clear that $H_U 1 > 0$ on some $U \cap B(z, r)$, $r > 0$.

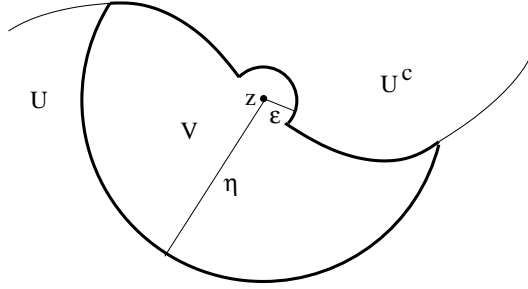
(3) \Rightarrow (1): We may assume without loss of generality that $|f| \leq 1$. We fix $\eta_0 > 0$, choose $0 < \eta \leq \eta_0$ such that $|f - f(z)| < \eta_0$ on $\partial U \cap B(z, \eta)$, and define $S := T_{B(z, \eta)^c}$. Clearly,

$$T_{U^c \setminus B(z, \varepsilon)} \downarrow T_{U^c \setminus \{z\}} = T_{U^c} = 0 \quad P^z\text{-almost surely}$$

as $\varepsilon \downarrow 0$. So there exists $0 < \varepsilon < \eta$ such that $T := T_{U^c \setminus B(z, \varepsilon)}$ satisfies

$$P^z(T < S) > 1 - \eta.$$

Let us consider the open set $V := (U \cap B(z, \eta)) \cup B(z, \varepsilon)$:



Defining $g := 1_{\partial V \cap B(z, \eta)}$ we have

$$(9.1) \quad H_V g(x) = P^x(X_{T_{V^c}} \in \partial V \cap B(z, \eta)) = P^x(T < S), \quad x \in V,$$

and we know that $H_V g$ is harmonic on V . In particular, it is continuous on $B(z, \varepsilon)$! So there exists $0 < \delta < \varepsilon$ such that, for every $x \in B(z, \delta)$,

$$P^x(T < S) = H_V g(x) > 1 - \eta.$$

Let us fix $x \in U \cap B(z, \delta)$. If $\omega \in \Omega$ such that $X_0(\omega) = x$ and $T(\omega) < S(\omega)$, then

$$X_{T_{U^c}}(\omega) \in U^c \cap B(z, \eta)$$

and therefore $|f(X_{T_{U^c}}) - f(z)| < \eta_0$. Thus, using (9.1) and $|f| \leq 1$,

$$|H_U f(x) - f(z)| \leq E^x(|f(X_{T_{U^c}}) - f(z)|; T < S) + 2\eta \leq \eta_0 + 2\eta \leq 3\eta_0.$$

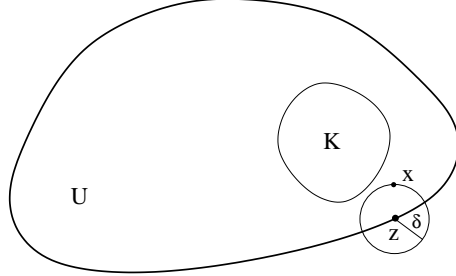
(2) \Rightarrow (3): Let s be a barrier for z . Having shown that $T_{(U \cap B(z, r))^c} = 0$ P^z -a.s. for some $r > 0$, we immediately get that $T_{U^c} = 0$ P^z -a.s. (since $T_{B(z, r)^c} > 0$ P^z -a.s.). Therefore we may assume without loss of generality that s is a strictly positive superharmonic function on all of U .

Assume that $P^z(T_{U^c} > 0) > 0$ (so that, in fact, $P^z(T_{U^c} > 0) = 1$ by Blumenthal's 0–1–law). If (K_n) is a sequence of compact sets such that $K_n \uparrow U$, then $T_{K_n} \wedge T_{U^c} = T_{K_n \cup U^c} \downarrow T_{\mathbb{R}^d} = 0$. So there exists a compact subset K of U such that

$$\alpha := P^z(T_K < T_{U^c}) > 0.$$

Since $\beta := \inf s(K) > 0$ and $\lim_{x \rightarrow z} s(x) = 0$, there exists $\delta > 0$ such that $\overline{B}(z, \delta)$ does not intersect K and

$$(9.2) \quad s < \frac{\alpha\beta}{2} \quad \text{on } U \cap \overline{B}(z, \delta).$$



By continuity of the paths and the strong Markov property,

$$P^x(T_K < T_{U^c}) \geq \alpha \quad \text{for some } x \in U \cap S(z, \delta).$$

Indeed, defining $S := T_{B(z, \delta)^c}$ we clearly have $P^z(S < T_K) = 1$,

$$\{S < T_K\} \cap \{T_K < T_{U^c}\} = \{S < T_K\} \cap \{T_K \circ \theta_S < T_{U^c} \circ \theta_S\} \cap \{S < T_{U^c}\},$$

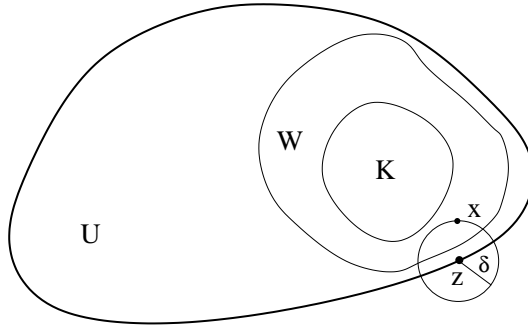
where $\{S < T_{U^c}\} \in \mathcal{F}_S^+$ by Proposition 3.2. Therefore, by Theorem 5.4

$$\alpha = P^z(T_K < T_{U^c}) = E^z(P^{X_S}(T_K < T_{U^c}); S < T_{U^c}).$$

So there exists $\omega \in \{S < T_{U^c}\}$ such that $P^{X_S(\omega)}(T_K < T_{U^c}) \geq \alpha$, where $x := X_S(\omega) \in U \cap S(z, \delta)$!

Moreover, there exists an open set W such that \overline{W} is a compact subset of U , $x \in W$, $K \subset W$, and

$$P^x(T_K < T_{W^c}) > \frac{\alpha}{2}.$$



Then, by Proposition 7.2,

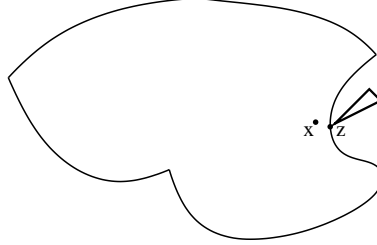
$$s(x) \geq E^x(s(X_{T_{(W \setminus K)^c}})) \geq \beta P^x(X_{T_{(W \setminus K)^c}} \in K) = \beta P^x(T_K < T_{W^c}) > \frac{\alpha\beta}{2}$$

contradicting (9.2).

COROLLARY 9.2. Let U be an open set in \mathbb{R}^d and $z \in \partial U$ such that U satisfies a weak exterior cone condition at z , that is, that there exists a convex cone C , a hyperplane H , and a neighborhood V of z such that $H \cap \overset{\circ}{C} \neq \emptyset$ and

$$V \cap (z + H \cap C) \subset U^c.$$

Then z is regular.



Proof. Corollary 5.7 and Theorem 9.1. □

10 Superharmonic functions, potentials

PROPOSITION 10.1. Let $s \geq 0$ be a hyperharmonic function on U and V, W be open subsets of U such that $\overline{W} \subset V$, $\overline{V} \subset U$. Then $H_V s \leq H_W s \leq s$ on W .

In particular, the functions $r \mapsto \int s d\sigma_{x,r}$, $x \in U$, are decreasing on $(0, \text{dist}(x, U^c))$.

Proof. We know from Proposition 7.2 that $H_V s \leq s$ on V and $H_W s \leq s$ on W . Let $S := T_{W^c}$, $T := T_{V^c}$. If $\omega \in \Omega$ such that $X_0(\omega) \in W$ and $T(\omega) < \infty$, then $S(\omega) < T(\omega)$ by the continuity of Brownian paths. So the strong Markov property (Corollary 5.5) implies that, for every $x \in W$,

$$\begin{aligned} H_V s(x) &= E^x(s \circ X_T; T < \infty) = E^x(E^{X_S}(s \circ X_T; T < \infty); S < \infty) \\ &= E^x((H_V s) \circ X_S; S < \infty) \leq E^x(s \circ X_S; S < \infty) = H_W s(x). \end{aligned}$$

□

PROPOSITION 10.2. Let U be connected and $s \geq 0$ be hyperharmonic on U . Then either $s > 0$ or s is identically zero.

Proof. Since s is lower semicontinuous, the set $\{s = 0\}$ is closed. If $x \in U$ and $r > 0$ such that $s(x) = 0$ and $\overline{B(x, r)} \subset U$, then $\int s d\lambda_{x,r} \leq s(x) = 0$, hence $s = 0$ $\lambda_{x,r}$ -almost everywhere on $B(x, r)$ and, in fact, $s = 0$ on $B(x, r)$, since s is lower semicontinuous. Thus the set $\{s = 0\}$ is open as well. Since U is connected, we conclude that $\{s = 0\} = \emptyset$ or $\{s = 0\} = U$. □

LEMMA 10.3. For every $y \in \mathbb{R}^d$, $G(\cdot, y)$ is superharmonic.

Proof. Let us fix $x, y \in \mathbb{R}^d$ and $r > 0$. Of course, $G(\cdot, y)$ is lower semicontinuous. So we have to show that

$$(10.1) \quad \int G(\cdot, y) d\sigma_{x,r} \leq G(x, y) \quad \text{and} \quad \int G(\cdot, y) d\sigma_{x,r} < \infty.$$

- a) If $r < |x - y|$, then $\int G(\cdot, y) d\sigma_{x,r} = G(x, y) < \infty$.
b) If $r = |x - y| > 0$, then, for every $n \in \mathbb{N}$,

$$\int G(\cdot, y) \wedge n d\sigma_{x,r} = \lim_{\rho \uparrow r} \int G(\cdot, y) \wedge n d\sigma_{x,\rho} \leq \liminf_{\rho \uparrow r} \int G(\cdot, y) d\sigma_{x,\rho} \leq G(x, y)$$

whence (10.1) holds.

- c) If $x = y$, then (10.1) holds trivially.

d) Let us now consider the (only interesting) case where $0 < |x - y| < r$. Let h be the Poisson integral for $f := G(\cdot, y)|_{B(x,r)}$ and $s := f - h$. Then $\lim_{x \rightarrow z} s(x) = 0$ for every $z \in S(x, r)$. By (a) and (c), we see that, for every $\xi \in B(x, r)$, there exists $\rho > 0$ such that $\overline{B(\xi, \rho)} \subset B(x, r)$ and $\int f d\sigma_{\xi, \rho} \leq f(\xi)$. Therefore our minimum principle (Proposition 6.1) implies that

$$\int G(\cdot, y) d\sigma_{x,r} = h(x) \leq f(x) = G(x, y) < \infty.$$

□

From now on let us suppose that $d \geq 3$. For every measure μ on \mathbb{R}^d , we define

$$G^\mu(x) := \int G(\cdot, y) d\mu(y) = \int \frac{d\mu(y)}{|x - y|^{d-2}} \quad (x \in \mathbb{R}^d).$$

G^μ is called *potential of μ* if $G^\mu \not\equiv \infty$.

- PROPOSITION 10.4.** 1. If $v \geq 0$ is hyperharmonic on \mathbb{R}^d , then $v \equiv \infty$ or v is superharmonic and $v < \infty$ λ -almost everywhere.
2. If μ is a measure on \mathbb{R}^d such that $G^\mu \not\equiv \infty$, then G^μ is superharmonic on \mathbb{R}^d and harmonic on $\mathbb{R}^d \setminus \text{supp}(\mu)$.

Proof. 1. Let $v \geq 0$ be hyperharmonic on \mathbb{R}^d and $v(x_0) < \infty$ for some $x_0 \in \mathbb{R}^d$. Then, for every $R > 0$,

$$\int v d\lambda_{x_0, R} \leq v(x_0) < \infty,$$

whence $\lambda(\{v = \infty\} \cap B(x_0, R)) = 0$. Therefore $v < \infty$ λ -almost everywhere.

We next fix $x \in \mathbb{R}^d$, $r > 0$, and define $R := |x - x_0| + 2r$. Then, for every $\tilde{x} \in B(x, r)$,

$$\int v d\lambda_{\tilde{x}, r} \leq \frac{R^d}{r^d} \int v d\lambda_{x_0, R} =: a < \infty.$$

By Proposition 10.1, $\int v d\sigma_{\tilde{x}, r} \leq a$. Thus v is superharmonic.

2. Let us now assume that $G^\mu(x_0) < \infty$. By Fatou's lemma, G^μ is lower semicontinuous. By Fubini's theorem, it is supermedian on \mathbb{R}^d and median on $\mathbb{R}^d \setminus \text{supp}(\mu)$. In particular, G^μ is hyperharmonic and we see from (1) that G^μ is superharmonic.

If $x \notin \text{supp}(\mu)$ and $r < \text{dist}(x, \text{supp}(\mu))/2$, then $G^\mu(\tilde{x}) = \int G^\mu d\sigma_{\tilde{x}, r} \leq a$, where a is defined as in (1). Applying Proposition 6.4, we finally conclude that G^μ is harmonic on $\mathbb{R}^d \setminus \text{supp}(\mu)$. □

LEMMA 10.5. *Let $f \geq 0$ be a bounded Borel measurable function on \mathbb{R}^d with compact support. Then $G^{f\lambda} \in \mathcal{C}_0(\mathbb{R}^d)$, that is, $G^{f\lambda}$ is a continuous potential vanishing at infinity.*

There exists a strictly positive function $f_0 \in \mathcal{C}_0(\mathbb{R}^d)$ such that $v_0 := G^{f_0\lambda} \in \mathcal{C}_0(\mathbb{R}^d)$.

Proof. There exists $M > 0$ and a ball $B = B(0, R)$ such that $f \leq M$ and $f = 0$ on B^c . Since

$$G^{f\lambda} + G^{(M1_B - f)\lambda} = MG^{1_B\lambda},$$

where both $G^{f\lambda}$ and $G^{(M1_B - f)\lambda}$ are lower semicontinuous, it suffices to show that $G^{1_B\lambda} \in \mathcal{C}_0(\mathbb{R}^d)$. If $|x| > 2R$, then $\text{dist}(x, B) > |x|/2$ and therefore $G^{1_B\lambda}(x) \leq (2/|x|)^{d-2}\lambda(B)$. So $G^{1_B\lambda}$ tends to zero at infinity. Since $x \mapsto |x|^{2-d}$ is locally integrable, we easily see that $G^{1_B\lambda}$ is bounded and that

$$\int_{B(0, \varepsilon)} |x|^{2-d} dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Moreover, the decomposition

$$G^{1_B\lambda} = G^{1_{B \setminus B(x, \varepsilon)}\lambda} + G^{1_{B \cap B(x, \varepsilon)}\lambda},$$

where the first term is harmonic and hence continuous on $B(x, \varepsilon)$, proves the continuity of $G^{1_B\lambda}$.

To finish the proof it suffices to choose a sequence (f_n) in $\mathcal{C}_0(\mathbb{R}^d)$ such that, for every $n \in \mathbb{N}$,

$$f_n > 0 \text{ on } B(0, n), \quad f_n \leq 2^{-n}, \quad G^{f_n\lambda} \leq 2^{-n},$$

and to take $f_0 = \sum_{n=1}^{\infty} f_n$. □

We shall finish this section proving that there is a close relationship between G and the Brownian semigroup $(P_t)_{t>0}$. The *potential kernel* of $(P_t)_{t>0}$ is defined by

$$Vf(x) := \int_0^{\infty} P_s f(x) ds = E^x \left(\int_0^{\infty} f(X_s) ds \right).$$

We note that, for every Borel measurable subset B of \mathbb{R}^d ,

$$V1_B(x) = E^x \left(\int_0^{\infty} 1_{\{X_s \in B\}} ds \right)$$

is the expected total time which Brownian motion, starting at x , spends in B .

PROPOSITION 10.6. *For every Borel measurable function $f \geq 0$ on \mathbb{R}^d ,*

$$Vf = \frac{\Gamma(\frac{d}{2} - 1)}{2\pi^{d/2}} G^{f\lambda}.$$

Proof. The substitution $s = |x|^2/(2t)$ shows that

$$\int_0^{\infty} (2\pi t)^{-d/2} e^{-\frac{|x|^2}{2t}} dt = \frac{|x|^{2-d}}{2\pi^{d/2}} \int_0^{\infty} s^{(d/2)-2} e^{-s} ds = \frac{\Gamma(\frac{d}{2} - 1)}{2\pi^{d/2}} |x|^{2-d}.$$

An application of Fubini's theorem finishes the proof. □

11 Optional sampling

As already in the last part of the preceding section we shall assume that $d \geq 3$ (localizing our considerations, the 2-dimensional case could be included as well).

For stopping times T , Borel measurable functions $f \geq 0$ on \mathbb{R}^d , and $x \in \mathbb{R}^d$, we define

$$(11.1) \quad P_T f(x) := E^x(f \circ X_T) := E^x(f \circ X_T; T < \infty)$$

that is, the measure $P_T(x, \cdot)$ is the distribution of X_T (the position at time T) with respect to P^x .

LEMMA 11.1. *Let T be a stopping time, $x \in \mathbb{R}^d$, and $f \geq 0$ be a Borel measurable function on \mathbb{R}^d . Then*

$$P_T V f(x) = E^x \left(\int_T^\infty f(X_s) ds \right).$$

Proof. By the strong Markov property (Proposition 5.3),

$$\begin{aligned} P_T V f(x) &= E^x(V f(X_T)) = E^x \left(E^{X_T} \left(\int_0^\infty f(X_s) ds \right) \right) \\ &= E^x \left(\left(\int_0^\infty f(X_s) ds \right) \circ \theta_T \right) = E^x \left(\int_0^\infty f(X_{T+s}) ds \right) = E^x \left(\int_T^\infty f(X_s) ds \right). \end{aligned}$$

□

Using Proposition 10.6 and the continuous superharmonic function $v_0 = G^{f_0 \lambda}$ from Lemma 10.5, we get the following result.

COROLLARY 11.2. *Let S and T be stopping times such that $S \leq T$. Then, for every $x \in \mathbb{R}^d$,*

$$S = T \text{ } P^x\text{-almost surely} \quad \text{if and only if} \quad P_S v_0(x) = P_T v_0(x).$$

LEMMA 11.3. *Let $v \geq 0$ be a hyperharmonic on \mathbb{R}^d . Then, for every $t > 0$, the function $P_t v$ is hyperharmonic on \mathbb{R}^d , and $P_t v \uparrow v$ as $t \downarrow 0$.*

Proof. It clearly suffices to consider the case $v > 0$. For the moment, let us fix $t > 0$. By Fubini's theorem, the function $P_t v$ is supermedian. Since it is lower semicontinuous, we conclude that it is hyperharmonic.

Moreover, since v is supermedian and $P_t(x, \cdot) = \int \varphi(r) \sigma_{x,r} dr$ for some function φ , we see that

$$P_t v(x) = \int_0^\infty \sigma_{x,r}(v) \varphi(r) dr \leq \int_0^\infty v(x) \varphi(r) dr = v(x).$$

The inequality $P_t v \leq v$ implies that, for all $s \geq 0$,

$$P_{s+t} v = P_s P_t v \leq P_s v,$$

whence $t \mapsto P_t v$ is decreasing. If $0 \leq a < v(x)$, then there exists a ball $B := B(x, r)$ such that $a < v$ on B . Given $\varepsilon > 0$, there exists $t > 0$ such that $P_t(x, B) > 1 - \varepsilon$ and therefore

$$P_t v(x) > a P_t(x, B) > (1 - \varepsilon)a.$$

This shows that $\sup_{t>0} P_t v(x) \geq v(x)$. \square

COROLLARY 11.4. *Every superharmonic function $v \geq 0$ on \mathbb{R}^d is the limit of an increasing sequence of positive superharmonic functions on \mathbb{R}^d .*

Proof. It follows from Lemma 11.3 that every function $v_n := P_{1/n}(v \wedge n)$ is superharmonic and $v_n \uparrow v$ as $n \uparrow \infty$. \square

The next result is our last preparation for the optional sampling theorem.

LEMMA 11.5. *Let $v \geq 0$ be hyperharmonic on \mathbb{R}^d , $x \in \mathbb{R}^d$, $0 < s < t < \infty$, and $A \in \mathcal{F}_s^+$. Then $E^x(v(X_t); A) \leq E^x(v(X_s); A)$.*

Proof. By Corollary 5.2 and by Lemma 11.3,

$$\begin{aligned} E^x(v(X_t); A) &= E^x(v(X_{t-s}) \circ \theta_s; A) \\ &= E^x(E^{X_s}(v(X_{t-s})); A) = E^x(P_{t-s}v(X_s); A) \leq E^x(v(X_s); A). \end{aligned}$$

\square

THEOREM 11.6 (Optional sampling). *Let $v \geq 0$ be a superharmonic function on \mathbb{R}^d and S, T stopping times such that $S \leq T$. Then $P_T v \leq P_S v \leq v$.*

Proof. By Corollary 11.4, we may assume without loss of generality that v is bounded and continuous. Let (S_n) and (T_n) be the canonical approximating sequences for S and T (see the end of Section 3). Clearly, $S_n \leq T_n$ for every $n \in \mathbb{N}$. By continuity of Brownian paths and the trivial inclusion $\{T < \infty\} \subset \{S < \infty\}$, it is sufficient to show that, for all $m, n \in \mathbb{N}$,

$$(11.2) \quad E^x(v(X_{T_n}); T_n \leq m) \leq E^x(v(X_{S_n}); T_n \leq m).$$

So we fix $m, n \in \mathbb{N}$. Let

$$(11.3) \quad R_i := (S_n + i2^{-n}) \wedge T_n \quad (i = 0, 1, \dots, m2^n).$$

By Lemma 3.1, R_0, \dots, R_{m2^n} are stopping times. Moreover,

$$(11.4) \quad S_n = R_0 \leq R_1 \leq \dots \leq R_{m2^n} = T_n \quad \text{on } \{T_n \leq m\}.$$

For the moment, we fix $1 \leq i \leq m2^n$, $1 \leq j < m2^{-n}$, and define $R := R_{i-1}$, $\tilde{R} := R_i$,

$$A_j := \{R < \tilde{R}\} \cap \{R = j2^{-n}\}.$$

Our Definition (11.3) implies that $\tilde{R} = (j+1)2^{-n}$ on A_j . For every $0 < \varepsilon < 2^{-n}$,

$$A_j = \{R < \tilde{R}\} \cap \{R \geq j2^{-n}\} \cap \{R < j2^{-n} + \varepsilon\},$$

where $\{R < \tilde{R}\}, \{R \geq j2^{-n}\} \in \mathcal{F}_R^+$ by Proposition 3.2 and therefore $A_j \in \mathcal{F}_{j2^{-n}+\varepsilon}$ by definition of \mathcal{F}_R^+ . Thus $A_j \in \mathcal{F}_{j2^{-n}}^+$ and we conclude by Lemma 11.5 that

$$E^x(v(X_{\tilde{R}}); A_j) = E^x(v(X_{(j+1)2^{-n}}); A_j) \leq E^x(v(X_{j2^{-n}}); A_j) = E^x(v(X_R); A_j).$$

Trivially $v(X_{\tilde{R}}) = v(X_R)$ on $\{T_n \leq m\} \setminus (A_1 \cup A_2 \cup \dots \cup A_{m2^{-n}})$. Therefore the previous estimate shows that

$$E^x(v(X_{\tilde{R}}); T_n \leq m) \leq E^x(v(X_R); T_n \leq m).$$

In view of (11.4) this proves (11.2). \square

REMARK 11.7. Using the *resolvent* $(V_\alpha)_{\alpha>0}$ of the semigroup $(P_t)_{t>0}$, defined by

$$V_\alpha f(x) := \int_0^\infty e^{-\alpha t} P_t f(x) dt,$$

it is possible to show that, for every superharmonic function $v \geq 0$ on \mathbb{R}^d , there exists a (not necessarily increasing) sequence (f_n) of positive bounded Borel measurable functions on \mathbb{R}^d such that

$$V f_n \uparrow v \quad \text{as } n \uparrow \infty.$$

With this knowledge, our optional sampling theorem becomes an immediate consequence of Lemma 11.1.

COROLLARY 11.8. *Let $v \geq 0$ be a superharmonic function on \mathbb{R}^d and let (T_n) be a sequence of stopping times such that $T_n \downarrow T$. Then $P_{T_n} v \uparrow P_T v$.*

Proof. By optional sampling, for every $n \in \mathbb{N}$,

$$P_{T_n} v \leq P_{T_{n+1}} v \leq P_T v.$$

Fix $x \in \mathbb{R}^d$. By the lower semicontinuity of v and by Fatou's lemma,

$$P_T v(x) = E^x(f \circ X_T) \leq E^x(\liminf_{n \rightarrow \infty} v \circ X_{T_n}) \leq \liminf_{n \rightarrow \infty} E^x(v \circ X_{T_n}) = \liminf_{n \rightarrow \infty} P_{T_n} v.$$

\square

The next proposition will be used to deal with successive stoppings.

PROPOSITION 11.9. *Let S, T be stopping times. Then*

$$P_S P_T = P_{S+T \circ \theta_S}.$$

Proof. By the strong Markov property (Corollary 5.4), for every Borel measurable function $f \geq 0$ on \mathbb{R}^d and for every $x \in \mathbb{R}^d$,

$$\begin{aligned} P_S P_T f(x) &= E^x(E^{X_S}(f \circ X_T)) \\ &= E^x(f \circ X_T \circ \theta_S) = E^x(f \circ X_{S+T \circ \theta_S}) = P_{S+T \circ \theta_S} f(x). \end{aligned}$$

\square

12 Balayage

Given a K_σ -subset A of \mathbb{R}^d and a Borel measurable function $f \geq 0$ on \mathbb{R}^d , we define functions $P_A f, \hat{P}_A f$ by

$$P_A f(x) = P_{D_A} f(x) = E^x(f \circ D_A), \quad \hat{P}_A f(x) = P_{T_A} f(x) = E^x(f \circ T_A)$$

(see (11.1)).

THEOREM 12.1. *Let $v \geq 0$ be a superharmonic function on \mathbb{R}^d and A be a K_σ -subset³ of \mathbb{R}^d . Then the following holds:*

1. $\hat{P}_A v \leq P_A v \leq v$ on \mathbb{R}^d , $\hat{P}_A v = P_A v = v$ on $\overset{\circ}{A}$, $\hat{P}_A v = P_A v$ on $\mathbb{R}^d \setminus A$.
In particular, $\hat{P}_A v = P_A v$ if A is open.
2. The function $P_A v$ is supermedian and harmonic on $\mathbb{R}^d \setminus \bar{A}$.
3. The function $\hat{P}_A v$ is superharmonic (and harmonic on $\mathbb{R}^d \setminus \bar{A}$).
4. For every $x \in \mathbb{R}^d$,

$$(12.1) \quad \hat{P}_A v(x) = \liminf_{y \rightarrow x} P_A v(y),$$

that is, $\hat{P}_A v$ is the greatest lower semicontinuous minorant of $P_A v$.

Proof. 1. Since $0 \leq D_A \leq T_A$, the inequalities $\hat{P}_A v \leq P_A v \leq v$ are an immediate consequence of Theorem 11.6. The equalities follow from the fact that, for every $x \in \overset{\circ}{A}$, $T_A = D_A = 0$ P^x -almost surely and that, for every $x \in \mathbb{R}^d \setminus A$, $T_A = D_A$ P^x -almost surely.

2. Fix $x \in \mathbb{R}^d$ and define $S := T_{B(x,r)^c}$ (so that $P_S f(x) = \int f d\sigma_{x,r}$). Since $D_A \leq S + D_A \circ \theta_S$ and $T_A \leq S + T_A \circ \theta_S$, we obtain by Theorem 11.6 and Proposition 11.9 that

$$(12.2) \quad P_S P_A v \leq P_A v, \quad P_S \hat{P}_A v \leq \hat{P}_A v.$$

So $P_A v$ and $\hat{P}_A v$ are supermedian. If $\overline{B(x,r)}$ and \bar{A} are disjoint, then $D_A = S + D_A \circ \theta_S$ P^x -almost surely and therefore $P_S P_A v(x) = P_A v(x)$. This shows that $P_A v$ is harmonic on \bar{A}^c .

3. The definitions of D_A and T_A implies immediately that, for every $t > 0$,

$$T_A \leq t + D_A \circ \theta_t \leq t + T_A \circ \theta_t$$

and

$$t + T_A \circ \theta_t \downarrow T_A \quad \text{as } t \downarrow 0.$$

By Theorem 11.6, Corollary 11.8, and Proposition 11.9, we see that

$$(12.3) \quad P_t \hat{P}_A v \leq P_t P_A v \leq \hat{P}_A v \quad \text{and} \quad P_t \hat{P}_A v \uparrow \hat{P}_A v \quad \text{as } t \downarrow 0.$$

³For Borel sets see Corollary 12.9.

Since the functions $P_t \hat{P}_A v$ are lower semicontinuous (every function $P_t \hat{P}_A(v \wedge n)$, $n \in \mathbb{N}$, is continuous!), this shows that $\hat{P}_A v$ is lower semicontinuous as well. So $\hat{P}_A v$ is superharmonic.

4. Fix $x \in \mathbb{R}^d$. Since $\hat{P}_A v$ is lower semicontinuous and $\hat{P}_A v \leq P_A v$, we see that

$$\hat{P}_A v(x) = \liminf_{y \rightarrow x} \hat{P}_A v(y) \leq \liminf_{y \rightarrow x} P_A v(y).$$

To prove the converse inequality, we fix $a < \liminf_{y \rightarrow x} P_A v(y)$. There exists a continuous bounded function f on \mathbb{R}^d such that $0 \leq f \leq P_A v$ and $a < f(x)$. Then $P_t f \leq P_t P_A v$ for every $t > 0$ and therefore (using (12.3))

$$f = \lim_{t \downarrow 0} P_t f \leq \lim_{t \downarrow 0} P_t P_A v = \hat{P}_A v.$$

In particular, $a < f(x) \leq \hat{P}_A v(x)$ and therefore $\liminf_{y \rightarrow x} P_A v(y) \leq \hat{P}_A v(x)$. \square

LEMMA 12.2. *Let $v \geq 0$ be superharmonic and continuous on \mathbb{R}^d . If K is a compact subset of \mathbb{R}^d and U_n , $n \in \mathbb{N}$, are open neighborhoods of K such that $\overline{U}_n \downarrow K$, then*

$$D_{U_n} \uparrow D_K \quad \text{and} \quad P_{U_n} v \downarrow P_K v.$$

Proof. Immediate (see the proof of Proposition 4.1). \square

PROPOSITION 12.3. *Let A, B be K_σ -subsets of \mathbb{R}^d . Then, for every superharmonic function $v \geq 0$ on \mathbb{R}^d ,*

$$P_{A \cup B} v + P_{A \cap B} v \leq P_A v + P_B v.$$

Proof. Fix a superharmonic function $v \geq 0$ on \mathbb{R}^d and a point $x \in \mathbb{R}^d$. Since trivially $D_{A \cap B} \geq D_A \vee D_B$ and $D_{A \cup B} = D_A \wedge D_B$, we have $P_{A \cap B} v \leq P_{D_A \vee D_B} v$ by Theorem 11.6 and

$$\begin{aligned} P_{A \cup B} v(x) + P_{A \cap B} v(x) &\leq E^x(v \circ X_{D_A \wedge D_B}) + E^x(v \circ X_{D_A \vee D_B}) \\ &= E^x(v \circ X_{D_A}) + E^x(v \circ X_{D_B}) = P_A v(x) + P_B v(x). \end{aligned}$$

\square

Given a probability measure μ on \mathbb{R}^d , we define

$$c(A) := \inf \left\{ \int P_U v_0 d\mu : U \text{ open neighborhood of } A \right\} \quad (A \subset \mathbb{R}^d).$$

PROPOSITION 12.4. 1. *If $A_n \uparrow A$, then $c(A_n) \uparrow c(A)$.*

2. *If (K_n) is a sequence of compact subsets of \mathbb{R}^d and $K_n \downarrow K$, then $c(K_n) \downarrow c(K)$.*

Proof. 1. Consider subsets A_n, A of \mathbb{R}^d such that $A_n \uparrow A$. Obviously, the sequence $(c(A_n))$ is increasing and $c(A_n) \leq c(A)$ for every $n \in \mathbb{N}$. To prove that $\sup c(A_n) \geq c(A)$ it suffices to consider the case where $c(A_n) < \infty$ for every $n \in \mathbb{N}$. Fix $\varepsilon > 0$. Then there exist open neighborhoods U_n of A_n , $n \in \mathbb{N}$, such that

$$c(U_n) < c(A_n) + 2^{-n}\varepsilon.$$

Defining

$$V_n := U_1 \cup \dots \cup U_n,$$

we obtain an *increasing* sequence (V_n) of open sets. We claim that

$$(12.4) \quad c(V_n) > c(A_n) + (1 - 2^{-n})\varepsilon \quad \text{for every } n \in \mathbb{N}.$$

This is clearly true for $n = 1$. So let us assume that (12.4) holds for some $n \in \mathbb{N}$. Then, by Proposition 12.3,

$$\begin{aligned} c(V_{n+1}) + c(V_n \cap U_{n+1}) &\leq c(V_n) + c(U_{n+1}) \\ &\leq c(A_n) + (1 - 2^{-n})\varepsilon + c(A_{n+1}) + 2^{-(n+1)}\varepsilon, \end{aligned}$$

where $A_n \subset U_n \subset V_n$ and $A_n \subset A_{n+1} \subset U_{n+1}$, whence $c(A_n) \leq c(V_n \cap U_{n+1})$. Therefore

$$c(V_{n+1}) < c(A_{n+1}) + (1 - 2^{-(n+1)})\varepsilon.$$

This proves (12.4). Obviously, $U := \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} V_n$ is an open neighborhood of A . By Corollary 11.8, $P_{V_n}v_0 \uparrow P_Uv_0$. Therefore

$$c(A) \leq c(U) = \sup_n c(V_n) \leq \sup c(A_n) + \varepsilon.$$

2. Immediate consequence of Proposition 12.2. □

Using Choquet's capacitability theorem and Corollary 11.2, we now obtain the following:

THEOREM 12.5. *Let μ be a probability measure on \mathbb{R}^d and A be a Borel subset of \mathbb{R}^d . Then there exists an increasing sequence (K_n) of compact subsets of A and a decreasing sequence (U_n) of open neighborhoods of A such that*

$$D_{K_n} \downarrow D_A, \quad D_{U_n} \uparrow D_A \quad P^\mu\text{-almost surely.}$$

In particular,

$$D_A = D_{\bigcup K_n} = D_{\bigcap U_n} \quad P^\mu\text{-almost surely.}$$

Given $x \in \mathbb{R}^d$ and a Borel subset A of \mathbb{R}^d , we have

$$(12.5) \quad T_A = T_{A \setminus \{x\}} = D_{A \setminus \{x\}} \quad P^x\text{-almost surely,}$$

since $T_{\{x\}} = \infty$ almost surely. By Theorem 12.5, we may therefore extend our definitions of P_A and \hat{P}_A to arbitrary Borel subsets A of \mathbb{R}^d . Using Corollary 11.8, we then obtain the following result:

COROLLARY 12.6. For every Borel set A in \mathbb{R}^d and for every superharmonic function $v \geq 0$ on \mathbb{R}^d ,

$$P_A v = \sup\{P_K v : K \text{ compact subset of } A\}.$$

THEOREM 12.7. For every Borel set A in \mathbb{R}^d and for every continuous superharmonic function $v \geq 0$ on \mathbb{R}^d ,

$$\begin{aligned} P_A v &= \sup\{P_K v : K \text{ compact subset of } A\} \\ &= \inf\{P_U v : U \text{ open neighborhood of } A\} \\ &= \inf\{w : w \geq 0 \text{ superharmonic on } \mathbb{R}^d, w \geq v \text{ on } A\}. \end{aligned}$$

Proof. The first equality holds by Corollary 12.6. Since v is continuous, the second follows immediately from Theorem 12.5.

To prove the final one, we first consider any superharmonic function $w \geq 0$ majorizing v on A . For every compact subset K of A , $P_K v \leq P_K w \leq w$, where the first inequality holds, since $X_{D_K} \in K$ on the set $\{D_K < \infty\}$. Therefore

$$P_A v = \sup\{P_K v : K \text{ compact subset of } A\} \leq w.$$

On the other hand, for every open neighborhood U of A , the function $P_U v$ is a positive superharmonic function majorizing v on A . Thus

$$\begin{aligned} &\inf\{w : w \geq 0 \text{ superharmonic on } \mathbb{R}^d, w \geq v \text{ on } A\} \\ &\leq \inf\{P_U v : U \text{ open neighborhood of } A\} = P_A v. \end{aligned}$$

□

THEOREM 12.8. Let A be a Borel subset of \mathbb{R}^d . Then there exists a K_σ -subset \tilde{A} of A such that, for every $x \in \mathbb{R}^d$,

$$(12.6) \quad T_{\tilde{A}} = T_A \quad P^x\text{-almost surely.}$$

In particular, for every Borel measurable function $f \geq 0$ on \mathbb{R}^d , the function $\hat{P}_A f$ is lower semicontinuous and the function $P_A f$ is Borel measurable.

Proof. Let A be a Borel measurable subset of \mathbb{R}^d . By Theorem 12.1, Corollary 12.6, and (12.5), we see that, for every $x \in \mathbb{R}^d$,

$$\hat{P}_A v_0(x) = \sup\{\hat{P}_K v_0(x) : K \text{ compact subset of } A \setminus \{x\}\}.$$

Since the functions $\hat{P}_K v$, K compact subset of A , are lower semicontinuous, there exists an increasing sequence (K_n) of compact subsets of A such that

$$\hat{P}_A v_0 = \sup_{n \in \mathbb{N}} \hat{P}_{K_n} v_0.$$

Defining $\tilde{A} := \bigcup_{n=1}^{\infty} K_n$ we see that $\hat{P}_A v_0 = \hat{P}_{\tilde{A}} v_0$. By Corollary 11.2, we conclude that

$$T_A = T_{\tilde{A}} \quad \text{almost surely.}$$

For every Borel measurable function $f \geq 0$ on \mathbb{R}^d , the function $\hat{P}_A f = \hat{P}_{\tilde{A}} f$ is lower semicontinuous and $P_A f = 1_A f + 1_{A^c} \hat{P}_A f$ is Borel measurable. □

COROLLARY 12.9. *Let $v \geq 0$ be a superharmonic function on \mathbb{R}^d and A be a Borel measurable subset of \mathbb{R}^d . Then the statements of Theorem (12.1) hold.*

Proof. Only statement (4) requires some additional reasoning. Fix $x \in \mathbb{R}^d$. As before we immediately obtain the inequality $\hat{P}_A v(x) \leq \liminf_{y \rightarrow x} P_A v(y)$. To prove the converse inequality we choose a K_σ -set A' contained in A such that $T_{A'} = T_A$ almost surely. Taking $\mu := P_{X_t}^x$ we have $\theta_t(P^x) = P^\mu$ whence $T_{A'} \circ \theta_t = T_A \circ \theta_t$ P^x -almost surely, and this allows us to argue as in the proof of Proposition 12.1 (the details are left to the reader). \square

A subset A of \mathbb{R}^d is called *polar* if there exists a superharmonic function $v \geq 0$ on \mathbb{R}^d such that

$$v = \infty \quad \text{on } A.$$

We note that the set

$$\{v = \infty\} = \bigcap_{n=1}^{\infty} \{v > n\}$$

is a G_δ -set. So every polar set is contained in a polar G_δ -set. Considering $x \in \mathbb{R}^d$ such that $v(x) < \infty$, the inequalities $\lambda_{x,r}(v) \leq v(x)$, $r > 0$, show that the set $\{v = \infty\}$ is a Lebesgue-null set, whence a set which certainly has no interior points.

PROPOSITION 12.10. *For every Borel set A in \mathbb{R}^d the following statements are equivalent:*

1. A is polar.
2. $T_A = \infty$ P^x -almost surely for some $x \in \mathbb{R}^d$.
3. $T_A = \infty$ P^x -almost surely for every $x \in \mathbb{R}^d$.

Proof. The function $\hat{P}_A 1$ is superharmonic. So we know by Proposition 10.2 that $\hat{P}_A 1 = 0$ if and only if $\hat{P}_A 1(x) = 0$ for some $x \in \mathbb{R}^d$. Since

$$\hat{P}_A 1(x) = P^x(T_A < \infty) \quad \text{for every } x \in \mathbb{R}^d,$$

we see that (2) and (3) are equivalent.

(1) \Rightarrow (2): Let $v \geq 0$ be a superharmonic function on \mathbb{R}^d such that $v = \infty$ on A . Choose a point $x \in \mathbb{R}^d$ such that $v(x) < \infty$. Then, for every $n \in \mathbb{N}$ and every compact subset K of A ,

$$nP_K 1(x) \leq P_A v(x) \leq v(x) < \infty.$$

By Theorem 12.5, we see that $nP_A 1(x) \leq v(x)$ for every $n \in \mathbb{N}$, that is, $P_A 1(x) = 0$ and $\hat{P}_A 1(x) = 0$.

(3) \Rightarrow (1): Since $T_{\mathbb{R}^d} = 0$, we know that $A \neq \mathbb{R}^d$, and hence we may choose a point $x \in A^c$. Then $P_A 1(x) = \hat{P}_A 1(x) = 0$. By Theorem 12.7, there exist superharmonic functions $w_n \geq 0$ on \mathbb{R}^d , $n \in \mathbb{N}$, majorizing 1 on A such that $w_n(x) < 2^{-n}$. Define $w := \sum_{n=1}^{\infty} w_n$. Then w is hyperharmonic on \mathbb{R}^d , $w = \infty$ on A , and $w(x) < 1$. By Lemma 10.2, w is superharmonic on \mathbb{R}^d . Thus A is polar. \square

We define the *balayage* ε_x^A of the Dirac measure ε_x , $x \in \mathbb{R}^d$, on a subset A of \mathbb{R}^d by

$$\varepsilon_x^A(B) := P^x(X_{T_A} \in B) = \hat{P}_A 1_B(x) \quad (B \text{ Borel subset of } \mathbb{R}^d),$$

that is, ε_x^A is the distribution of X_{T_A} (the position of Brownian motion at time T_A) with respect to P^x . Of course, for every Borel function $f \geq 0$ on \mathbb{R}^d ,

$$\hat{P}_A f(x) = \int f d\varepsilon_x^A.$$

Given a measure $\mu \geq 0$ on \mathbb{R}^d , we define the *balayage of μ on A* by

$$\mu^A := \int \varepsilon_x^A d\mu(x),$$

that is, for every Borel function $f \geq 0$ on \mathbb{R}^d ,

$$\mu^A(f) := \int \varepsilon_x^A(f) d\mu(x).$$

PROPOSITION 12.11. *For every Borel measurable subset A of \mathbb{R}^d , there exists a K_σ -set \tilde{A} such that $\tilde{A} \subset A$ and*

$$\mu^{\tilde{A}} = \mu^A \quad \text{for every measure } \mu \text{ on } \mathbb{R}^d.$$

Proof. Theorem 12.8. □

13 Fine topology and base of sets

By definition, the *fine topology* on \mathbb{R}^d is the coarsest topology on \mathbb{R}^d such that all superharmonic functions $v \geq 0$ on \mathbb{R}^d are finely continuous. Clearly, open balls in \mathbb{R}^d are finely open, since

$$B(y, r) = \{G(\cdot, y) > r^{2-d}\} \quad (y \in \mathbb{R}^d, r > 0).$$

Therefore the fine topology is finer than (that is, at least as fine as) the Euclidean topology (it is easily seen that it is, in fact, much finer).

We intend to show that the fine topology is closely related to the base operation defined by balayage as follows. For every Borel subset A of \mathbb{R}^d , the *base* $b(A)$ of A is the set

$$b(A) := \{x \in \mathbb{R}^d : \varepsilon_x^A = \varepsilon_x\}.$$

We note that, of course, $\overset{\circ}{A} \subset b(A) \subset \bar{A}$.

PROPOSITION 13.1. *For every Borel subset A of \mathbb{R}^d ,*

$$b(A) = \{x \in \mathbb{R}^d : T_A = 0 \text{ } P^x\text{-almost surely}\} = \{\hat{P}_A v_0 = v_0\}.$$

In particular, $b(A)$ is a G_δ -set which is finely closed.

Proof. Fix $x \in \mathbb{R}^d$. If $\varepsilon_x^A = \varepsilon_x$, then obviously $\hat{P}_A v_0(x) = v_0(x)$. By Corollary 11.2, the equality $\hat{P}_A v_0(x) = v_0(x)$ implies that $T_A = 0$ P^x -almost surely, which in turn trivially implies that $\varepsilon_x^A = \varepsilon_x$. \square

COROLLARY 13.2. *For every open subset U of \mathbb{R}^d , the set of all irregular boundary points is the intersection of ∂U with the base of U^c .*

Proof. Proposition 13.1 and Theorem 9.1. \square

We may define the base for an arbitrary subset A of \mathbb{R}^d by

$$b(A) = \bigcap_{B \text{ Borel}, A \subset B} b(B).$$

LEMMA 13.3. *The mapping b is additive, that is, for all subsets A_1, A_2 of \mathbb{R}^d ,*

$$(13.1) \quad b(A_1 \cup A_2) = b(A_1) \cup b(A_2).$$

Proof. Since $T_{A_1 \cup A_2} = T_{A_1} \wedge T_{A_2}$, it follows from Proposition 13.1 that (13.1) holds, if A_1, A_2 are Borel sets.

Consider now arbitrary subsets A_1, A_2 of \mathbb{R}^d . Since the mapping $A \mapsto b(A)$ is obviously increasing, we see that

$$b(A_1) \cup b(A_2) \subset b(A_1 \cup A_2).$$

If $x \notin b(A_1) \cup b(A_2)$, there are Borel sets B_1, B_2 such that $A_1 \subset B_1$, $A_2 \subset B_2$ and $x \notin b(B_1) \cup b(B_2)$. Thus $x \notin b(B_1 \cup B_2)$, whence $x \notin b(A_1 \cup A_2)$. \square

THEOREM 13.4. *For every subset A of \mathbb{R}^d , the union of A and $b(A)$ is the fine closure of A .*

Proof. Since b is additive, there is a topology τ on \mathbb{R}^d such that, for every subset A of \mathbb{R}^d , the union of A and $b(A)$ is the closure of A with respect to the topology τ . For every closed subset A of \mathbb{R}^d , the base $b(A)$ of A is contained in A , and hence A is closed with respect to the topology τ . Thus τ as well is finer than the Euclidean topology on \mathbb{R}^d . We have to show that τ is the fine topology.

a) We note first that, by Theorem 12.7,

$$P_A v_0 = \inf\{w : w \geq 0 \text{ superharmonic on } \mathbb{R}^d, w \geq v_0 \text{ on } A\} \leq v_0.$$

So the function $P_A v_0$ is finely upper semicontinuous and $\{v_0 - P_A v_0 = 0\}$ is finely closed. Since $P_A v_0 = v_0$ on A and $P_A v_0 = \hat{P}_A v_0$ on A^c , we see that

$$A \subset \{P_A v_0 = v_0\} = A \cup b(A),$$

and hence the fine closure of A is contained in the τ -closure of $A \cup b(A)$ of A . This shows that the fine topology is finer than the topology τ .

b) Consider an arbitrary superharmonic function $v \geq 0$ on \mathbb{R}^d and fix $a \in \mathbb{R}$. Then the set $\{v > a\}$ is open (with respect to the Euclidean topology) whence τ -open. We intend to prove that the set $\{v < a\}$ is τ -open as well. This will show

that v is τ -continuous and that hence, by definition of the fine topology, the fine topology is coarser than the topology τ .

So it remains to prove that the set $\{v \leq a\}$ is τ -closed. To that end we fix $x \in \mathbb{R}^d \setminus \{v \geq a\}$. Then $a > 0$ and, for every compact subset K of $\{v \geq a\}$,

$$aP_K 1 \leq P_K v \leq v(x).$$

Applying Corollary 12.6, we see that

$$a\hat{P}_{\{v \geq a\}} 1(x) = aP_{\{v \geq a\}} 1(x) \leq v(x) < a$$

and hence $x \notin b(\{v \geq a\})$. Thus $b(\{v \geq a\}) \subset \{v \geq a\}$, that is, the set $\{v \geq a\}$ is τ -closed. \square

COROLLARY 13.5. *For every Borel subset A of \mathbb{R}^d and for every $x \in \mathbb{R}^d$, the measure ε_x^A is supported by the fine closure $A \cup b(A)$ of A .*

Proof. Let us define $T := T_A$ and consider $\omega \in \Omega$ such that $T(\omega) < \infty$. If $X_T(\omega)$ is not contained in A , then $T(\omega) = (T + T \circ \theta_T)(\omega)$ whence $T \circ \theta_T(\omega) = 0$. If $y := X_T(\omega) \notin b(A)$, then $P^y(T = 0) = 0$ by Blumenthal's 0–1–law (Proposition 5.6). Moreover, $\{X_T \notin A \cup b(A)\} \in \mathcal{F}_T^+$ by Proposition 3.3. Therefore the strong Markov property (Theorem 5.4) implies that

$$\begin{aligned} P^x(X_T \notin A \cup b(A), T < \infty) &= P^x(T \circ \theta_T = 0, X_T \notin A \cup b(A), T < \infty) \\ &= E^x(P^{X_T}(T = 0); X_T \notin A \cup b(A), T < \infty) = 0, \end{aligned}$$

\square

COROLLARY 13.6. *A subset V of \mathbb{R}^d is finely open if and only if, for every $x \in V$,*

$$T_{V^c} > 0 \quad P^x\text{-almost surely.}$$

To illustrate the possibilities of our probabilistic approach let us state the following:

PROPOSITION 13.7. *Let A and B be Borel subsets of \mathbb{R}^d , $B \subset A$, and $x \in \mathbb{R}^d$. Then the following statements are equivalent:*

1. $x \notin (b(A) \setminus b(B)) \cap B$.
2. $\varepsilon_x^B = \varepsilon_x^A|_B + (\varepsilon_x^A|_{B^c})^B$.
3. $\varepsilon_x^A|_B \leq \varepsilon_x^B$.

Proof. If $x \in (b(A) \setminus b(B)) \cap B$, then $\varepsilon_x^A = \varepsilon_x$, whence $\varepsilon_x^A|_B = \varepsilon_x$, whereas ε_x^B is a probability measure which is not ε_x . Therefore (3) does not hold. The implication (2) \Rightarrow (3) is trivial.

It remains to show that (1) \Rightarrow (2). So let us suppose that $x \notin (b(A) \setminus b(B)) \cap B$, that is, $x \in B^c \cup b(A)^c \cup b(B)$. We may assume without loss of generality that A and B

are K_σ -sets. Indeed, by Proposition 12.11, there exist K_σ -sets \tilde{A}, \tilde{B} such that $\tilde{A} \subset A$, $\tilde{B} \subset B$ and $\mu^{\tilde{A}} = \mu^A$, $\mu^{\tilde{B}} = \mu^B$ for all measures μ on \mathbb{R}^d . Enlarging \tilde{B} if necessary we may assume that $\varepsilon_x^A(\tilde{B}) = \varepsilon_x^A(B)$. Then $\varepsilon_x^{\tilde{A}}|_{\tilde{B}} = \varepsilon_x^A|_B$ and $(\varepsilon_x^{\tilde{A}}|_{\tilde{B}^c})^{\tilde{B}} = (\varepsilon_x^A|_{B^c})^B$. Moreover, $x \in \tilde{B}^c \cup b(A) \cup b(\tilde{B}^c)$, since $\tilde{B} \subset B$, $b(\tilde{A}) = b(A)$, and $b(B) = b(\tilde{B})$.

It is easily verified that

$$T_B = T_A + D_B \circ \theta_{T_A} \quad P^x\text{-almost surely.}$$

So the strong Markov property (Proposition 5.3) implies that, for every Borel measurable function $f \geq 0$ on \mathbb{R}^d ,

$$\varepsilon_x^B(f) = E^x(f \circ X_{T_B}) = E^x(f \circ X_{D_B} \circ \theta_{T_A}) = E^x(E^{X_{T_A}}(f \circ X_{D_B})) = \varepsilon_x^A(w),$$

if we define

$$w(y) := E^y(f \circ X_{D_B}) \quad (y \in \mathbb{R}^d).$$

If $y \in B$, then $D_B = 0$ P^y -almost surely and therefore $w(y) = f(y)$. If $y \notin B$, then $D_B = T_B$ P^y -almost surely and therefore $w(y) = \varepsilon_y^B(f)$. Thus

$$\varepsilon_x^A(w) = \int_B f(y) d\varepsilon_x^A(y) + \int_{B^c} \varepsilon_y^B(f) d\varepsilon_x^A(y) = (\varepsilon_x^A|_B + (\varepsilon_x^A|_{B^c})^B)(f).$$

□