Choquet theory, boundaries and applications

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Contents

1 The simplicial spaces H(U) (classical case)

To begin with let us consider classical potential theory, where U, V, W will always denote open sets in $\mathbb{R}^d$, $d \geq 1$, unless specified otherwise. As usual, let $\mathcal{H}(U)$ denote the linear space of all harmonic functions on U, that is,

$$\mathcal{H}(U) := \{h \in \mathcal{C}^2(U) \colon \Delta h = 0\}.$$

EXAMPLES 1.1. a) If f is affinely linear on $\mathbb{R}^d$, then $f \in \mathcal{H}(\mathbb{R}^d)$ (if $d = 1$ the converse is true as well).

b) For every $d \geq 3$, let G denote the *Green function* on $\mathbb{R}^d$, that is,

$$G(x, y) := |x - y|^{2-d} \qquad (x, y \in \mathbb{R}^d).$$

For all $y \in \mathbb{R}^d$, $G(\cdot, y) \in \mathcal{H}(\mathbb{R}^d \setminus \{y\})$.

c) If $d = 2$ and $y \in \mathbb{R}^2$, then $(x \mapsto \ln |x - y|) \in \mathcal{H}(\mathbb{R}^2 \setminus \{y\})$.

For the moment, let us fix a relatively compact open set U in $\mathbb{R}$, $d \geq 1$, let

$$H(U) := \{h \in \mathcal{C}(\overline{U}) \colon h|_U \in \mathcal{H}(U)\},$$

and, for every $x \in \overline{U}$, let $\mathcal{M}_x(H(U))$ denote the set of all *representing measures for x with respect to $H(U)$*, that is,

$$\mathcal{M}_x(H(U)) := \{\mu \colon \mu \text{ measure on } \overline{U},\ \mu(h) = h(x) \text{ for every } h \in H(U)\}.$$

Further, let $Ch_{H(U)}\overline{U}$ denote the *Choquet boundary of $\overline{U}$ with respect to $H(U)$*,

$$Ch_{H(U)}\overline{U} := \big\{x \in \overline{U} \colon \mathcal{M}_x(H(U)) = \{\varepsilon_x\}\big\}.$$

Our first aim is the following result.

THEOREM 1.2. *$H(U)$ is simplicial, that is, for every $x \in \overline{U}$, there exists a unique measure $\mu_x \in \mathcal{M}_x(H(U))$ such that $\mu(\overline{U} \setminus \mathrm{Ch}_{H(U)}\overline{U}) = 0$. Moreover, $\mathrm{Ch}_{H(U)}\overline{U}$ is the set U_r of all regular boundary points of U and, for every $x \in U$, μ_x is the harmonic measure μ_x^U.*

Here U_r and μ_x^U are related to the solution of the *Dirichlet problem*: Given $f \in \mathcal{C}(\partial U)$, find $h \in H(U)$ such that $h|_{\partial U} = f$.

If $x \in \mathbb{R}^d$ and $r > 0$, then the solution h to the Dirichlet problem for the open ball $B(x, r)$ and a function $f \in \mathcal{C}(\partial B(x, r))$ can be obtained by the *Poisson integral*:

$$h(y) = H_{B(x,r)}f(y) := r^{d-2}(r^2 - |y - x|^2) \int |y - z|^{-d} f(z)\, d\sigma_{x,r}(z), \tag{1.1}$$

where $\sigma_{x,r}$ denotes normed surface measure on the sphere $\partial B(x, r)$. Moreover, we note that, for every lower bounded Borel measurable $f \colon \partial B(x, r) \to (-\infty, \infty]$, the function $H_{B(x,r)}f$ is either harmonic on $B(x, r)$ or identically ∞. For details see, for example, [4, pp. 3/4] or [1].

In particular,

$$\mathcal{H}(U) = \{h \in \mathcal{C}(U)\colon H_{B(x,r)}h = h, \text{ whenever } \overline{B(x,r)} \subset U\}. \tag{1.2}$$

It is well known that there are sets U, where for many functions f the Dirichlet problem does not admit a solution. However, there is always a *generalized solution* $H_U f$ which is harmonic on U and tends to f at the set U_r of *regular boundary points*. Since the mapping $f \mapsto H_U f$ is positive and linear, the mappings $f \mapsto H_U f(x)$, $x \in X$, define *harmonic measures* μ_x^U.

The generalized solution $H_U f$, the set U_r, and the harmonic measures μ_x^U are most easily described probabilistically using Brownian motion $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ (see the lectures "Brownian motion and potential theory" given in 2006). Let τ_U denote the (first) *exit time* for U, that is,

$$\tau_U(\omega) := \inf\{t > 0\colon X_t(\omega) \notin U\}.$$

Then, for all $f \in \mathcal{C}(\partial U)$ and $x \in U$,

$$H_U f(x) = \int f(X_{\tau_U})\, dP^x = E^x(f(X_{\tau_U})),$$

where $X_{\tau_U}(\omega) := X_{\tau_U(\omega)}(\omega)$ and P^x is the probability measure for Brownian motion starting at x. Moreover, U_r is the set of all points $z \in \partial U$ such that Brownian motion starting at z immediately hits U^c, that is,

$$U_r = \{z \in \partial U\colon \tau_U = 0 \ P^z - \text{a.s.}\},$$

and μ_x^U is the distribution of the first exit from U for Brownian motion starting at x, that is,

$$\mu_x^U = P^x_{X_{\tau_U}} \qquad (x \in U).$$

If $h \in H(U)$, then $H_U(h|_{\partial U}) = h|_U$ and therefore

$$\mu_x^U \in \mathcal{M}_x(H(U)) \qquad (x \in U). \tag{1.3}$$

Obviously, the measures μ_x^U, $x \in U$, are supported by ∂U and therefore (1.3) implies that

$$Ch_{H(U)}\overline{U} \subset \partial U. \tag{1.4}$$

In fact, $P^x_{X_{\tau_U}} \in \mathcal{M}_x(H(U))$ for all $x \in \overline{U}$, and $P^x_{X_{\tau_U}} \neq \varepsilon_x$ if $x \in \partial U \setminus U_r$. Thus even

$$Ch_{H(U)}\overline{U} \subset U_r. \tag{1.5}$$

EXAMPLES 1.3. If $U := B(0,1) \setminus \{0\}$ in $\mathbb{R}^d$, $d \geq 2$, then 0 is an irregular point of U (the only one), since $\tau_U = \tau_{B(0,1)}$ almost surely. If $d \geq 3$ and U is obtained removing a spine S from $B(0,1)$ which is sufficiently thin at the origin, then $0 \in \partial U \setminus U_r$.

2 Analytic solution to the generalized Dirichlet problem

Analytically, the generalized solutions $H_U f$ are obtained using *superharmonic functions.* A function $s\colon U \to (-\infty, +\infty]$ is superharmonic (on U), if it is lower semicontinuous and the average of s on all spheres $\partial B(x,r)$ with $\overline{B(x,r)} \subset U$ is finite and at most $s(x)$. Obviously, the set $\mathcal{S}(U)$ of all superharmonic functions on U is a convex cone which contains $\mathcal{H}(U)$ and is $\wedge$-stable, that is, the minimum $s \wedge t$ is superharmonic on U, whenever $s, t \in \mathcal{S}(U)$.

In the following we shall need some additional well-known facts from classical potential theory. In order to keep these notes as self-contained as possible, we shall always give a proof or a reference to results which are proven later on in the more general framework of harmonic kernels (see Example 7.2).

An important ingredient in the discussion of generalized solutions is the following *minimum principle.*

PROPOSITION 2.1. *Let U be relatively compact and let $s \in \mathcal{S}(U)$ such that $s \ge 0$ at ∂U, that is, $\liminf_{x\to z} s(x) \ge 0$ for every $z \in \partial U$. Then $s \ge 0$ on U.*

More generally, $s \ge 0$ on U for every lower semicontinuous $s\colon U \to (-\infty, \infty]$ such that $s \ge 0$ at ∂U and, for every $x \in U$, there exists $r > 0$ with $\overline{B(x,r)} \subset U$ and $\sigma_{x,r}(s) \le s(x)$.

This is a special case of Proposition 15.2. However, in our classical case, there is such a nice and simple proof that we should give it.

Proof of Proposition 2.1. Let $s\colon U \to (-\infty, \infty]$ be lower semicontinuous such that $s \ge 0$ at ∂U and let us suppose that $\alpha := \inf s(U) < 0$. Then $K := \{s = \alpha\}$ is a compact subset of U. There exists a point in K having minimal distance to the boundary ∂U. If $r > 0$ such that $\overline{B(x,r)} \subset U$ and $\sigma_{x,r}(s) \le s(x)$, then $\partial B(x,r) \subset K$. Since $\partial B(x,r)$ contains points having a strictly smaller distance to ∂U than x, this is impossible. $\square$

COROLLARY 2.2. *$\mathcal{S}(U)$ is the set of all lower semicontinuous $s\colon U \to (-\infty, \infty]$ such that $H_{B(x,r)} s \le s$ and $H_{B(x,r)} s$ is harmonic on $B(x,r)$, whenever $\overline{B(x,r)} \subset U$. Moreover,*

$$\mathcal{S}(U) = \Big\{ s \;\Big|\; s\colon U \to (-\infty,\infty] \text{ l.s.c. such that, for all } x \in U \text{ and } \varepsilon > 0, \text{ there exists } r \in (0,\varepsilon) \text{ with } \overline{B(x,r)} \subset U,\ \sigma_{x,r}(s) \le s(x),\ \sigma_{x,r}(s) < \infty \Big\}.$$

Proof. Trivially, $\mathcal{S}(U)$ is contained in the set $\tilde{\mathcal{S}}(U)$ on the right side of the displayed formula.

Let us now consider $s \in \tilde{\mathcal{S}}(U)$ and $V := B(x,r)$ such that $\overline{B(x,r)} \subset U$. Let $f \in \mathcal{C}(\partial V)$ such that $f \le s$ on ∂V and

$$v := s|_V - H_V f.$$

Since $H_V f$ tends to f at ∂V, we know that $v \ge 0$ at ∂V. Moreover, let $y \in V$ and $\varepsilon := r - |y - x|$. By definition of $\tilde{\mathcal{S}}(U)$, there exists $0 < \rho < \varepsilon$ such that

$\sigma_{y,\rho}(s) \le s(x)$ and $\sigma_{y,\rho}(s) < \infty$. Clearly, $\overline{B(y,\rho)} \subset V$ and $\sigma_{y,\rho}(H_V f) = H_V f(y)$ by (1.1) (the solution to the Dirichlet problem for V and $(H_V f)|_{\partial V}$ is $(H_V f)|_V$). Hence, by Proposition 2.1, $v \ge 0$. This implies that $H_V s \le s$. Moreover, $H_V s$ is harmonic, since $H_V s(y) = \sigma_{y,\rho}(H_V s) \le \sigma_{y,\rho}(s) < \infty$.

Thus s has the properties described at the beginning of Corollary 2.2. Finally, any such function is, of course, contained in $\mathcal{S}(U)$. □

An easy consequence of Corollary 2.2 is the fact that, for every $y \in \mathbb{R}^d$, the function $s\colon x \mapsto G(x,y)$ ($s\colon x \mapsto -\ln|x-y|$, respectively) is superharmonic on $\mathbb{R}^d$, if $d \ge 3$ ($d = 2$, respectively). It suffices to note that, by (1.2), $\sigma_{x,r}(s) = s(x)$, whenever $x \ne y$, $0 < r < |x-y|$ and, of course, $\sigma_{x,r}(x) \le \infty = s(x)$, if $x = y$, $r > 0$.

Moreover, Corollary 2.2 and (1.2) imply that

$$\mathcal{H}(U) = \mathcal{S}(U) \cap (-\mathcal{S}(U)). \tag{2.1}$$

Since we want to discuss the generalized Dirichlet problem for open sets which are not necessarily relatively compact, we shall need a corresponding minimum principle. Given $g\colon U \to [-\infty,\infty]$, let us say that $g \ge 0$ *at* ∞, if $\liminf_{|x|\to\infty} g(x) \ge 0$ and $g \le 0$ *at* ∞, if $\limsup_{|x|\to\infty} g(x) \le 0$. Let us note that both properties hold formally, if U is relatively compact (using $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$).

PROPOSITION 2.3. *Let $s \in \mathcal{S}(U)$ such that $s \ge 0$ at ∂U and $s \ge 0$ at ∞. Then $s \ge 0$ on U.*

Proof. Let $\varepsilon > 0$ and $\tilde{s} := s + \varepsilon$. Then there exists a relatively compact set V such that $V \subset U$ and $\tilde{s} > 0$ on $U \setminus V$. Obviously, $\tilde{s} \ge 0$ at ∂V (if $z \in \partial V \cap U$, then $\liminf_{x\to z, x\in V} \tilde{s}(x) \ge \tilde{s}(z) > 0$). Thus, by Proposition 2.1, $\tilde{s} \ge 0$ on V. This shows that $s + \varepsilon = \tilde{s} \ge 0$ on U and hence $s \ge 0$ on U. □

We now define, for every $f \in \mathcal{C}_0(\partial U)$,[1]

$$\begin{aligned} \overline{H}_U f &:= \inf\{s \in \mathcal{S}(U)\colon s \ge f \text{ at } \partial U \text{ and } s \ge 0 \text{ at } \infty\}, \\ \underline{H}_U f &:= \sup\{t \in -\mathcal{S}(U)\colon t \le f \text{ at } \partial U \text{ and } t \le 0 \text{ at } \infty\}, \end{aligned}$$

where, of course, $t \le f$ at ∂U means that $\limsup_{x\to z} t(x) \le f(z)$ for every $z \in \partial U$. Clearly, $\underline{H}_U f = -\overline{H}_U(-f)$. By Proposition 2.3, $\underline{H}_U f \le \overline{H}_U f$. Moreover, it is immediately seen that

$$\overline{H}_U(f+g) \le \overline{H}_U f + \overline{H}_U g \quad \text{and} \quad \underline{H}_U f + \underline{H}_U g \le \underline{H}_U(f+g). \tag{2.2}$$

whenever $f, g \in \mathcal{C}(\partial U)$. Further, the mapping $f \mapsto \overline{H}_U f$ is positively homogeneous, $\overline{H}_U f \ge 0$, if $f \ge 0$, and

$$-1 \le \underline{H}_U f \le \overline{H}_U f \le 1, \qquad \text{whenever } |f| \le 1. \tag{2.3}$$

If the Dirichlet problem for $f \in \mathcal{C}_0(\partial U)$ admits a solution h vanishing at ∞, then $h \le \underline{H}_U f$ and $\overline{H}_U f \le h$ by Proposition 2.3 and hence $h = \underline{H}_U f = \overline{H}_U f$. Studying

[1] Obviously, we could define $\overline{H}_U f$ and $\underline{H}_U f$ for every $\mathcal{S}^+(\mathbb{R}^d)$-bounded function f on ∂U.

reduced functions we shall see very soon that, in any case, f is *resolutive*, that is, $\underline{H}_U f = \overline{H}_U f$ (see Corollary 2.6), and then

$$H_U f := \underline{H}_U f = \overline{H}_U f$$

will be called the *generalized solution* to the Dirichlet problem.

Let us assume for simplicity that $d \geq 3$ and let us recall the definition of *reduced* functions. Given $A \subset \mathbb{R}^d$ and $u \in \mathcal{S}^+(\mathbb{R}^d)$, let

$$R_u^A := \inf\{s \in \mathcal{S}^+(\mathbb{R}^d) \colon s \geq u \text{ on } A\}$$

(if $d \leq 2$, we would, for example, replace $\mathbb{R}^d$ by an open ball $B(0,R)$, $R > 0$). Clearly, $R_u^A \leq u$ on $\mathbb{R}^d$ and $R_u^A = u$ on A. Moreover, the *reduced function* R_u^A is harmonic outside the closure of A (see Proposition 17.2). This leads to the following.

PROPOSITION 2.4. *Let $s \in \mathcal{S}^+(X) \cap \mathcal{C}_0(X)$. Then $R_s^{U^c} = \underline{H}_U(s|_U) = \overline{H}_U(s|_U)$ on U.*

Proof. Let $f := s|_U$. Since $\mathcal{S}^+(\mathbb{R}^d)|_U \subset \mathcal{S}(U)$, we trivially have $R_s^{U^c} \geq \overline{H}_U f$. Moreover, $R_s^{U^c}$ is harmonic on U and $R_s^{U^c} \leq s$ at ∂U. Therefore $R_s^{U^c} \leq \underline{H}_U f$. Since $\underline{H}_U s \leq \overline{H}_U s$, the proof is finished. □

It will be useful to have the following approximation result (see also Proposition 9.4 in connection with Examples 9.2).

LEMMA 2.5. *There exists a sequence (s_m) in $\mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d)$ such that the linear hull of $\{s_m \colon m \in \mathbb{N}\}$ is dense in $\mathcal{C}_0(\mathbb{R}^d)$ with respect to uniform convergence.*

Proof. The convex cone $\mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(X)$ is $\wedge$-stable and linearly separating, since it contains the functions $G(\cdot, y) \wedge 1$, $y \in \mathbb{R}^d$. So the result follows from a version of the theorem of Stone-Weierstraß. □

COROLLARY 2.6. *Every function $f \in \mathcal{C}_0(\partial U)$ is resolutive. In particular, the mapping $f \mapsto H_U f$ is linear and positive. It defines harmonic measures μ_x^U, $x \in U$, on ∂U:*

$$\mu_x^U(f) = H_U f(x) \qquad (f \in \mathcal{C}_0(\partial U)).$$

Proof. Using (2.2) it is quickly verified that the set $\mathcal{R}(U)$ of all resolutive functions in $\mathcal{C}_0(\partial U)$ is a linear space. By Proposition 2.4 and Lemma 2.5, $\mathcal{R}(U)$ is dense in $\mathcal{C}_0(\partial U)$.

To finish the proof it suffices to show that $\mathcal{R}(U)$ is closed with respect to uniform convergence. So let us suppose that $f \in \mathcal{R}(U)$, $g \in \mathcal{C}_0(\partial U)$ and $\varepsilon > 0$ such that $|f - g| \leq \varepsilon$. By (2.3) and (2.2), $-\varepsilon \leq \overline{H}_U(f-g) \leq H_U f + \overline{H}_U(-g)$ and hence

$$-\underline{H}_U g \leq H_U f + \varepsilon. \tag{2.4}$$

Replacing f, g by $-f, -g$, we have as well $-\underline{H}_U(-g) \leq H_U(-f) + \varepsilon$, that is,

$$\overline{H}_U g \leq -H_U f + \varepsilon. \tag{2.5}$$

Combining (2.4) and (2.5) we see that $\overline{H}_U g - \underline{H}_U g \leq 2\varepsilon$. □

REMARK 2.7. By (2.3), $\mu_x^U(\partial U) \leq 1$ ($\mu_x^U(\partial U) = 1$ if U is relatively compact).

3 Swept measures and regular boundary points

Again let A be an arbitrary subset of $\mathbb{R}^d$, let $u \in \mathcal{S}^+(\mathbb{R}^d)$, and

$$\hat{R}_u^A(x) := \liminf_{y\to x} R_u^A(y) \qquad (x \in \mathbb{R}^d).$$

Clearly, the *swept function* $\hat{R}_u^A$ is the greatest lower semicontinuous minorant of R_u^A, $\hat{R}_u^A = u$ on the interior of A, and $\hat{R}_u^A = R_u^A$ on $\mathbb{R}^d \setminus \overline{A}$ (where R_u^A is harmonic).[2] Moreover, $\hat{R}_u^A \in \mathcal{S}^+(\mathbb{R}^d)$ (see Proposition 17.3).

By Proposition 2.4 and Proposition 17.4, for all $s, t \in \mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(X)$,

$$(3.1) \qquad R_{s+t}^{U^c} = R_s^{U^c} + R_t^{U^c} \quad \text{and} \quad \hat{R}_{s+t}^{U^c} = \hat{R}_s^{U^c} + \hat{R}_t^{U^c}.$$ [3]

This implies that, for every $x \in \mathbb{R}^d$, there exists a unique measure $\varepsilon_x^{U^c}$ such that

$$(3.2) \qquad \varepsilon_x^{U^c}(s) = \hat{R}_s^{U^c}(x) \qquad \text{for every } s \in \mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d)$$

(cf. also Proposition 9.5).

PROPOSITION 3.1. *Let $s \in \mathcal{S}^+(\mathbb{R}^d)$ and $s_n \in \mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that $s_n \uparrow s$. Then*

$$R_{s_n}^{U^c} \uparrow R_s^{U^c} \quad \textit{and} \quad \hat{R}_{s_n}^{U^c} \uparrow \hat{R}_s^{U^c}.$$

Proof. Trivially, for every $n \in \mathbb{N}$, $R_{s_n}^{U^c} \le R_{s_{n+1}}^{U^c} \le R_s^{U^c}$. For the moment we fix $x \in U$ and $\varepsilon > 0$. Let $n \in \mathbb{N}$ and let h_n be the restriction of $R_{s_n}^{U^c}$ on U. By Proposition 2.4, $h_n = \overline{H}_U s_n$, so there exists $v_n \in \mathcal{S}(U)$ such that $v_n \ge s_n$ at ∂U, $v_n \ge 0$ at ∞, and

$$(3.3) \qquad v_n(x) < h_n(x) + 2^{-n}\varepsilon.$$

We know that $h_n \in \mathcal{H}^+(U)$ and $v_n - h_n \in \mathcal{S}^+(U)$. Now let

$$v := \Big(\sup_{n\in\mathbb{N}} h_n + \sum_{n=1}^{\infty}(v_n - h_n)\Big) \wedge (s|_U).$$

Then $v \in \mathcal{S}^+(U)$, $v \ge v_n \wedge (s|_U)$ for every $n \in \mathbb{N}$, and hence $v \ge \sup s_n = s$ at ∂U. Taking $v := s$ on U^c we obtain that $v \in \mathcal{S}^+(\mathbb{R}^d)$ and hence $R_s^{U^c} \le v$. Moreover, by (3.3), $v(x) \le \sup_n h_n(x) + \varepsilon = \sup_n R_{s_n}^{U^c}(x) + \varepsilon$. Thus $R_s^{U^c} = \sup_n R_{s_n}^{U^c}$ and, by Corollary 17.4, $\hat{R}_s^{U^c} = \sup_n \hat{R}_{s_n}^{U^c}$. □

Every $s \in \mathcal{S}^+(\mathbb{R}^d)$ is the increasing limit of a sequence (s_n) in $\mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d)$ (see Corollary 17.6 and its proof). Hence, by Proposition 3.1 and (3.2),

$$(3.4) \qquad \varepsilon_x^{U^c}(s) = \hat{R}_s^{U^c}(x) \quad \text{for every } s \in \mathcal{S}^+(\mathbb{R}^d).$$

Of course, $\varepsilon_x^{U^c} = \varepsilon_x$, if $x \in \mathbb{R}^d \setminus \overline{U}$, and $\varepsilon_x^{U^c} = \mu_x^U$ is supported by ∂U, if $x \in U$.

The next lemma will imply that, for every $z \in \partial U$, there exists a sequence (x_n) in U such that $x_n \to z$ and $\varepsilon_{x_n}^{U^c} \to \varepsilon_z^{U^c}$ (see Proposition 3.3). In particular, $\varepsilon_z^{U^c}$ is supported by ∂U.

[2] It can be shown that even $\hat{R}_u^A = R_u^A$ on A^c.

[3] In fact, for *all* $A \subset \mathbb{R}^d$, the mapping $s \mapsto \hat{R}_s^A$ is additive on $\mathcal{S}^+(\mathbb{R}^d)$.

LEMMA 3.2. *Let A be an arbitrary subset of $\mathbb{R}^d$, z a point in the boundary of A, and $v \in \mathcal{S}^+(\mathbb{R}^d)$ such that v is continuous at z. Then*

$$\hat{R}_v^A(z) = \liminf_{x \to z, x \in A^c} \hat{R}_v^A(x) = \liminf_{x \to z, x \in A^c} R_v^A(x). \tag{3.5}$$

Proof. By the lower semicontinuity of $\hat{R}_v^A$,

$$\hat{R}_v^A(z) \le \liminf_{x \to z, x \in A^c} \hat{R}_v^A(x) \le \liminf_{x \to z, x \in A^c} R_v^A(x).$$

Moreover, $\hat{R}_v^A(z) = \liminf_{x \to z} R_v^A(x)$ is the minimum of $\liminf_{x \to z, x \in A^c} R_v^A(x)$ and $\liminf_{x \to z, x \in A} R_v^A(x)$, where

$$\liminf_{x \to z, x \in A^c} R_v^A(x) \le \liminf_{x \to z, x \in A^c} v(x) = v(z) = \liminf_{x \to z, x \in A} v(x) = \liminf_{x \to z, x \in A} R_v^A(x).$$

Thus $\hat{R}_v^A(z) = \liminf_{x \to z, x \in A^c} R_v^A(x)$. □

PROPOSITION 3.3 (Köhn-Sieveking)**.** *For every $z \in \partial U$, there exists a sequence $(x_n) \in U$ such that $x_n \to z$ and $\varepsilon_{x_n}^{U^c} \to \varepsilon_z^{U^c}$.*

Proof. We choose a sequence (s_m) according to Lemma 2.5, where we may assume without loss of generality that $s_m \le 2^{-m}$. Then $s := \sum_{m=1}^{\infty} s_m \in \mathcal{S}^+(U) \cap \mathcal{C}_0(\mathbb{R}^d)$.

Let us fix $z \in \partial U$. By (3.4) and Lemma 3.2, there exists a sequence (x_n) in U such that $x_n \to z$ and $\varepsilon_{x_n}^{U^c}(s) \to \varepsilon_z^{U^c}(s)$. Knowing that $\liminf_{n \to \infty} \varepsilon_{x_n}^{U^c}(s_m) \ge \varepsilon_z^{U^c}(s_m)$ for every $m \in \mathbb{N}$, we see that $\lim_{n \to \infty} \varepsilon_{x_n}^{U^c}(s_m) = \varepsilon_z^{U^c}(s_m)$ for every $m \in \mathbb{N}$ and therefore $\lim_{n \to \infty} \varepsilon_{x_n}^{U^c} = \varepsilon_z^{U^c}$. □

A boundary point $z \in \partial U$ is *regular* provided,

$$\lim_{x \to z} H_U f(x) = f(z) \qquad \text{for every } f \in \mathcal{C}_0(\partial U). \tag{3.6}$$

By Proposition 2.4, Lemma 2.5, and Proposition 3.3, the set U_r of all regular points $z \in \partial U$ can be characterized by

$$U_r := \{z \in \partial U : \varepsilon_z^{U^c} = \varepsilon_z\} = \{z \in \overline{U} : \varepsilon_z^{U^c} = \varepsilon_z\}. \tag{3.7}$$

REMARK 3.4. Let us assume for a moment that U is relatively compact (this is the only situation, where we have defined $H(U)$ until now). Then we obtain that $\varepsilon_z^{U^c} \in \mathcal{M}_x(H(U))$. Since clearly $\varepsilon_x^{U^c} = \mu_x^U \in \mathcal{M}_x(H(U))$, if $x \in U$, this proves that the Choquet boundary $Ch_{H(U)}\overline{U}$ is contained in the set U_r of regular points $z \in \partial U$.

LEMMA 3.5. *Let U be relatively compact, $x \in \overline{U}$, and let ν be a measure in $\mathcal{M}_x(H(U))$ which is supported by $\mathrm{Ch}_{H(U)}\overline{U}$. Moreover, let $s \in \mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d)$ such that there is a sequence (h_n) in $H(U)$ which is increasing to $\hat{R}_s^{U^c}$ on $\overline{U}$. Then $\nu(s) = \varepsilon_x^{U^c}(s)$.*

Proof. Since $\hat{R}_s^{U^c} = s$ on the set U_r containing $Ch_{H(U)}\overline{U}$, we see that

$$\nu(s) = \nu(\hat{R}_s^{U^c}) = \lim_{n \to \infty} \nu(h_n) = \lim_{n \to \infty} h_n(x) = \hat{R}_s^{U^c}(x) = \varepsilon_x^{U^c}(s).$$

□

4 Some properties of Newtonian potentials

For a proof of Theorem 1.2 it will hence be helpful to show that the functions s in $\mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d)$ having the properties in Lemma 3.5 separate measures on ∂U. To that end we recall that there is a *Green function* G on $\mathbb{R}^d$,

$$G(x,y) := |x-y|^{2-d} \qquad (x, y \in \mathbb{R}^d),$$

and that, for every measure μ on $\mathbb{R}^d$, the *Newtonian potential* G^μ is defined by

$$G^\mu(x) := \int G(x,y)\, d\mu(y) = \int |x-y|^{2-d}\, d\mu(y) \qquad (x \in \mathbb{R}^d).$$

PROPOSITION 4.1. *If $G^\mu \not\equiv \infty$, then $G^\mu \in \mathcal{S}^+(\mathbb{R}^d)$ and G^μ is harmonic outside the support of μ.*

Proof. Let $v := G^\mu$. By Fatou's lemma, v is lower semicontinuous. By Fubini's theorem, for all $x \in \mathbb{R}^d$, $r > 0$, and $y \in B(x,r)$, $\mu_y^{B(x,r)}(v) \le v(y)$, where we have equality, if $\overline{B(x,r)}$ does not intersect the support of μ.

Let us now suppose that $v(x_0) < \infty$ for some $x_0 \in \mathbb{R}^d$. For every $r > 0$, the inequality $\sigma_{x_0,r}(v) < v(x_0) < \infty$ implies that $v < \infty$ $\sigma_{x_0,r}$-almost everywhere. Therefore $v < \infty$ on a dense subset of $\mathbb{R}^d$. Fixing $x \in \mathbb{R}^d$ and $r > 0$, we thus know that $v(y) < \infty$ for some $y \in B(x,r)$. A quick look at (1.1) shows that $\sigma_{x,r} = \mu_x^{B(x,r)} \le c\mu_y^{B(x,r)}$ for some $c > 0$ and hence $\sigma_{x,r}(v) \le c\mu_y^{B(x,r)}(v) \le cv(y) < \infty$. $\square$

We shall frequently use that, due to the symmetry of G and Fubini's theorem,

$$\int G^\sigma\, d\tau = \int G^\tau\, d\sigma \tag{4.1}$$

for all measures σ, τ on $\mathbb{R}^d$.

PROPOSITION 4.2 (Continuity principle of Evans-Vasilesco)**.** *Let A be a closed set in $\mathbb{R}^d$ and let μ be a measure on A. If $x_0 \in A$ such that $G^\mu|_A$ is continuous at x_0, then G^μ is continuous at x_0.*

Proof. G^μ being lower semicontinuous, the statement holds trivially if $G^\mu(x_0) = \infty$. So let us suppose that $G^\mu(x_0) < \infty$. Then, in particular, $\mu(\{x_0\}) = 0$. Hence, fixing $\varepsilon > 0$, we may find $r > 0$ such that $\nu := 1_{B(r,x_0)}\mu$ satisfies $G^\nu(x_0) < 2^{2-d}\varepsilon$. On the ball $B(r, x_0)$, the function $G^{\mu-\nu}$ is harmonic and hence continuous. Therefore our assumption on G^μ implies that the restriction of $G^\nu = G^\mu - G^{\mu-\nu}$ on A is continuous at x_0. So there exists $\delta > 0$ such that

$$G^\nu < 2^{2-d}\varepsilon \quad \text{on } A \cap B(x_0, 2\delta) \quad \text{and} \quad |G^{\mu-\nu} - G^{\mu-\nu}(x_0)| < \varepsilon \quad \text{on } B(x_0,\delta).$$

Finally, let us fix $x \in B(x_0, \delta)$. Choosing a point $y_x \in A$ minimizing the distance between x and points in A, we obtain that $|x - y_x| \le |x - x_0| < \delta$ and, for all $y \in A$,

$$|y_x - y| \le |y_x - x| + |x - y| \le 2|x - y|,$$

hence $y_x \in B(x_0, 2\delta)$ and

$$G^\nu(x) = \int |x-y|^{2-d}\, d\nu(y) \le 2^{d-2} \int |x-y_x|^{2-d}\, \nu(dy) = 2^{d-2} G^\nu(y_x) < \varepsilon.$$

Thus

$$|G^\mu(x) - G^\mu(x_0)| \le |G^{\mu-\nu}(x) - G^{\mu-\nu}(x_0)| + G^\nu(x) + G^\nu(x_0) < 3\varepsilon.$$

□

We shall need that every $G^\mu < \infty$ is the sum of continuous G^{μ_m}, $m \in \mathbb{N}$.

COROLLARY 4.3. *Let μ be a measure on $\mathbb{R}^d$ such that $G^\mu < \infty$. Then there exist measures μ_m on $\mathbb{R}^d$, $m \in \mathbb{N}$, such that $\mu = \sum_{m=1}^\infty \mu_m$, every G^{μ_m} is continuous, and the supports of the measures μ_m are compact and pairwise disjoint.*

Proof. By Lusin's theorem, there exists a sequence (K_m) of pairwise disjoint compact sets in $\mathbb{R}^d$ such that, defining

$$\mu_m := 1_{K_m}\mu, \qquad m \in \mathbb{N},$$

the functions $G^\mu|_{K_m}$ are continuous and $\mu = \sum_{m=1}^\infty \mu_m$.

Let us fix $m \in \mathbb{N}$. The functions $G^{\mu_m}|_{K_m}$ and $G^{\mu-\mu_m}|_{K_m}$ are lower semicontinuous, their sum is the continuous function $G^\mu|_{K_m}$. Therefore $G^{\mu_m}|_{K_m}$ is continuous. Thus, by Proposition 4.2, G^{μ_m} is continuous at each point $x_0 \in K_m$. On the complement of K_m, the function G^{μ_m} is harmonic and hence continuous as well. □

Let λ^d denote Lebesgue measure on $\mathbb{R}^d$.

PROPOSITION 4.4. *Let μ, ν be measures on $\mathbb{R}^d$ such that $G^\mu = G^\nu \not\equiv \infty$. Then $\mu = \nu$.*

Proof. a) It can be shown that $\Delta G^\rho = -\kappa_d \rho$, whenever ρ is a measure on $\mathbb{R}^d$ such that $G^\rho \not\equiv \infty$. This trivially implies our uniqueness statement.

b) Alternatively, let us first consider the case, where $\mu = f\lambda^d$ and $\nu = g\lambda^d$, $f, g \in \mathcal{C}^+(\mathbb{R}^d)$. Clearly, $G^\mu = G^\nu$ then implies that

$$G^{(f - f\wedge g)\lambda^d} = G^{(g - f\wedge g)\lambda^d}.$$

The left side is harmonic on the open set $V := \{g > f\}$, whereas the right side is not, unless V is empty. Thus $g \le f$. By symmetry, $f \le g$ as well.

To discuss the general case, let $\varphi\colon \mathbb{R}^d \to [0,\infty)$ with compact support. Then $\mu * \varphi, \nu * \varphi \in \mathcal{C}^+(\mathbb{R}^d)$ and

$$G^{(\mu*\varphi)\lambda^d} = G(\cdot,0) * (\mu * \varphi) = G^\mu * \varphi = G^\nu * \varphi = G(\cdot,0) * (\nu * \varphi) = G^{(\nu*\varphi)\lambda^d}.$$

So $\mu * \varphi = \nu * \varphi$ by the preceding part. Thus $\mu = \nu$. □

LEMMA 4.5. *Let U be an open set in $\mathbb{R}^d$ and $s, t \in \mathcal{S}^+(U)$ such that $s \le t$ λ^d-a.e. Then $s \le t$.*

Proof. Let $x \in U$. If $r > 0$ such that $\overline{B(x,r)} \subset U$, then the average $\lambda_{x,r}(s)$ of s on $B(x,r)$ is at most $s(x)$ and, since s is lower semicontinuous, $\lambda_{x,r}(s)$ tends to $s(x)$ as r tends to zero. Similarly for t. Clearly, $\lambda_{x,r}(s) \le \lambda_{x,r}(t)$ by our assumption on s, t. Thus $s(x) \le t(x)$. □

LEMMA 4.6. *The set of all G^μ, where $\mu = 1_K \lambda^d$, K compact set in $\mathbb{R}^d$, separates finite measures on $\mathbb{R}^d$.*

Proof. Let σ, τ be finite measures on $\mathbb{R}^d$, $\sigma \neq \tau$. Then $\lambda^d(\{G^\sigma \neq G^\tau\}) > 0$ by Proposition 4.4 and Lemma 4.5. So there exists a compact set K such that $\lambda^d(K) > 0$ and $G^\sigma > G^\tau$ on K or $G^\sigma < G^\tau$ on K. Thus, by (4.1),

$$\sigma(G^{1_K \lambda^d}) = \int_K G^\sigma \, d\lambda^d \neq \int_K G^\tau \, d\lambda^d = \tau(G^{1_K \lambda^d}).$$

□

We want to show that, for every closed set A in $\mathbb{R}^d$ and every measure μ on $\mathbb{R}^d$ satisfying $G^\mu \not\equiv \infty$,

$$\hat{R}^A_{G^\mu} = G^{\mu^A}, \qquad \text{where } \mu^A := \int \varepsilon^A_x \, d\mu(x).\text{[4]} \tag{4.2}$$

To that end we need a few preparations involving sweeping on open sets. Let us first note the following. If V is an open set in $\mathbb{R}^d$, then

$$R^V_u = \hat{R}^V_u$$

and hence $R^V_u \in \mathcal{S}^+(\mathbb{R}^d)$. Indeed, it suffices to note that $R^V_u = u$ on V and therefore $\hat{R}^V_u = u$ on V. Since $\hat{R}^V_u \in \mathcal{S}^+(\mathbb{R}^d)$, this implies that $\hat{R}^V_u \ge R^V_u$. The converse inequality is trivial.

PROPOSITION 4.7. *Let V and V_n, $n \in \mathbb{N}$, be open sets in $\mathbb{R}^d$.*

(*i*) *If $V_n \uparrow V$ and (s_n) is a sequence in $\mathcal{S}^+(\mathbb{R}^d)$ such that $s_n \uparrow s \in \mathcal{S}^+(\mathbb{R}^d)$, then $R^{V_n}_{s_n} \uparrow R^V_s$.*

(*ii*) *The mapping $s \mapsto R^V_s$ is additive on $\mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}^+_0(\mathbb{R}^d)$.*

Proof. i) Let us suppose that $V_n \uparrow V$ and let $s_s, s \in \mathcal{S}^+(\mathbb{R}^d)$, $s_n \uparrow s$. Obviously, for every $n \in \mathbb{N}$, $R^{V_n}_{s_n} \le R^{V_{n+1}}_{s_{n+1}} \le R^V_s$. Moreover, $\tilde{s} := \sup_{n\in\mathbb{N}} R^{V_n}_{s_n} \in \mathcal{S}^+(\mathbb{R}^d)$ and $\tilde{s} = s$ on V, hence $\tilde{s} \ge R^V_s$.

ii) We may choose V_n such that $\overline{V_n} \subset V_{n+1}$, $n \in \mathbb{N}$, and $V_n \uparrow V$. Since $R^{V_n}_s \le R^{\overline{V}_n}_s \le R^{V_{n+1}}_s$, (i) implies that

$$\lim_{n\to\infty} R^{\overline{V}_n}_s = R^V_s \qquad \text{for every } s \in \mathcal{S}^+(\mathbb{R}^d).$$

The proof is finished, since the mappings $s \mapsto R^{\overline{V}_n}_s$ are additive by (3.1) (applied to the open set $\mathbb{R}^d \setminus \overline{V}_n$). □

[4] In fact, (4.2) holds for *any* subset A of $\mathbb{R}^d$.

In particular, for every $y \in \mathbb{R}^d$, there exists a unique measure ε_y^V such that

$$\varepsilon_y^V(s) = R_s^V(y) \qquad \text{for every } s \in \mathcal{S}^+(\mathbb{R}^d).$$

PROPOSITION 4.8. *For all $x, y \in \mathbb{R}^d$ and open sets V in $\mathbb{R}^d$,*

$$R^V_{G(\cdot,x)}(y) = R^V_{G(\cdot,y)}(x).$$

Proof. By symmetry, it suffices to show that $R^V_{G(\cdot,x)}(y) \geq R^V_{G(\cdot,y)}(x)$. By the minimum principle,

$$R^V_{G(\cdot,x)} = G(\cdot, x), \qquad \text{if } x \in V. \tag{4.3}$$

Let $y \in \mathbb{R}^d$ and $\nu := \varepsilon_y^V$. Then, for every $x \in \mathbb{R}^d$, using (4.1) for the first equality,

$$G^\nu(x) = \int G(\cdot, x)\, d\nu = R^V_{G(\cdot,x)}(y).$$

In particular, by (4.3), $G^\nu = G(y, \cdot) = G(\cdot, y)$ on V. Thus $G^\nu \geq R^V_{G(\cdot,y)}$, that is, for every $x \in \mathbb{R}^d$, $R^V_{G(\cdot,x)}(y) = G^\nu(x) \geq R^V_{G(\cdot,y)}(x)$. □

If $s \in \mathcal{S}^+(\mathbb{R}^d)$ and A, B are subsets of $\mathbb{R}^d$, it is easily seen that $R_s^{A\cup B} \leq R_s^A + R_s^B$ and hence $\hat{R}_s^{A\cup B} \leq \hat{R}_s^A + \hat{R}_s^B$ by Corollary 17.4.

LEMMA 4.9. *For all $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, and $s \in \mathcal{S}^+(\mathbb{R}^d)$,*

$$\hat{R}_s^A = \hat{R}_s^{A\setminus\{x\}}.$$

Proof. We have $\hat{R}_s^A \leq \hat{R}_s^{A\setminus\{x\}} + \hat{R}_s^{\{x\}}$, where $\hat{R}_s^{\{x\}} = 0$, since

$$R_s^{\{x\}} \leq \inf_{\alpha>0} \alpha G(\cdot, x) = \infty \cdot 1_{\{x\}}.$$

Thus $\hat{R}_s^A \leq \hat{R}_s^{A\setminus\{x\}}$. The converse inequality holds trivially. □

PROPOSITION 4.10. *Let us suppose that A is a closed set in $\mathbb{R}^d$ and let μ be a measure on $\mathbb{R}^d$ such that $G^\mu \neq \infty$. Then*

$$\hat{R}^A_{G^\mu} = G^{\mu^A}. \tag{4.4}$$

Proof. a) Let $x \in A^c$ and $\varepsilon > 0$. Given $s \in \mathcal{S}^+(\mathbb{R}^d)$ with $s \geq G(\cdot, x)$ on A, the set

$$V := \{s + \varepsilon > G(\cdot, x)\}$$

is an open neighborhood of A (not containing x) such that $R^V_{G(\cdot,x)} \leq s + \varepsilon$. Thus, by Proposition 4.8, for every $y \in \mathbb{R}^d$,

$$\begin{aligned} R^A_{G(\cdot,x)}(y) &= \inf\{R^V_{G(\cdot,x)}(y) \colon V \text{ open } \supset A\} \\ &= \inf\{R^V_{G(\cdot,y)}(x) \colon V \text{ open } \supset A\} \geq R^A_{G(\cdot,y)}(x). \end{aligned}$$

b) Let $x, y \in \mathbb{R}^d$. We claim that

$$\hat{R}^A_{G(\cdot,x)}(y) = \hat{R}^A_{G(\cdot,y)}(x). \tag{4.5}$$

To that end we define $A' := A \setminus \{x\}$ and consider $z \in \mathbb{R}^d$. By (a), Lemma 4.9, and (3.4),

$$\begin{aligned} \hat{R}^{A'}_{G(\cdot,x)}(z) &\geq R^{A'}_{G(\cdot,x)}(z) \geq R^{A'}_{G(\cdot,z)}(x) \geq \hat{R}^{A'}_{G(\cdot,z)}(x) \\ &= \hat{R}^A_{G(\cdot,z)}(x) = \varepsilon^A_x(G(\cdot,z)) = G^{\varepsilon^A_x}(z). \end{aligned}$$

Since the function $G^{\varepsilon^A_x}$ is lower semicontinuous, we obtain that $\hat{R}^A_{G(\cdot,x)} \geq G^{\varepsilon^A_x}$. In particular, $\hat{R}^A_{G(\cdot,x)}(y) \geq G^{\varepsilon^A_x}(y) = \hat{R}^A_{G(\cdot,y)}(x)$. By symmetry,

$$\hat{R}^A_{G(\cdot,x)}(y) = \hat{R}^A_{G(\cdot,y)}(x). \tag{4.6}$$

3) Finally, let μ be a measure on $\mathbb{R}^d$ such that $G^\mu \not\equiv \infty$, and let $x \in \mathbb{R}^d$. Then, by (4.6) and the definition of μ^A (see (4.2)),

$$\begin{aligned} \hat{R}^A_{G^\mu}(x) &= \varepsilon^A_x(G^\mu) = \int \hat{R}^A_{G(\cdot,y)}(x)\,\mu(dy) = \int \hat{R}^A_{G(\cdot,x)}(y)\,\mu(dy) \\ &= \int \varepsilon^A_y(G(\cdot,x))\,\mu(dy) = \mu^A(G(\cdot,x)) = G^{\mu^A}(x). \end{aligned}$$

□

5 Proof of the classical result on H(U)

Now we are prepared to prove the main result on $H(U)$ in the classical case (see Theorem 1.2).

THEOREM 5.1. *Let U be a relatively compact open set in $\mathbb{R}^d$. Then $H(U)$ is simplicial, $\mathrm{Ch}_{H(U)}\overline{U} = U_r$, and $\varepsilon^{U^c}_x$, $x \in \overline{U}$, is* the *representing measure for x which is supported by $\mathrm{Ch}_{H(U)}\overline{U}$.*

Proof. Let K be a compact set in $\mathbb{R}^d$, $\rho := 1_K\lambda^d$, and $v := G^\rho$. By Proposition 4.10,

$$\hat{R}^{U^c}_v = G^\sigma, \qquad \text{where } \sigma := \rho^{U^c}.$$

By Proposition 4.2, there exist measures σ_m on $\mathbb{R}^d$, $m \in \mathbb{N}$, such that $\sigma = \sum_{m=1}^\infty \sigma_m$ and every G^{σ_m} is continuous on $\mathbb{R}^d$. Moreover, every G^{σ_m} is harmonic on U, since σ is supported by U^c. So we obtain that

$$h_n := \sum_{m=1}^n G^{\sigma_m}|_{\overline{U}} \in H(U) \quad \text{and} \quad h_n \uparrow G^\sigma = \hat{R}^{U^c}_v.$$

Now let us fix $x \in \overline{U}$ and a measure $\nu \in \mathcal{M}_x(H(U))$ which is supported by $Ch_{H(U)}\overline{U}$. By Lemma 3.5, $\nu(v) = \varepsilon^{U^c}_x(v)$. By Lemma 4.6, $\nu = \varepsilon^{U^c}_x$. Thus $H(U)$ is simplicial and $\varepsilon^{U^c}_x$ is the representing measure for x which is supported by $Ch_{H(U)}\overline{U}$.

We already know that $Ch_{H(U)}\overline{U} \subset U_r$ (see (1.5)). On the other hand, if $x \in U_r$, then $\varepsilon_x = \varepsilon^{U^c}_x$, hence ε_x is supported by $Ch_{H(U)}\overline{U}$, that is, $x \in Ch_{H(U)}\overline{U}$. □

6 Heat equation

In this section we will give an introduction to the potential theory of the heat equation $\sqcap u = 0$ on $\mathbb{R} \times \mathbb{R}$. The heat operator $\sqcap$ on $\mathbb{R} \times \mathbb{R}$ is defined by

$$\sqcap := \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}.$$

It will turn out that, for example, for the set V obtained from the open square $(0,1) \times (0,1)$ by removing the horizontal line segments $(0,1) \times \{1 - 2^{-n}\}$, $n \in \mathbb{N}$, the Choquet boundary of $\overline{V}$ with respect to the space $H^{\sqcap}(V)$ of all $h \in \mathcal{C}(\overline{V})$ such that $h|_V \in \mathcal{C}^2(V)$ and $\sqcap h = 0$ on V is a proper subset of the set of all points which are regular with respect to $\sqcap$ (see Example 8.4). Nevertheless, as we shall prove much later, all spaces $H^{\sqcap}(W)$, W relatively compact open in $\mathbb{R} \times \mathbb{R}$, are simplicial.

For every open subset W of $\mathbb{R} \times \mathbb{R}$, let

$$\mathcal{H}^{\sqcap}(W) := \{h \in \mathcal{C}^2(W) \colon \sqcap h = 0 \text{ on } W\}.$$

Simple examples of functions in $\mathcal{H}^{\sqcap}(\mathbb{R}\times\mathbb{R})$ are the functions $ax+b$, $\exp(ax+a^2t/2)$, $\sin(ax)\exp(-a^2t/2)$. We define

$$g_{\sqcap}(x,t) = \begin{cases} \dfrac{1}{\sqrt{2\pi t}} \exp(-\dfrac{x^2}{2t}), & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

LEMMA 6.1. *On* $(\mathbb{R} \times \mathbb{R}) \setminus \{0\}$ *the function* $g_{\sqcap}$ *is* C^∞ *and satisfies* $\sqcap g_{\sqcap} = 0$. *On* $\mathbb{R} \times (0, \infty)$,

$$\frac{\partial g_{\sqcap}}{\partial x} = -\frac{x}{t} g_{\sqcap} \quad \textit{and} \quad \frac{\partial^2 g_{\sqcap}}{\partial x^2} = (\frac{x^2}{t^2} - \frac{1}{t}) g_{\sqcap} = 2\frac{\partial g_{\sqcap}}{\partial t}. \tag{6.1}$$

Proof. The proof of (6.1) is straightforward. The C^∞-property of g on $(\mathbb{R}\times\mathbb{R})\setminus\{0\}$ follows from the fact that, for every $a > 0$, the function $\exp(-a/t)$ tends to zero very quickly as $t > 0$ tends to zero. □

Throughout this section let U be an (open) *standard rectangle*, that is,

$$U := (a,b) \times (\alpha, \beta),$$

where $a, b, \alpha, \beta \in \mathbb{R}$, $a < b$, $\alpha < \beta$. Moreover, let

$$\partial_{\sqcap} U := \partial U \setminus \{(x,\beta) \colon a < x < b\}, \qquad P := \{(a,\alpha),(b,\alpha)\}.$$

For every $\gamma \in \mathbb{R}$, let T^γ denote the reflection at the line vertical line $\{\gamma\} \times \mathbb{R}$, that is,

$$T_\gamma(x,t) := (2\gamma - x, t), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Let

$$U_\alpha := \mathbb{R} \times (\alpha, \infty) \quad \text{and} \quad Y_\alpha := \partial U_\alpha = \mathbb{R} \times \{\alpha\}.$$

We shall first consider the Dirichlet problem for the upper half-plane U_α. For every $f \in \mathcal{B}_b(Y_\alpha)$, that is, for every bounded Borel measurable function f on Y_α, let

$$H_\alpha f(x,t) := \int_{-\infty}^{\infty} g_{\sqcap}(x-y, t-\alpha) f(y,\alpha)\, dy, \qquad (x,t) \in U_\alpha.$$

PROPOSITION 6.2. *For every $f \in \mathcal{B}_b(Y_\alpha)$, the following holds:*

1. *$H_\alpha f \in \mathcal{C}^\infty(U_\alpha)$ and $\sqcap\, H_\alpha f = 0$.*
2. *For every $z^* \in Y_\alpha$,*

$$\liminf_{z\to z^*} f(z) \le \liminf_{z\to z^*} H_\alpha f(z) \le \limsup_{z\to z^*} H_\alpha f(z) \le \limsup_{z\to z^*} f(z).$$

3. *If (f_m) is a uniformly bounded sequence in $\mathcal{B}_b(Y_\alpha)$ which converges pointwise to f, then $(H_\alpha f_m)$ converges pointwise to $H_\alpha f$.*
4. *If $\gamma \in \mathbb{R}$ such that $f \circ T^\gamma = -f$, then $H_\alpha f = 0$ on $\{\gamma\} \times (\alpha, \infty)$.*

Proof. We may assume without loss of generality that $\alpha = 0$. To prove (1) we only have to justify that partial derivatives commute with the integral. To that end we fix $r > 1$ and define

$$\varphi(y) := \begin{cases} \sqrt{\frac{r}{2\pi}}, & |y| < 2r, \\ \sqrt{\frac{r}{2\pi}} \exp\left(-\frac{y^2}{8r}\right), & |y| \ge 2r. \end{cases}$$

Let $x \in (-r, r)$ and $t \in (1/r, r)$. If $|y| \ge 2r$, then $|y-x| \ge 2r - r = r$, $|y| \le 2|y-x|$, and $-(x-y)^2/(2t) \le -y^2/(8r)$. So

$$g_{\sqcap}(x-y,t) \le \varphi(y) \qquad \text{for every } y \in \mathbb{R}.$$

It is now easily verified that all partial derivatives of $g_{\sqcap}$ on $(-r,r) \times (1/r, r)$ are bounded by some function $p(y)\varphi(y)$, where $p(y)$ is a polynomial. Since all these functions are integrable, our claim follows.

The remaining statements are easily verified. □

Given $\gamma \in \mathbb{R}$, let

$$U^\gamma := (\mathbb{R} \setminus \{\gamma\}) \times (\alpha, \beta), \quad S^\gamma := (\mathbb{R} \setminus \{\gamma\}) \times [\alpha, \beta], \quad Y^\gamma := \{\gamma\} \times [\alpha, \beta].$$

For all $(x,t) \in S^\gamma$ and $f \in \mathcal{B}_b(Y^\gamma)$, we define

$$H^\gamma f(x,t) := \int_\alpha^b \Big|\frac{\partial g_{\sqcap}}{\partial x}(x-\gamma, s-\alpha)\Big| f(\gamma, s)\, ds.$$

PROPOSITION 6.3. *For all $\gamma \in \mathbb{R}$ and $f \in \mathcal{B}_b(Y^\gamma)$, the following holds:*

1. *$H^\gamma f$ is continuous on S^γ, $H^\gamma f = 0$ on $(\mathbb{R} \setminus \{\gamma\}) \times \{\alpha\}$.*

2. *On the open set U^γ, the function $H^\gamma f$ is $\mathcal{C}^\infty$ and satisfies $\sqcap H^\gamma f = 0$.*

3. $\limsup_{z\to(\gamma,\alpha)} |H^\gamma f(z)| \le \limsup_{z\to(\gamma,\alpha)} |f(z)|$.

4. *For every $t \in (\alpha, \beta]$,*

$$\liminf_{z\to(\gamma,t)} f(z) \le \liminf_{z\to(\gamma,t)} H^\gamma f(z) \le \limsup_{z\to(\gamma,t)} H^\gamma f(z) \le \limsup_{z\to(\gamma,t)} f(z). \tag{6.2}$$

5. *If (f_m) is a uniformly bounded sequence in $\mathcal{B}_b(Y^\gamma)$ which converges pointwise to f, then $(H^\gamma f_m)$ converges pointwise to $H^\gamma f$.*

6. *$H^\gamma 1(x,t) = 2G(|x-\gamma|/\sqrt{t-s})$, where $G(\xi) := \frac{1}{\sqrt{2\pi}}\int_\xi^\infty e^{-\eta^2/2}\,d\eta$.*

Proof. $(H^\gamma f)|_{U^\gamma} \in \mathcal{C}^\infty(U^\gamma)$ and $\sqcap H^\gamma f = 0$ on U^γ follow as the corresponding property in Proposition 6.2. The other statements are immediate consequences of Lebesgue's convergence theorem and the identity

$$\begin{aligned} H^\gamma f(x,t) &= \frac{1}{\sqrt{2\pi}}\int_\alpha^t \frac{|x-\gamma|}{(t-s)^{3/2}}\, e^{-\frac{(x-\gamma)^2}{2(t-s)}} f(\gamma,s)\,ds \\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{|x-\gamma|}{\sqrt{t-\alpha}}}^\infty e^{-\eta^2/2} f\big(\gamma, t-\frac{(x-\gamma)^2}{\eta^2}\big). \end{aligned}$$

□

Now let

$$U_{a,b} := (a,b)\times(\alpha,\beta), \quad S_{a,b} := (a,b)\times[\alpha,\beta], \quad Y_{a,b} := Y^a \cup Y^b,$$

and, for every $f \in \mathcal{B}_b(Y_{a,b})$,

$$f^a := f|_{Y^a}, \qquad f^b := f|_{Y^b}.$$

We define $V\colon \mathcal{B}_b(Y_{a,b}) \to \mathcal{B}_b(Y_{a,b})$ by

$$Vf(a,t) := H^b f^b(a,t) \quad \text{and} \quad Vf(b,t) := H^a f^a(b,t).$$

Then $V1 \le 2G((b-a)/\sqrt{\beta-\alpha}) < 1$. Therefore the inverse of $I+V$ on $\mathcal{B}_b(Y_{a,b})$ exists,

$$(I+V)^{-1} = \sum_{n=0}^\infty (-V)^n.$$

Given $f \in \mathcal{B}_b(Y_{a,b})$, we define

$$H_{a,b}f(x,t) := H^a\big((I+V)^{-1}f\big)^a(x,t) + H^b\big((I+V)^{-1}f\big)^b(x,t) \qquad (x,t)\in U.$$

PROPOSITION 6.4. *For every $f \in \mathcal{B}_b(Y_{a,b})$, the following holds:*

1. *$H_{a,b}f$ is continuous on $S_{a,b}$, $H_{a,b}f = 0$ on $(a,b)\times\{\alpha\}$.*

2. *On the open set $U_{a,b}$, the function $H_{a,b}f$ is $\mathcal{C}^\infty$ and satisfies $\sqcap H_{a,b}f = 0$.*

3. *For every $c \in \{a, b\}$,*

$$\limsup_{z\to(c,\alpha)} |H_{a,b}f(z)| \le \limsup_{z\to(c,\alpha)} |f(z)|,$$

4. *For all $z^* = (a, t)$ and $z^* = (b, t)$ with $t \in (\alpha, \beta]$,*

$$\liminf_{z\to z^*} f(z) \le \liminf_{z\to z^*} H_{a,b}f(z) \le \limsup_{z\to z^*} H_{a,b}f(z) \le \limsup_{z\to z^*} f(z). \tag{6.3}$$

5. *If (f_m) is a uniformly bounded sequence in $\mathcal{B}_b(Y_{a,b})$ which converges pointwise to f, then $(H_{a,b}f_m)$ converges pointwise to $H_{a,b}f$.*

Proof. (1)-(3) are immediate consequences of (1)-(3) in Proposition 6.3.

4) To prove (6.3) let $g := (I + V)^{-1}f$ and $z = (a, t)$ with $t \in (\alpha, \beta]$. By (6.2) in Proposition 6.3,

$$\liminf_{z\to z^*} g(z) \le \liminf_{z\to z^*} H^a g^a(z) \le \limsup_{z\to z^*} H^a g^a(z) = \lim_{z\to z^*} g(z).$$

Moreover,

$$\lim_{z\to z^*} V g(z) = H^b g^b(z^*) = \lim_{z\to z^*} H^b g^b(z).$$

Since $V g + g = f$ and $H_{a,b}f = H^a f^a + H^b f^b$, (6.3) holds. Similarly, if $z^* = (b, t)$.

5) Let (f_m) be a uniformly bounded sequence in $\mathcal{B}_b(Y_{a,b})$ such that $f_m \to f$. For every $m \in \mathbb{N}$, let $g_m := (I + V)^{-1}f_m$. Then the sequence (g_m) is uniformly bounded, say $|g_m| \le M$ for every $m \in \mathbb{N}$. Moreover, (g_m) converges pointwise to $g := (I + V)^{-1}f$. Indeed, $|V^k f_m| \le M\|V\|^k$ for all $k, m \in \mathbb{N}$ and $\lim_{m\to\infty} V^k f_m = V^k f$ for every $k \in \mathbb{N}$, and hence

$$\lim_{m\to\infty} g_m = \lim_{m\to\infty} \sum_{k=0}^{\infty} (-V)^k f_m = \sum_{k=0}^{\infty} \lim_{m\to\infty} (-V)^k f_m = \sum_{k=0}^{\infty} (-V)^k f = g.$$

Thus $\lim_{m\to\infty} H_{a,b}f_m = H_{a,b}f$.

□

Finally, let $f \in \mathcal{B}_b(\partial_\sqcap U)$. There exists a unique function f_α on Y_α such that $f_\alpha = f$ on $(a, b) \times \{\alpha\}$ and $f_\alpha \circ T^\gamma = -f_\alpha$ for $\gamma = a$ and $\gamma = b$. For all points (x, t) in $(a, b) \times (\alpha, \beta]$, let

$$\boxed{H_U^\sqcap f(x, t) := H_\alpha f_\alpha(x, t) + H_{a,b}(f|_{Y_{a,b}})(x, t).}$$

PROPOSITION 6.5. *For every $f \in \mathcal{B}_b(\partial_\sqcap U)$, the following holds:*

1. *$H_U^\sqcap f$ is continuous on $(a, b) \times (\alpha, \beta]$.*

2. *On the open set U, the function $H_U^\sqcap f$ is $\mathcal{C}^\infty$ and satisfies $\sqcap H_U^\sqcap f = 0$.*

3. *For every* $z^* \in \partial_\sqcap U \setminus P$,

$$\liminf_{z\to z^*} f(z) \le \liminf_{z\to z^*} H_{a,b}f(z) \le \limsup_{z\to z^*} H_{a,b}f(z) \le \limsup_{z\to z^*} f(z).$$

4. *If* $z^* \in \partial_\sqcap U$ *such that* $\lim_{z\to z^*} f(z) = 0$, *then* $\lim_{z\to z^*} H_U^\sqcap f(z) = 0$.

5. *If* (f_m) *is a uniformly bounded sequence in* $\mathcal{B}_b(\partial_\sqcap U)$ *which converges pointwise to* f, *then* $(H_U^\sqcap f_m)$ *converges pointwise to* $H_U^\sqcap f$.

Proof. Immediate consequence of Proposition 6.2 and Proposition 6.4. □

LEMMA 6.6. *Let* $g \in C^2(U)$, $\sqcap g \le 0$, $\liminf_{z\to z^*} g(z) \ge 0$ *for every* $z^* \in \partial_\sqcap U$. *Then* $g \ge 0$.

Proof. Suppose that there is a point $z_1 \in U$ such that $g(z_1) < 0$. We fix $\tilde\beta \in (\alpha, \beta)$ such that $z_1 \in \tilde U := (a,b) \times (\alpha, \tilde\beta)$ and define

$$\tilde w(x,t) := \frac{1}{\tilde\beta - t}, \qquad (x,t) \in \tilde U.$$

Then $\sqcap \tilde w = -(\tilde\beta - t)^{-2} < 0$ and $\lim_{z\to z^*} \tilde w(z) = \infty$ for every $z^* \in \partial\tilde U \setminus \partial_\sqcap \tilde U$. We choose $\varepsilon > 0$ such that $g(z_1) + \varepsilon\tilde w(z_1) < 0$ and consider

$$\tilde g := g|_{\tilde U} + \varepsilon \tilde w.$$

Obviously, $g \in \mathcal{C}^2(\tilde U)$ and $\sqcap \tilde g < 0$. Moreover, $\tilde g(z_0) < 0$ and $\liminf_{z\to z^*} \tilde g \ge 0$ for every $z^* \in \partial \tilde U$. Hence there exists a point $z_0 \in \tilde U$ such that $\tilde g(z_0) = \inf \tilde g(\tilde U)$. Then

$$\frac{\partial \tilde g}{\partial t}(z_0) = 0 \quad \text{and} \quad \frac{\partial^2 \tilde g}{\partial x}(z_0) \ge 0.$$

So $\sqcap \tilde g(z_0) \ge 0$, a contradiction. □

LEMMA 6.7. *Let* $z_0 = (x_0, t_0) \in U$ *and* $M > 0$. *Then there exists* $h \in \mathcal{H}^+(U)$ *such that* $h(x_0) \le 1$ *and* $\lim_{z\to z^*} h(z) \ge M$ *for every* $z^* \in P = \{(a,\alpha),(b,\alpha)\}$.

Proof. Let $\gamma := \sqrt{t_0 - \alpha}/2$, $\delta := (\alpha/M)^2$, and, for all $(x,t) \in U$,

$$h(x,t) := \frac{\gamma}{\sqrt{t - (\alpha - \delta)}} \left(\exp\Big(-\frac{(x-a)^2}{2(t-(\alpha-\delta))}\Big) + \exp\Big(-\frac{(x-b)^2}{2(t-(\alpha-\delta))}\Big) \right),$$

Then $h \in \mathcal{H}^+(U)$, $h(x_0,t_0) \le 2\gamma/\sqrt{t-(\alpha-\delta)} \le 1$, and, for every $z^* \in P$, $\lim_{z\to z^*} h(z) \ge \gamma/\sqrt{\delta} = M$. □

PROPOSITION 6.8. *Let* $g \in \mathcal{C}^2(U)$ *be bounded,* $\sqcap g \le 0$, $\liminf_{z\to z^*} g(z) \ge 0$ *for every* $z^* \in \partial_\sqcap U \setminus P$. *Then* $g \ge 0$.

Proof. Let $z_0 \in U$, $\varepsilon > 0$, and $K > 0$ such that $|g| \le K$. We take $M := K/\varepsilon$, choose $h \in \mathcal{H}^+(U)$ according to Lemma 6.7, and define

$$\tilde{g} := g + \varepsilon h.$$

Then $\sqcap \tilde{g} = \sqcap g \le 0$ and $\lim_{z\to z^*} \tilde{g}(z) \ge 0$ for every $z^* \in \partial_\sqcap U$. By Lemma 6.6, $\tilde{g} \ge 0$. In particular, $g(z_0) \ge \tilde{g}(z_0) - \varepsilon h(z_0) \ge -\varepsilon$. $\square$

COROLLARY 6.9. *Let $h \in \mathcal{H}_b^\sqcap(U)$ and $\lim_{z\to z^*} h(z) = 0$ for every $z^* \in \partial_\sqcap U \setminus P$. Then $h = 0$.*

PROPOSITION 6.10. *$H_U^\sqcap$ is a kernel, $H_U^\sqcap 1 = 1$. Moreover, for all $f \in \mathcal{C}(\partial_\sqcap U)$ and $z^* \in \partial_\sqcap U$,*

$$\lim_{z\to z^*} H_U^\sqcap f(z) = f(z^*). \tag{6.4}$$

Proof. Let $f \in \mathcal{B}_b(\partial_\sqcap U)$. Then $H_U^\sqcap f \in \mathcal{H}_b^\sqcap(U)$. By Proposition 6.5, for every $z^* \in \partial_\sqcap U \setminus P$,

$$\liminf_{z\to z^*} H_U^\sqcap f(z) \ge \liminf_{z\to z^*} f(z) \ge 0.$$

By Lemma 6.8, $H_U^\sqcap f \ge 0$. It is easily verified that $f \mapsto H_U^\sqcap f$ is linear and σ-continuous (that is, $H_U^\sqcap f_n \uparrow H_U^\sqcap f$ if $f_n \uparrow f$).

Let us suppose now that f is continuous and let $h := H_U^\sqcap f$. Then

$$\lim_{z\to z^*} H_U^\sqcap f(z) = f(z^*) \qquad \text{for every } z^* \in \partial_\sqcap U \setminus P.$$

In particular, $H_U^\sqcap 1 = 1$ by Corollary 6.9.

Finally, let $z^* \in P$ and $\tilde{f} := f - f(z^*)$. By Proposition 6.5, $\lim_{z\to z^*} H_U^\sqcap \tilde{f}(z) = 0$. Since $H_U^\sqcap \tilde{f} = H_U^\sqcap f - f(z^*) H_U^\sqcap 1 = H_U^\sqcap f - f(z^*)$, we conclude that $\lim_{z\to z^*} H_U^\sqcap f(z) = f(z^*)$, and the proof is finished. $\square$

Let

$$H^\sqcap(U) := \{h \in \mathcal{C}(\overline{U})\colon h|_U \in \mathcal{H}^\sqcap(U)\}.$$

COROLLARY 6.11.

1. *For every $f \in \mathcal{C}(\partial_\sqcap U)$, there exists a unique function $h_f \in H^\sqcap(U)$ such that $h_f = f$ on $\partial_\sqcap U$. The mapping $f \mapsto h_f$ is linear and positive. In particular, $H^\sqcap(U)$ is isomorphic to $C(\partial_\sqcap U)$ and $\mathrm{Ch}_{H^\sqcap(U)} \overline{U} = \partial_\sqcap U$.*

2. *Let $\beta' \in (\beta, \infty)$ and $U' := (a,b) \times (\alpha, \beta')$. Then, for every $f \in \mathcal{C}(\partial_\sqcap U')$,*

$$\big(H_{U'}^\sqcap f\big)|_U = H_U^\sqcap (f|_{\partial_\sqcap U}).$$

3. *For every point $t := (x,t) \in U$, the measure $H_U^\sqcap(z,\cdot)$ is supported by the set of all $z^* = (y,s) \in \partial U$ such that $s \le t$.*

Proof. Easy consequence of Proposition 6.5 and Corollary 6.9. (The second statement also follows directly from the definitions.) $\square$

REMARK 6.12. 1. For every open subset W of $\mathbb{R}\times\mathbb{R}$, $\mathcal{H}^{\cap}(W) \subset \mathcal{C}^{\infty}(U)$. Indeed, if $U \subset W$ and $h \in \mathcal{H}^{\cap}(W)$, then $h|_U = H_U^{\cap}(h|_{\partial_{\cap} U})$ by Corollary 6.9, where the right side is contained in $\mathcal{C}^{\infty}(U)$.

2. For all $f \in \mathcal{B}_b(\partial_{\cap} U)$ and $z^* \in \partial_{\cap} U$,

$$\liminf_{z\to z^*} f(z) \le \liminf_{z\to z^*} H_U^{\cap} f(z) \le \limsup_{z\to z^*} H_U^{\cap} f(z) \le \limsup_{z\to z^*} f(z).$$

Indeed, to prove the first inequality it suffices to observe that there exists a function $\tilde f \in \mathcal{C}(\partial U)$ such that $\tilde f \le f$ and $\lim_{z\to z^*} \tilde f(z) = \liminf_{z\to z^*} f(z)$ and to use that (6.4) holds for the function $\tilde f$. The last inequality follows from the first one replacing f by $-f$.

To show that functions which are harmonic with respect to heat equation fit into a very general frame, let us introduce the definition of a *family of harmonic kernels*.

7 Families of harmonic kernels

Let X be a locally compact space with countable base and let $\mathcal{U}$ be a base for the topology of X consisting of relatively compact open sets in X. Let $(H_U)_{U\in\mathcal{U}}$ be a family of kernels on X such that each H_U is a *sweeping kernel for* U, that is, $H_U(x, X\setminus \partial U) = 0$ for every $x \in U$ and $H_U(x,\cdot) = \varepsilon_x$ for every $x \in U^c$.

For every open subset W of X, let $\mathcal{U}(W)$ denote the set of all $U \in \mathcal{U}$ with $\overline{U} \subset W$, let ${}^*\mathcal{H}(W)$ denote the set of all *hyperharmonic* functions on W, that is,

$${}^*\mathcal{H}(W) := \{v|\ v\colon W \to (-\infty,\infty],\ v \text{ is l.s.c.},\ H_U v \le v \text{ for every } U \in \mathcal{U}(W)\},$$

let $\mathcal{S}(W)$ denote the set of all *superharmonic* functions on W, that is,

$$\mathcal{S}(W) := \{s \in {}^*\mathcal{H}(W)\colon (H_U s)|_U \in \mathcal{C}(U) \text{ for every } U \in \mathcal{U}(W)\},$$

and let $\mathcal{H}(W)$ denote the set of all *harmonic* functions on W, that is

$$\mathcal{H}(W) := \{h \in \mathcal{C}(W)\colon H_U h = h \text{ for every } U \in \mathcal{U}(W)\}.$$

Of course,

$$\mathcal{H}(W) = \mathcal{S}(W) \cap (-S(W)) = {}^*\mathcal{H}(W) \cap (-{}^*\mathcal{H}(W)).$$

For all numerical functions f on X and all subsets A of X, the *reduced function* R_f^A is defined by

$$R_f^A := \inf\{u \in {}^*\mathcal{H}^+(X)\colon u \ge f \text{ on } A\}$$

(writing R_f instead of R_f^X we have of course $R_f^A = R_{1_A f}$).

Given a filter $\mathcal{F}$ on $U \in \mathcal{U}$ which converges to a point $z \in \partial U$, we say that $\mathcal{F}$ is *regular* (with respect to U), if $\lim_{x,\mathcal{F}} H_U(x,\cdot) = \varepsilon_z$, that is, if $\lim_{\mathcal{F}} H_U f = f(z)$ for every $f \in \mathcal{K}(X)$, that is, for every continuous real function on X having compact support.

Definition 7.1. $(H_U)_{U\in\mathcal{U}}$ is a *family of harmonic kernels*, if the following axioms are satisfied:

(H_1) For every $x \in X$, $\lim_{U \downarrow x} H_U 1(x) = 1$.

(H_2) $H_V H_U = H_U$ for all $V, U \in \mathcal{U}$ with $\overline{V} \subset U$.

(H_3) For all $U \in \mathcal{U}$ and $f \in \mathcal{B}(X)$ which are bounded on ∂U, the function $H_U f$ is continuous on U.

(H_4) For every $x \in U \in \mathcal{U}$, there exists an *Evans function* w, that is, a function $w \in {}^*\mathcal{H}^+(U)$ such that $w(x) < \infty$ and $\lim_{\mathcal{F}} w = \infty$ for every non-regular ultrafilter $\mathcal{F}$ on U.

(H_5) There exists a strictly positive continuous real function in $\mathcal{S}(X)$ and ${}^*\mathcal{H}^+(X)$ is linearly separating, that is, given $x, y \in X$, $x \neq y$, and $\lambda \geq 0$, there exists a function $v \in {}^*\mathcal{H}^+(X)$ such that $v(x) \neq \lambda v(y)$.

EXAMPLE 7.2. 1. Of course, classical potential theory provides examples. Let X be an open set $\mathbb{R}^d$, $d \geq 1$, such that, if $d = 1$, $X \neq \mathbb{R}$ and, if $d = 2$, $\mathbb{R}^2 \setminus X$ contains a closed disc $\overline{B(x_0, r_0)}$, $x_0 \in \mathbb{R}^2$, $r_0 > 0$.[5] Let $\mathcal{U}$ be the set of all open balls $B(x, r)$ such that $\overline{B(x, r)} \subset X$ and let $H_{B(x,r)}$ denote the kernels given by the corresponding Poisson integrals.

By (1.1) and (1.2), (H_1) – (H_3) hold. Moreover, (H_4) holds trivially, since every filter on $U \in \mathcal{U}$ is regular. Clearly $1 \in \mathcal{S}(X)$, even $1 \in \mathcal{H}(X)$. Thus, to prove (H_5), we only have to separate points in X by functions in ${}^*\mathcal{H}^+(X)$. If $d \geq 3$, this is achieved by the functions $x \mapsto |x - y|^{2-d}$, $y \in X$. If $d = 1$ and $x_0 \in \mathbb{R} \setminus X$, it suffices to consider the functions $x \mapsto (x - x_0)^+$ and $x \mapsto (x_0 - x)^+$ (which are harmonic on X). In the case $d = 2$, the functions $x \mapsto \ln|x - y| - \ln(r_0/2)$, $y \in B(x_0, r_0/2)$, are contained in $\mathcal{H}^+(X)$ and separate the points of X.

2. Moreover, the following is obvious. If $(H_U)_{U \in \mathcal{U}}$ is a family of harmonic kernels on X and if Y is an open set in X, then the restrictions of the harmonic kernels H_U on Y, $U \in \mathcal{U}$ with $\overline{U} \subset Y$, yield a family of harmonic kernels on Y.

Given a family $(H_U)_{U \in \mathcal{U}}$ of harmonic kernels on X, a relatively compact open subset W of X, and $f \in \mathcal{B}_b(\partial W)$, the *upper solution and the lower solution* to the corresponding Dirichlet problem are defined by

$$\begin{aligned} \overline{H}_W f &:= \inf\{s \in \mathcal{S}(W) \colon \liminf_{z \to z^*} s(z) \geq f(z^*) \text{ for every } z^* \in \partial W\}, \\ \underline{H}_W f &:= \sup\{t \in -\mathcal{S}(W) \colon \limsup_{z \to z^*} t(z) \leq f(z^*) \text{ for every } z^* \in \partial W\}. \end{aligned}$$

If $\overline{H}_W f = \underline{H}_W f$, the function f is called *resolutive*. It can be shown (but we shall not use it) that every $f \in \mathcal{B}_b(\partial W)$ is resolutive and $H_W f := \overline{H}_W f = \underline{H}_W f \in \mathcal{H}(W)$. By definition, the set of all regular points of W is the set

$$W_r := \{z^* \in \partial W \colon \lim_{z \to z^*} \overline{H}_W f(z) = \lim_{z \to z^*} \underline{H}_W f(z) = f(z^*) \text{ for every } f \in \mathcal{C}(\partial W)\}.$$

For every relatively compact open set W in X, we define a function space $H(W)$ by

$$H(W) := \{h \in \mathcal{C}(\overline{W}) \colon h|_W \in \mathcal{H}(W)\}.$$

[5] In fact, it would be sufficient that $\mathbb{R}^2 \setminus X$ is non-polar.

8 Heat equation, continued

Let us now return to the heat equation on $\mathbb{R} \times \mathbb{R}$, where we take

$$\mathcal{U} := \{(a,b) \times (\alpha,\beta) \colon a,b,\alpha,b \in \mathbb{R}, a < b, \alpha < \beta\},$$

and, for all $U \in \mathcal{U}$, $z \in \mathbb{R} \times \mathbb{R}$, and Borel subsets B of $\mathbb{R} \times \mathbb{R}$,

$$H_U^{\sqcap}(z,B) := \begin{cases} H_U^{\sqcap}(1_B|_{\partial_\sqcap U})(z), & z \in U, \\ \varepsilon_z(B), & z \in U^c. \end{cases}$$

THEOREM 8.1. *1. $(H_U^{\sqcap})_{U\in\mathcal{U}}$ is a family of harmonic kernels.*

2. For every open subset W of $\mathbb{R} \times \mathbb{R}$, $\mathcal{H}(W) = \mathcal{H}^{\sqcap}(W)$.

3. If $s \in \mathcal{C}^2(W)$ and $\sqcap\, s \le 0$, then $s \in \mathcal{S}(W)$.

Proof. 1. Of course, (H_1) holds, since $H_U^{\sqcap}1 = 1$ for every $U \in \mathcal{U}$. Given $U \in \mathcal{U}$ and $f \in \mathcal{B}_b^+(\mathbb{R} \times \mathbb{R})$, we know that $h := (H_U^{\sqcap} f)|_U \in \mathcal{H}^{\sqcap}(U) \subset \mathcal{C}^\infty(U)$. This implies (H_3). Moreover, for every $V \in \mathcal{U}(U)$, $H_V^{\sqcap} h = h$ by Corollary 6.9. Therefore (H_2) is satisfied. Since $1 \in \mathcal{H}(\mathbb{R} \times \mathbb{R})$ and the positive harmonic functions e^{ax+a^2t}, $a \in \mathbb{R}$, separate the points in $\mathbb{R} \times \mathbb{R}$, (H_5) holds. For the moment we postpone the proof of (H_4).

Let W be an open subset of $\mathbb{R}\times\mathbb{R}$, $s \in \mathcal{C}^2(W)$, and $\sqcap\, s \le 0$. By Proposition 6.10 and Corollary 6.8, $s - H_U s \ge 0$ for every $U \in \mathcal{U}$. Therefore $s \in \mathcal{S}(W)$.

In particular, every function in $\mathcal{H}^{\sqcap}(W) \subset \mathcal{H}(W)$. Conversely, let $h \in \mathcal{H}(W)$. By Corollary 6.9, $h = H_U^{\sqcap} h \in \mathcal{H}(U) \subset \mathcal{H}^{\sqcap}(U)$ for every $U \in \mathcal{U}$. Thus $h \in \mathcal{H}^{\sqcap}(W)$.

Finally, let us fix again $U = (a,b) \times (\alpha,\beta) \in \mathcal{U}$ and define $w(x,t) := (\beta - t)^{-1}$, $(x,t) \in U$. Then $\sqcap\, w > 0$ and hence w is superharmonic. Let $\mathcal{F}$ be a non-regular ultrafilter on U. By Proposition 6.10, $\mathcal{F}$ cannot converge to a point $z^* \in \partial_\sqcap U$. So $\mathcal{F}$ converges to a point $z^* \in \partial U \setminus \partial_\sqcap U$ and hence $\lim_{\mathcal{F}} w = \infty$. □

LEMMA 8.2. *Let $U \in \mathcal{U}$ and $s \in \mathcal{S}(U)$ such that $\liminf_{z\to z^*} s(z) \ge 0$ for every $z^* \in \partial_\sqcap U$. Then $s \ge 0$.*

Proof. We fix $z \in U$. As before let $w(x,t) := (\beta - t)^{-1}$, $(x,t) \in U$. Let $\varepsilon > 0$ and $\tilde{s} := s + \varepsilon w$. Then $\tilde{s} \in \mathcal{S}(U)$ and $\liminf_{z\to z^*} \tilde{s}(z) > 0$ for every $z^* \in \partial U$. So there exist $a < \tilde{a} < \tilde{b} < b$ and $\alpha < \tilde{\alpha} < \tilde{\beta} < \beta$, such that defining $\tilde{U} := (\tilde{a},\tilde{b}) \times (\tilde{\alpha},\tilde{\beta})$ we have $s > 0$ on $\partial_\sqcap \tilde{U}$. Then $\tilde{s} \ge H_{\tilde{U}}^{\sqcap} \tilde{s} \ge 0$ on $\tilde{U}$. In particular, $\tilde{s}(x) > 0$. Thus $s \ge 0$. □

PROPOSITION 8.3. *Let $U \in \mathcal{U}$. For every $f \in \mathcal{C}(\partial U)$,*

$$\overline{H}_U f = \underline{H}_U f = H_U^{\sqcap} f. \tag{8.1}$$

In particular, $U_r = \partial_\sqcap U$.

Proof. Let $f \in \mathcal{C}(\partial U)$. By Lemma 8.2, $\overline{H}_U f \geq \underline{H}_U f$. By Proposition 6.5 and Theorem 8.1, $H_U^{\cap} f \in \mathcal{S}(U) \cap (-\mathcal{S}(U))$ and $\lim_{z \to z^*} H_U^{\cap} f(z) = f(z^*)$ for every $z^* \in \partial_{\cap} U$. Defining w as before we see that, for every $\varepsilon > 0$,

$$H_U^{\cap} f + \varepsilon w \geq \overline{H}_U f \quad \text{and} \quad \underline{H}_U f \geq H_U^{\cap} f - \varepsilon w.$$

Therefore (8.1) holds and the identity $U_r = \partial_{\cap} U$ follows from Proposition 6.11. □

EXAMPLE 8.4. An example, where, contrary to the potential theory associated with the Laplace operator, the Choquet boundary and the set of regular points do not coincide, can be obtained in the following way: Let

$$V := \bigcup_{n=1}^{\infty} V_n, \qquad \text{where } V_n := (0,1) \times (1 - 2^{-n+1}, 1 - 2^{-n}),\ n \in \mathbb{N},$$

and let $U := (0,1) \times (0,1)$. We claim that $H(V) = H(U)$ and hence

$$Ch_{H(V)} \overline{U} = \partial_{\cap} U, \qquad \text{whereas } U_r = \partial U.$$

Indeed, trivially $H(V) \subset H(U)$. Conversely, let $h \in H(V)$. There exists $g \in H(U)$ such that $g = h$ on $\partial_{\cap} U$. Then $h = g$ on $\overline{V}_1$ by Corollary 6.9. By induction, we obtain that $h = g$ on every $\overline{V}_n$, $n \in \mathbb{N}$, and therefore $h = g \in H(U)$. Thus $H(V) = H(U)$ whence $Ch_{H(V)} \overline{U} = \partial_{\cap} U$.

To see that $U_r = \partial U$ we fix $f \in \mathcal{C}(\partial V)$ such that (without loss of generality) $0 \leq f \leq 1$. For every $(x,t) \in V_n$, $n \in \mathbb{N}$, let

$$h(x,t) := H_{V_n} f(x,t), \qquad w(x,t) := \frac{1}{t - 2^{n-1}}.$$

Then, for every $\varepsilon > 0$, $s := h + \varepsilon w \in \mathcal{S}(V)$, $t := h - \varepsilon w \in -\mathcal{S}(V)$,

$$\liminf_{z \to z^*} s(z) \geq f(z^*) \geq \limsup_{z \to z^*} t(z) \qquad \text{for every } z^* \in \partial V,$$

and therefore

$$h + \varepsilon w \geq \overline{H}_V f, \qquad \underline{H}_V f \geq h - \varepsilon w.$$

By Lemma 8.2, this shows that $\overline{H}_V f = \underline{H}_V f = h$. Since $\bigcup_{n=1}^{\infty} (V_n)_r = \partial_{\cap} U$, we conclude that $\partial_{\cap} U \subset U_r \subset \partial U$.

Finally, let $z^* \in \partial U \setminus \partial_{\cap} U$, that is, $z = (x, \beta)$, where $x \in (a,b)$. Given $\varepsilon \in (0,1)$, there exists $0 < \eta < \min(x - a, b - x, \beta - \alpha)$ such that

$$|f(z) - f(z^*)| < \varepsilon \text{ for all } z \in \partial V \cap \big([x - \eta, x + \eta] \times [\beta - \eta, \beta]\big).$$

There exists $0 < \delta < \varepsilon$ such that

$$H_U(z, [x - \eta, x + \eta] \times \{0\}) > 1 - \varepsilon \quad \text{for all } z \in (x - \delta, x + \delta) \times (0, \delta).$$

We fix $n \in \mathbb{N}$ and $z = (y, s) \in V_n$ such that such that $2^{-n+1} < \delta$ and $|y - x| < \delta$. Let $A_n := [x - \eta, x + \eta] \times \{1 - 2^{-n+1}\}$. By Corollary 6.11 and translation invariance, $H_{V_n}(z, A_n) > 1 - \varepsilon$. Since $H_{V_n} 1 = 1$, this implies that

$$0 \leq H_{V_n}(1 - 1_{A_n}) f(z) \leq H_{V_n}(1 - 1_{A_n})(z) < \varepsilon. \tag{8.2}$$

Moreover, since $|f - f(z^*)| < \varepsilon$ on A_n, we know that

$$H_{V_n} 1_{A_n} |f - f(z^*)| \le \varepsilon H_{V_n} 1_{A_n} \le \varepsilon. \tag{8.3}$$

Combining (8.2) and (8.3) we see that

$$|H_{V_n} f(z) - f(z^*)| = H_{V_n} |f(z) - f(z^*)| < 3\varepsilon.$$

So $z^* \in V_r$ and we have shown that $V_r = \partial U$.

Let us finally mention the following immediate consequence of Corollary 6.11.

PROPOSITION 8.5. *Let W be an open subset of $\mathbb{R} \times \mathbb{R}$, $t \in \mathbb{R}$, $A := \mathbb{R} \times (t, \infty)$. Then, for every $v \in {}^*\mathcal{H}^+(W)$, the function $1_A v$ is hyperharmonic on W.*

9 Function cones

Let X be a locally compact space with countable base. Given a set $\mathcal{F}$ of numerical functions on X, we define

$$W(\mathcal{F}) \ := \ \{f_1 \wedge \cdots \wedge f_n \colon n \in \mathbb{N}, f_j \in \mathcal{F}\}.$$

So $\mathcal{F}$ is $\wedge$-stable provided $W(\mathcal{F}) = \mathcal{F}$.

We say that $\mathcal{F}$ is *linearly separating*, if, for every pair of points $x \ne y$ in X and every $\lambda \ge 0$, there exists $f \in \mathcal{F}$ such that $f(x) \ne \lambda f(y)$.

A real function f on X is said to be *dominated* by a real function $g \ge 0$ on X, $f \in o(g)$, if all sets $\{|f| > \varepsilon g\}$, $\varepsilon > 0$, are relatively compact. If $g > 0$, this means that f/g vanishes at infinity. If $\mathcal{G}$ is a set of positive real functions on X, we denote by $o(\mathcal{G})$ the set $\bigcup_{g \in \mathcal{G}} o(g)$.

Definition 9.1. A convex cone $S \subset \mathcal{C}(X)$ is called a *function cone* (on X), if S satisfies the following conditions:

(F_1) There exists a strictly positive $s_0 \in S$.

(F_2) S^+ is linearly separating.

(F_3) S is *adapted*, that is, $S \subset o(S^+)$.

If, moreover, S is a vector space, then S is called a *function space*.

EXAMPLES 9.2. 1. If X is compact, then (F_3) holds trivially and hence $\mathcal{C}(X)$ is a function cone.

2. More generally, in any case $\mathcal{C}_0(X)$ is a function cone. To verify (F_3), it suffices to note that, given a function $f \in \mathcal{C}_0(X)$, the function $g := \sum_{n=1}^\infty 2^{-n} \wedge |f|$ is contained in $C_0(X)$ (it is a uniform limit of functions in $\mathcal{C}_0(X)$) and $f \in o(g)$, since, for every $n \in \mathbb{N}$, there exists a compact set K in X such that $|f| \le 2^{-n}$ on K^c and hence $g \ge n|f|$ on K^c.

3. If $d \ge 3$, then $\mathcal{S}^+(\mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d)$ is a function cone. Indeed, it suffices to recall that the functions $G(\cdot, y) \wedge 1$, $y \in \mathbb{R}^d$, are strictly positive, linearly separate points, and that (F_3) follows exactly as in the previous example.

REMARK 9.3. Let S be a function cone. Then S^+ and $W(S)$ are function cones. Moreover, $S - S = S^+ - S^+$ and $W(S) - W(S) = W(S^+) - W(S^+)$. The vector space

$$\mathcal{C}_S(X) := \{f \in \mathcal{C}(X) \colon |f| \le s \text{ for some } s \in S^+\}$$

of all S-*bounded* continuous functions on X is a function space containing $\mathcal{K}(X)$. More generally, every convex cone T such that $S^+ \subset T \subset \mathcal{C}_S(X)$ is a function cone. In particular,

$$S_\sigma := \{f \in C_S(X) \colon f = \sum_{n=1}^{\infty} s_n, s_n \in S^+\}$$

is a function cone. Further, for every closed set F in X, the convex cone $S|_F$ is a function cone.

PROPOSITION 9.4. *Let S be a function cone, $s_0 \in S$, $s_0 > 0$, $f \in \mathcal{K}^+(X)$, and $\varepsilon > 0$. Then there exist $s, t \in W(S^+)$ such that*

$$0 \le s - t \le f \le s - t + \varepsilon s_0. \tag{9.1}$$

Proof. There exists $t_0 \in S$, $t_0 > 0$, such that $s_0 \in o(t_0)$ and $t_0 \le s_0$ on $\operatorname{supp}(f)$. Then

$$T := \{\frac{s}{t_0} \colon s \in W(S^+), s \le \alpha s_0 \text{ for some } \alpha > 0\}$$

is a $\wedge$-stable, linearly separating convex cone in $\mathcal{C}_0(X)$. By a version of the approximation theorem of Stone-Weierstrass, $T - T$ is dense in $\mathcal{C}_0(X)$ with respect to uniform convergence. In particular, there exist $s, s' \in W(S^+)$ such that

$$\left|\frac{f}{t_0} - \frac{s - s'}{t_0}\right| \le \frac{\varepsilon}{2}.$$

Then $t := s \wedge (s' + (\varepsilon/2)t_0) \in W(S^+)$ and $0 \le s - t \le f \le s - t + \varepsilon t_0$. Since $t_0 \le s_0$ on $\operatorname{supp}(f)$, (9.1) follows. □

PROPOSITION 9.5. *Let S be a function cone and T be a positive linear form on $W(S^+) - W(S^+)$. Then there exists a unique measure μ on X such that $T(s) = \int s\, d\mu$ for every $s \in W(S^+) - W(S^+)$.*

Proof. By Proposition 9.4, for every $f \in \mathcal{K}^+(X)$,

$$\begin{aligned} Lf : &= \sup\{T(t) \colon t \in W(S^+) - W(S^+),\ t \le f\} \\ &= \inf\{T(t) \colon t \in W(S^+) - W(S^+),\ t \ge f\}. \end{aligned}$$

Therefore L defines a positively homogeneous, additive functional on $\mathcal{K}^+(X)$, that is, a measure μ on X.

Let $s \in W(S^+)$. Then

$$\begin{aligned} \int s\, d\mu &= \sup\{\int f\, d\mu \colon f \in \mathcal{K}^+(X), f \le s\} \\ &= \sup\{L(f) \colon f \in \mathcal{K}^+(X), f \le s\} \le T(s). \end{aligned}$$

For the converse inequality, we choose $t \in S^+$ such that $s/t \in \mathcal{C}_0(X)$. Given $\varepsilon > 0$, we have $f := (s - \varepsilon t)^+ \in \mathcal{K}^+(X)$ and $s - \varepsilon t \le f \le s$, hence

$$T(s) - \varepsilon T(t) = T(s - \varepsilon\, t) \le L(f) = \int f\, d\mu \le \int s\, d\mu.$$

Therefore $T(s) = \int s\, d\mu$.

By Proposition 9.4, the measure μ is uniquely determined. □

Let S be a function cone and let $\mathcal{M}(S)$ denote the set of all $\mu \in \mathcal{M}(X)$ such that every $s \in S$ is integrable. By Proposition 9.5, $\mathcal{M}(S)$ can be identified with the set of all positive linear forms on $W(S^+) - W(S^+)$.

S determines a partial ordering on $\mathcal{M}(S)$:

$$\mu \prec \nu \iff \mu(s) \le \nu(s) \quad \text{for all } s \in W(S).$$

A function $s \in S_\sigma$ is called *strict*, if any two measures $\mu, \nu \in \mathcal{M}(S)$ coincide provided $\mu \prec \nu$ and $\mu(s) = \nu(s)$.

PROPOSITION 9.6. *Let S be a $\wedge$-stable function cone and $s_0 \in S$, $s_0 > 0$.*

1. *Then there exists a sequence in S^+ which is bounded by s_0 and separates the measures in $\mathcal{M}(S)$.*
2. *There exists a strict $s \in S_\sigma$ such that $s \le s_0$.*

Proof. There exists a sequence (f_n) in $\mathcal{K}^+(X)$ separating $\mathcal{M}^+(X)$. We may suppose that $f_n \le s_0$ for every $n \in \mathbb{N}$. By Proposition 9.4, there exists a sequence $(s_m)_{m\ge 1}$ in S^+ such that, for all $k, n \in \mathbb{N}$,

$$|f_n - (s_{n_k} - s_{n'_k})| \le \frac{1}{k}\, s_0 \tag{9.2}$$

for some $n_k, n'_k \in \mathbb{N}$. We may suppose that $s_m \le s_0$ for every $m \in \mathbb{N}$. Then

$$s := \sum_{m=0}^{\infty} 2^{-(m+1)} s_m \in S_\sigma, \qquad s \le s_0.$$

If $\mu, \nu \in \mathcal{M}(S)$ such that $\mu(s_m) = \nu(s_m)$ for all $m \ge 0$, then, by (9.2), $\mu(f_n) = \nu(f_n)$ for all $n \in \mathbb{N}$ and therefore $\mu = \nu$. So $(s_m)_{m\ge 0}$ separates $\mathcal{M}(S)$.

Finally, let $\mu, \nu \in \mathcal{M}(S)$ such that $\mu \prec \nu$ and $\mu(s) = \nu(s)$. Then $\mu(s_m) \le \nu(s_m)$ for all $m \ge 0$. So the equality $\mu(s) = \nu(s)$ implies that $\mu(s_m) = \nu(s_m)$ for all $m \ge 0$ and thus $\mu = \nu$. □

LEMMA 9.7. *Suppose that X is compact and that $\mathcal{F}$ is a convex set of lower semicontinuous functions $f\colon X \to (-\infty, \infty]$ such that, for every probability measure μ on X, there exists a function $f \in \mathcal{F}$ with $\mu(f) > 0$. Then $\mathcal{F}$ contains a strictly positive function.*

Proof. Let $\mathcal{G}$ denote the set of all functions $g \in \mathcal{C}(X)$ such that $g \leq \lambda f$ for some $\lambda \geq 0$ and $f \in \mathcal{F}$. Let $\mathcal{G}_0 := \{g \in \mathcal{C}(X)\colon g > 0\}$. We observe that both $\mathcal{G}$ and $\mathcal{G}_0$ are convex cones and that $\mathcal{G}_0$ is open with respect to the topology of uniform convergence on X.

We intend to show that $\mathcal{G} \cap \mathcal{G}_0 \neq \emptyset$. To that end let us suppose the contrary. Then there exists a continuous linear form $T \neq 0$ on $\mathcal{C}(X)$ and $\alpha \in \mathbb{R}$ such that

$$\sup_{g \in \mathcal{G}} T(g) \leq \alpha \leq \sup_{g \in \mathcal{G}_0} T(g).$$

Of course, $\alpha = 0$, since $\mathcal{G}$ and $\mathcal{G}_0$ are cones. Therefore T is positive, that is, T is a measure μ. Having $\mu(g) \leq 0$ for every $g \in \mathcal{G}$ we obtain, in particular, that $\mu(f) \leq 0$ for every $f \in \mathcal{F}$, a contradiction.

Thus $\mathcal{G}$ and hence $\mathcal{F}$ contain strictly positive functions. □

LEMMA 9.8. *Let $\mathcal{F}$ be a family of lower semicontinuous numerical functions on X. Then there exists a countable family $\mathcal{F}_0 \subset \mathcal{F}$ such that* $\sup \mathcal{F}_0 = \sup \mathcal{F}$*.*

Proof. By assumption, X has a countable base. Hence, for every $s \in \mathbb{Q}$, there exists a countable family $\mathcal{G}_s \subset \mathcal{F}$ such that

$$\bigcup_{f \in \mathcal{G}_s} \{f > s\} = \bigcup_{f \in \mathcal{F}} \{f > s\},$$

that is, $\{\sup \mathcal{G}_s > s\} = \{\sup \mathcal{F} > s\}$. Let $\mathcal{F}_0 := \bigcup_{s \in \mathbb{Q}} \mathcal{G}_s$. Then $\mathcal{F}_0 \subset \mathcal{F}$ is countable. If $x \in X$ and $a < \sup \mathcal{F}(x)$, then there exists $s \in \mathbb{Q}$ such that $a < s < \sup \mathcal{F}(x)$ and hence $\sup \mathcal{F}_0(x) \geq \sup \mathcal{G}_s(x) > s > a$. Thus $\sup \mathcal{F}_0 = \sup \mathcal{F}$. □

10 Minimal measures

For every $\nu \in \mathcal{M}(S)$, let

$$\mathcal{M}_\nu(S) := \{\mu \in \mathcal{M}(S)\colon \mu \prec \nu\}.$$

Given $x \in X$, we shall write $\mathcal{M}_x(S)$ instead of $\mathcal{M}_{\varepsilon_x}(S)$. We observe that

$$\mathcal{M}_x(S) = \{\mu \in \mathcal{M}(S)\colon \mu(s) \leq s(x) \quad \text{for all } s \in S\}.$$

For every $f \in \mathcal{C}_S(X)$, let $\hat{f}$, $\underset{\vee}{f}$ denote the *upper S-envelope* of f, the *lower S-envelope* of f, respectively, that is

$$\begin{aligned} \hat{f} &:= \inf\{s \in S\colon s \geq f\} = \inf\{s \in W(S)\colon s \geq f\}, \\ \underset{\vee}{f} &:= \sup\{s \in -S\colon s \leq f\} = \sup\{s \in -W(S)\colon s \leq f\}. \end{aligned}$$

Of course, $\hat{s} = s$ for all $s \in W(S)$. Moreover, let us note that, for all $\nu \in \mathcal{M}(S)$ and $f \in \mathcal{C}_S(X)$,

$$\mu(\underset{\vee}{f}) \geq \nu(\underset{\vee}{f}) \qquad \text{for every } \mu \in \mathcal{M}_\nu(S), \tag{10.1}$$

since $\mu(t) \geq \nu(t)$ for all $t \in -W(S)$ and the set of all $t \in -W(S)$ such that $t \leq f$ is increasingly filtered.

PROPOSITION 10.1. *For all $\nu \in \mathcal{M}(S)$ and $f \in \mathcal{C}_S(X)$,*

$$\{\mu(f) \colon \mu \in \mathcal{M}_\nu(S)\} = \big[\nu(\underset{\vee}{f}), \nu(\hat{f})\big]. \tag{10.2}$$

Proof. a) Let $\mu \in \mathcal{M}_\nu(S)$. If $s \in W(S)$, $f \leq s$, then $\mu(f) \leq \mu(s) \leq \nu(s)$. Therefore $\mu(f) \leq \nu(\hat{f})$. Moreover, $\mu(f) = -\mu(-f) \geq -\nu(\widehat{-f}) = \nu(\underset{\vee}{f})$.

b) Let

$$\gamma \in [\nu(\underset{\vee}{f}), \nu(\hat{f})] \quad \text{and} \quad \mathcal{F} := \mathbb{R}f.$$

Then $l \colon \lambda f \mapsto \lambda\gamma$ is linear on $\mathcal{F}$. The function $p \colon g \mapsto \nu(\hat{g})$ is sublinear on $\mathcal{C}_S(X)$. Further, $l \leq p$ on $\mathcal{F}$, since $l(f) = \gamma \leq \nu(\hat{f})$ and $l(-f) = -\gamma \leq -\nu(\underset{\vee}{f}) = \nu(\widehat{-f})$. By the theorem of Hahn-Banach, there exists a linear functional L on $\mathcal{C}_S(X)$ such that $L|_{\mathcal{F}} = l$ and, for every $g \in \mathcal{C}_S(X)$, $L(g) \leq \nu(\hat{g})$. If $g \leq 0$, then $L(g) \leq \nu(0) = 0$. So L is positive. By Proposition 9.5, there exists $\mu \in \mathcal{M}_S(X)$ such that $\mu = L$ on $\mathcal{C}_S(X)$. □

PROPOSITION 10.2. *For every $\nu \in \mathcal{M}(S)$, the following statements are equivalent:*

(i) *ν is minimal (with respect to $\prec$).*

(ii) *For every $f \in \mathcal{C}_S(X)$, $\nu(\hat{f}) = \nu(\underset{\vee}{f})$.*

(iii) *For every $s \in W(S)$, $\nu(s) = \nu(\underset{\vee}{s})$.*

(iv) *The set of all $s \in W(S)$ such that $\nu(s) = \nu(\underset{\vee}{s})$ separates $\mathcal{M}_\nu(S)$.*

(v) *There exists a strict $s \in (W(S^+))_\sigma$ such that $\nu(s) = \nu(\underset{\vee}{s})$.*

Proof. (i) ⇒ (ii): Proposition 10.1.

(ii) ⇒ (iii) ⇒ (iv) and (ii) ⇒ (v): Trivial by Proposition 9.6.

If $\mu \in \mathcal{M}_\nu(S)$ and $s \in W(S) \cup (W(S^+))_\sigma$ such that $\nu(s) = \nu(\underset{\vee}{s})$, then $\mu(s) = \nu(s)$, since, by (10.1), $\nu(\underset{\vee}{s}) \leq \mu(\underset{\vee}{s}) \leq \mu(s) \leq \nu(s)$. Therefore (iv) implies (i) and (v) implies (i). □

COROLLARY 10.3. *For every $\nu \in \mathcal{M}(S)$, there exists a minimal $\mu \in \mathcal{M}_\nu(S)$.*

Proof. By Proposition 9.6, there exists a strict $s \in (W(S^+))_\sigma$. By Proposition 10.1, there exists $\mu \in \mathcal{M}_\nu(S)$ such that $\mu(s) = \nu(\underset{\vee}{s})$. By Proposition 10.2, μ is minimal. □

The *Choquet boundary* of X with respect to S is defined by

$$Ch_S X := \{x \in X \colon \mathcal{M}_x(S) = \{\varepsilon_x\}\},$$

that is, $x \in Ch_S X$ if and only if ε_x is minimal.

COROLLARY 10.4. *For every $x \in X$, the following statements are equivalent:*

1. $x \in Ch_S X$.
2. *For every $s \in W(S)$, $s(x) = \underset{\vee}{s}(x)$.*
3. $\{s \in W(S)\colon s(x) = \underset{\vee}{s}(x)\}$ *separates $\mathcal{M}_x(S)$.*
4. *There exists a strict $s \in (W(S^+))_\sigma$ such that $s(x) = \underset{\vee}{s}(x)$.*

In particular, $Ch_S X$ is a G_δ-set and a measure $\mu \in \mathcal{M}(S)$ is minimal if and only if $\mu(X \setminus Ch_S X) = 0$.

Proof. The equivalences follow from Proposition 10.2. By Proposition 9.6, there exists a strict $s \in (W(S^+))_\sigma$. Since

$$Ch_S X = \{s = \underset{\vee}{s}\} = \bigcap_{m=1}^{\infty} \{s > \underset{\vee}{s} - \frac{1}{m}\},$$

we see that $Ch_S X$ is a G_δ-set. The proof is finished using Proposition 10.2 again. □

PROPOSITION 10.5. *If $s \in S$ such that $s \geq 0$ on $Ch_S X$, then $s \geq 0$ on X.*

Proof. Let $s \in S$, $s \geq 0$ on $Ch_S X$, and $x \in X$. By Proposition 10.3, there exists a minimal $\mu \in \mathcal{M}_x(S)$. By Corollary 10.4, μ is supported by $Ch_S X$. Thus $s(x) \geq \mu(s) \geq 0$. □

COROLLARY 10.6. *$Ch_S X = \emptyset$ if and only if $S = S^+$.*

Proof. If $Ch_S X = \emptyset$, then $S = S^+$ by Proposition 10.5. If $S = S^+$, then 0 is a minimal measure (the only minimal measure in $\mathcal{M}(S)$), it is contained in every $\mathcal{M}_x(S)$, and hence $Ch_S X = \emptyset$. □

11 Simplicial function cones

Again let S be a function cone on X.

Definition 11.1. *S is called* simplicial*, if, for every $x \in X$, there exists a* unique *minimal measure $\mu \in \mathcal{M}_x(S)$.*

Obviously, S is simplicial if $S = S^+$, that is, if $S \subset \mathcal{C}^+(X)$. Let

$$\hat{S} := \{s \in \mathcal{C}_S(X)\colon \mu(s) \leq s(x) \text{ for all } x \in X,\ \mu \in \mathcal{M}_x(S)\}$$

be the set of all continuous S-bounded S-*concave* functions and let

$$H(S) := \{h \in \mathcal{C}_S(X)\colon \mu(h) = h(x) \text{ for all } x \in X,\ \mu \in \mathcal{M}_x(S)\}$$

denote set of all continuous S-bounded H-*affine* functions.

It is immediately seen that $\hat{S}$ is a $\wedge$-stable function cone containing $W(S)$, $\hat{\hat{S}} = \hat{S}$, and

$$\hat{S} \cap (-\hat{S}) = H(S).$$

Let $\tilde{S}$ denote the set of all $s \in \hat{S}$ such that, for all $t \in -\hat{S}$ satisfying $t \leq s$, there exists $h \in H(S)$ with

$$t \leq h \leq s.$$

If $\mu \in \mathcal{M}(S)$ is minimal, then, for every $s \in \tilde{S}$,

$$\begin{aligned} \mu(s) = \mu(\underset{\vee}{s}) &= \sup\{\mu(t) \colon t \in -W(S), t \leq s\} \\ &\leq \sup\{\mu(h) \colon h \in H(S), h \leq s\} \leq \mu(s), \end{aligned}$$

and hence

$$\mu(s) = \sup\{\mu(h) \colon h \in H(S), h \leq s\}. \tag{11.1}$$

THEOREM 11.2. *Suppose that $\tilde{S}$ separates $\mathcal{M}(S)$. Then S is simplicial and the minimal measures $\mu_x \in \mathcal{M}_x(S)$, $x \in X$, satisfy*

$$\mu_x(s) = \sup\{h(x) \colon h \in H(S), h \leq s\} \qquad (s \in \tilde{S}). \tag{11.2}$$

If, in addition, $H(S) \subset W(S)$, then, for every $\nu \in \mathcal{M}(S)$, there exists a unique minimal measure μ_ν in $\mathcal{M}_\nu(S)$. It satisfies

$$\mu_\nu(s) = \sup\{\nu(h) \colon h \in H(S), h \leq s\} \qquad (s \in \tilde{S}). \tag{11.3}$$

Proof. Immediate consequence of Proposition 10.3 and (11.1). Indeed, it suffices to recall that $\mu(h) = h(x)$ for all $\mu \in \mathcal{M}_x(S)$, $x \in X$, $h \in H(S)$, and that $\mu(h) = \nu(h)$ for all $\mu \in \mathcal{M}_\nu(S)$, $\nu \in \mathcal{M}(S)$, and $h \in W(S) \cap (-W(S))$. $\square$

REMARK 11.3. It can be shown that $\tilde{S} = \hat{S}$, if S is simplicial. So, in fact, S is simplicial if and only if $\tilde{S}$ separates $\mathcal{M}(S)$.

THEOREM 11.4. *Let S_0 be a function cone such that $H(S) \subset S_0 \subset \hat{S}$ and $\tilde{S}_0$ separates $\mathcal{M}(S)$.*

Then S and S_0 are simplicial, $H(S_0) = H(S)$, $Ch_{S_0}X = Ch_S X$, and, for every $x \in X$, the minimal measures in $\mathcal{M}_x(S_0)$ and $\mathcal{M}_x(S)$ coincide.

Proof. $S_0 \subset \hat{S} \subset \mathcal{C}_S(X)$ implies that $\mathcal{C}_{S_0}(X) \subset \mathcal{C}_S(X)$ and therefore $\mathcal{M}(S) \subset \mathcal{M}(S_0)$. If $x \in X$ and $\mu \in \mathcal{M}_x(S)$, then $\mu(s) \leq s(x)$ for every $s \in S_0$ and hence $\mu \in \mathcal{M}_x(S_0)$. Thus $\hat{S}_0 \subset \hat{S}$ and

$$H(S) \subset S_0 \cap (-S_0) \subset H(S_0) = \hat{S}_0 \cap (-\hat{S}_0) \subset \hat{S} \cap (-\hat{S}) = H(S).$$

that is, $H(S_0) = H(S)$. Moreover, we see that $\tilde{S}_0 \subset \tilde{S}$. In particular, S is simplicial by Theorem 11.2.

Let us now assume that $x \in X$ and that μ is the minimal measure in $\mathcal{M}_x(S)$. We recall that then $\mu \in \mathcal{M}_x(S_0)$. Let μ_0 be a measure in $\mathcal{M}_x(S_0)$ which is minimal with respect to S_0 and let $s \in \tilde{S}_0$. Applying Lemma 11.1 to both S and S_0, we conclude that, for all $s \in \tilde{S}_0$,

$$\begin{aligned} \mu(s) = \mu(\underset{\vee}{s}) &= \sup\{h(x)\colon h \in H(S), h \le s\} \\ &= \sup\{\mu_0(h)\colon h \in H(S), h \le s\} = \mu_0(s). \end{aligned}$$

This implies that $\mu = \mu_0$. Thus S_0 is simplicial and the minimal measures in $\mathcal{M}_x(S)$ and $\mathcal{M}_x(S_0)$ coincide. In particular, $Ch_{S_0}X = Ch_S X$. □

COROLLARY 11.5. *If $H(S)$ is a function space and $\widetilde{H(S)}$ separates $\mathcal{M}(S)$, then $H(S)$ is simplicial, $Ch_{H(S)}X = Ch_S X$ and, for every $x \in X$, the minimal measures in $\mathcal{M}_x(S)$ and $\mathcal{M}_x(H(S))$ coincide.*

THEOREM 11.6. *Suppose that $\tilde{S}$ contains a function cone. Then S is simplicial and the minimal measures μ_x in $\mathcal{M}_x(S)$, $x \in X$, form a kernel D on X which is* idempotent*, that is, $D^2 = D$. Moreover,*

$$H(S) = \{h \in \mathcal{C}_S(X)\colon Dh = h\} \tag{11.4}$$

and $\mu D = \mu_x$ for all $x \in X$ and $\mu \in \mathcal{M}_x(S)$.

If, in addition, $H(S) \subset W(S)$, then, for all $\nu \in \mathcal{M}(S)$ and $\mu \in \mathcal{M}_\nu(S)$, μD is the minimal measure in $\mathcal{M}_\nu(S)$.

Proof. By Theorem 11.2, S is simplicial and the minimal measures μ_x, $x \in X$, satisfy

$$\mu_x(s) = \sup \mathcal{F}_s(x), \qquad s \in \tilde{S},$$

where $\mathcal{F}_s := \{h \in H(S)\colon h \le s\}$. Since each $\sup \mathcal{F}_s$, $s \in \tilde{S}$, is a lower semicontinuous function, this implies that the measures μ_x, $x \in X$, form a kernel D.

Next let $\mu \in \mathcal{M}(S)$. Then, for every $s \in W(S^+)$,

$$(\mu D)(s) = \int \mu_x(s)\, d\mu(x) \le \int s\, d\mu = \mu(s) < \infty. \tag{11.5}$$

Hence $\mu D \in \mathcal{M}(S^+) = \mathcal{M}(S)$ and we then have (11.5) for every $s \in W(S)$, that is, $\mu D \in \mathcal{M}(S)$ and $\mu D \prec \mu$. Further,

$$\mu D(X \setminus Ch_S X) = \int \mu_x(X \setminus Ch_S X)\, d\mu(x) = 0.$$

So μD is minimal by Corollary 10.4.

Let $x \in X$ and $\mu \in \mathcal{M}_x(S)$. Since μD is minimal in $\mathcal{M}_\mu(X) \subset \mathcal{M}_x(X)$, we see that $\mu D = \mu_x$. In particular, $\mu_x D = \mu_x$, that is, $D^2 = D$.

Since $\mu_x \in \mathcal{M}_x(S)$, we trivially have $\mu_x(h) = h(x)$ for all $h \in H(S)$ and $x \in X$, that is, $Dh = h$. Conversely, let us consider $h \in \mathcal{C}_S(X)$ such that $Dh = h$. Let $x \in X$ and $\mu \in \mathcal{M}_x(S)$. Then

$$h(x) = \mu_x(h) = (\mu D)(h) = \mu(Dh) = \mu(h).$$

Thus $h \in H(S)$.

Finally, let us assume that $H(S) \subset W(S)$ and let $\nu \in \mathcal{M}(S)$, $\mu \in \mathcal{M}_\nu(S)$. By Theorem 11.2, there exists a unique minimal measure $\mu_\nu \in \mathcal{M}_\nu(S)$. By our preceding considerations, μD is a minimal measure in $\mathcal{M}_\mu(S) \subset \mathcal{M}_\nu(S)$ and hence $\mu D = \mu_\nu$.

□

From now on we suppose that $\tilde{S}$ contains a function cone (the following results will lead to a rearrangement of the previous results).

LEMMA 11.7. *Let $s \in \hat{S}$, $t \in -\hat{S}$, and $r_0 \in S^+$ such that $t \leq s$ and $s, t \in o(r_0)$. Then, for every $\varepsilon > 0$, there exist $s_1 \in \hat{S}$ and $t_1 \in -\hat{S}$ such that*

$$t \leq t_1 \leq s_1 \leq t \quad \textit{and} \quad s_1 - t_1 \leq \varepsilon r_0. \tag{11.6}$$

Proof. There exists a compact subset Y of X such that $s - t \leq \varepsilon r_0$ on $X \setminus Y$. If $Y = \emptyset$, we may take $s_1 = s$, $t_1 = t$, and the proof is finished. So let us assume that $Y \neq \emptyset$, let $\delta := \inf r_0(Y)$ and

$$\mathcal{F} := \{\delta - (s_1 - t_1) \colon s_1 - t_1 \in W(S), t \leq t_1 \leq s_1 \leq s\}.$$

Then $\mathcal{F}$ is a convex subset of $\mathcal{C}(X)$.

Let μ be a probability measure on Y. Since $Ds = \check{s}$ and $\check{s}$ is the supremum of the increasingly filtered set of all $t_1 \in -W(S)$ with $t_1 \leq s$, there exists $t_1 \in -W(S)$ such that

$$t \leq t_1 \leq s \quad \text{and} \quad \mu(Ds - t_1) < \frac{\delta}{2}.$$

Similarly, since $Dt_1 = \hat{t}_1$, there exists $s_1 \in W(S)$ such that

$$t_1 \leq s_1 \leq s \quad \text{and} \quad \mu(s_1 - Dt_1) < \frac{\delta}{2}.$$

Clearly, $Dt_1 \leq Ds_1$. Hence we obtain that

$$\mu(s_1 - t_1) \leq \mu(s_1 - Dt_1) + \mu(Ds_1 - t_1) < \delta,$$

that is, $\mu(\delta - (s_1 - t_1)) > 0$.

Applying Lemma 9.7 to the restriction of $\mathcal{F}$ on Y, we conclude that there exist $s_1 \in W(S) \subset \hat{S}$ and $t_1 \in -W(S) \subset -\hat{S}$ such that

$$t \leq t_1 \leq s_1 \leq s \quad \text{and} \quad s_1 - t_1 < \delta \leq \varepsilon r_0 \quad \text{on } Y.$$

This finishes the proof, since $\delta \leq \varepsilon r_0$ on Y and $s_1 - t_1 \leq s - t \leq \varepsilon r_0$ on $X \setminus Y$. □

THEOREM 11.8. *$\tilde{S} = \hat{S}$, that is, for all $s \in \hat{S}$ and $t \in -\hat{S}$ such that $t \leq s$, there exists $h \in H(S)$ such that $t \leq h \leq s$.*

Proof. There exists $r_0 \in S^+$ such that $s, t \in o(r_0)$. Applying Lemma 11.7, we obtain a sequence $(s_n) \in \hat{S}$ and a sequence $(t_n) \in -\hat{S}$ such that, for every $n \in \mathbb{N}$,

$$t \le t_n \le t_{n+1} \le s_{n+1} \le s_n \le s \quad \text{and} \quad s_n - t_n \le \frac{1}{n} r_0.$$

Then (s_n) and (t_n) converge uniformly to a function $h \in \mathcal{C}_S(X)$. Moreover,

$$t_n \le Dt_n \le Ds_n \le s_n \qquad (n \in \mathbb{N})$$

and hence $Dh = h$. Thus $h \in H(S)$ by Theorem 11.6, $t \le h \le s$. □

12 Cones of potentials

Definition 12.1. A function cone on the space X (which is locally compact with countable base) is a *cone of potentials* if every $f \in \mathcal{C}_\mathcal{P}$ admits a smallest majorant in $\mathcal{P}$, that is, if

$$\inf\{p \in \mathcal{P} \colon p \ge f\} \in \mathcal{P}. \tag{12.1}$$

In the following let $\mathcal{P}$ be a cone of potentials. We define

$$\mathcal{W} := \{\sup p_n \colon p_n \in \mathcal{P}, p_n \le p_{n+1} \text{ for every } n \in \mathbb{N}\}.$$

For every $f \colon X \to \overline{\mathbb{R}}$ and $A \subset X$, let

$$R_f := \inf\{v \in \mathcal{W} \colon v \ge f\},$$

$$R_f^A := R_{1_A f} = \inf\{v \in \mathcal{W} \colon v \ge f \text{ on } A\}.$$

REMARK 12.2. Let $f \colon X \to [0, \infty)$ be upper semicontinuous. Then

$$R_f = \inf\{p \in \mathcal{P} \colon p \ge f\}.$$

Indeed, let $s, t \in \mathcal{P}$ such that $f \le s$, $s \in o(t)$, and $t > 0$. Let us consider $v \in \mathcal{W}$ such that $v \ge f$. There exist $p_n \in \mathcal{P}$ such that $p_n \uparrow v$. Let $\varepsilon > 0$. Then $\varepsilon t \ge s \ge f$ outside some compact subset K of X. Moreover, there exists $n \in \mathbb{N}$ such that $p_n + \varepsilon t > f$ on K. So $p_n + \varepsilon t \ge f$ on X. Since $p_n + \varepsilon t \in \mathcal{P}$ and $p_n + \varepsilon t \le v + \varepsilon t$, we see that $\inf\{p \in \mathcal{P} \colon p \ge f\} \le v$. Thus

$$\inf\{p \in \mathcal{P} \colon p \ge f\} \le \inf\{v \in \mathcal{W} \colon v \ge f\} = R_f.$$

The converse inequality is trivial.

LEMMA 12.3. *1. $\mathcal{P}$ is $\wedge$-stable.*

2. $\mathcal{W}$ is a convex cone such that $\mathcal{W} \cap \mathcal{C}_\mathcal{P}(X) = \mathcal{P}$.
In particular, there exist strict $p \in \mathcal{P}$.

3. If $\emptyset \ne \mathcal{F} \subset \mathcal{W}$ is increasingly filtered, then $\sup \mathcal{F} \in \mathcal{W}$.

4. If $\mathcal{F} \subset \mathcal{W}$, then $\widehat{\inf \mathcal{F}} \in \mathcal{W}$.

Proof. 1. Trivial by (12.1).

2. Obviously, $\mathcal{W}$ is a convex cone and $\mathcal{P} \subset \mathcal{W} \cap \mathcal{C}_{\mathcal{P}}(X)$. Conversely, if $f \in \mathcal{W} \cap \mathcal{C}_{\mathcal{P}}(X)$, then $f = R_f \in \mathcal{P}$ by (12.1). So, by Proposition 9.6, there are strict $p \in \mathcal{P}$.

3. Let $\mathcal{F} \neq \emptyset$ be an increasingly filtered subset of $\mathcal{W}$. By Lemma 9.8, there exists a sequence (w_n) in $\mathcal{F}$ such that $\sup w_n = \sup \mathcal{F}$. Since $\mathcal{F}$ is increasing filtered, we may assume that the sequence (w_m) is increasing. For every $n \in \mathbb{N}$, there exists an increasing sequence $(p_{n,m})_{m\in\mathbb{N}}$ in $\mathcal{P}$ such that $\sup_m p_{n,m} = w_n$. We define

$$q_n := R_{p_{1,n} \vee \dots \vee p_{n,n}}, \qquad n \in \mathbb{N}.$$

By (12.1), $q_n \in \mathcal{P}$ for every $n \in \mathbb{N}$. Since $p_{j,n} \le p_{j,n+1}$ for all $1 \le j \le n$, the sequence (q_n) is increasing and $\sup q_n = \sup w_n$. Thus $\sup \mathcal{F} \in \mathcal{W}$.

4. Finally, let $\mathcal{F}$ be an arbitrary subset of $\mathcal{W}$ and $s := \widehat{\inf \mathcal{F}}$. Since $s \ge 0$ and s is lower semicontinuous, there exists a sequence (f_n) in $\mathcal{K}^+(X)$ such that $f_n \uparrow s$. For every $n \in \mathbb{N}$, let $p_n := R_{f_n}$. Then (p_n) is an increasing sequence in $\mathcal{P}$ and $\sup p_n \ge \sup f_n = s$. Moreover, for every $n \in \mathbb{N}$, $f_n \le \inf \mathcal{F}$ and hence $p_n \le \inf \mathcal{F}$, $p_n \le s$. Hence $\sup p_n \le \widehat{\inf \mathcal{F}} = s$ and we see that $s = \sup p_n \in \mathcal{W}$. □

13 Simplicial cones generated by $\mathcal{P}$-dilations

Again let $\mathcal{P}$ be an arbitrary cone of potentials on X.

Definition 13.1. A *$\mathcal{P}$-dilation* is a kernel T on X such that, for every $p \in \mathcal{P}$,

$$Tp \in \mathcal{W} \quad \text{and} \quad Tp \le p. \tag{13.1}$$

The *base* of a $\mathcal{P}$-dilation T is the set

$$b(T) := \{x \in X : T(\cdot, x) = \varepsilon_x\}.$$

If T is a $\mathcal{P}$-dilation, then, of course, for every $v \in \mathcal{W}$,

$$Tv \in \mathcal{W} \quad \text{and} \quad Tv \le v. \tag{13.2}$$

For applications on harmonic spaces, let us note the following. If $A \subset X$ such that the mapping $p \mapsto \hat{R}_p^A$ is additive, then it defines a $\mathcal{P}$-dilation.

PROPOSITION 13.2. *Let T be a $\mathcal{P}$-dilation. Then $b(T)$ is a G_δ-set. If T is idempotent, that is, if $T^2 = T$, then $T1_{X\setminus b(T)} = 0$ and*

$$Tp = R_p^{b(T)} \qquad (p \in \mathcal{P}). \tag{13.3}$$

Proof. Let us suppose first that $p \in \mathcal{P}$ is strict. Then $b(T) = \{Tp = p\}$. Since the function Tp is lower semicontinuous and $\{Tp = p\} = \bigcap_{n=1}^\infty \{Tp > p - \frac{1}{n}\}$, we see that $b(T)$ is a G_δ-set.

Let us now assume that $T^2 = T$. Then $T(p - Tp) = Tp - T^2 p = 0$ and hence $T1_{X\setminus b(T)} = 0$. Finally, let us fix an arbitrary $p \in \mathcal{P}$. Then $Tp \in \mathcal{W}$ and $Tp = p$ on $b(T)$, hence $Tp \ge R_p^{b(T)}$. Conversely, if $v \in \mathcal{W}$ such that $v \ge p$ on $b(T)$, then $v \ge Tv \ge Tp$, since $T1_{X\setminus b(T)} = 0$, and therefore $R_p^{b(T)} \ge Tp$. □

Now let $\mathcal{T}$ be an arbitrary family of $\mathcal{P}$-dilations,

$$\begin{aligned} H(\mathcal{T}) &:= \{h \in \mathcal{C}_{\mathcal{P}}(X) \colon Th = h \text{ for every } T \in \mathcal{T}\}, \\ S(\mathcal{T}) &:= \{s \in \mathcal{C}_{\mathcal{P}}(X) \colon Ts \le s \text{ for every } T \in \mathcal{T}\}. \end{aligned}$$

Obviously,

$$H(\mathcal{T}) = S(\mathcal{T}) \cap (-S(\mathcal{T}))$$

and $S(\mathcal{T})$ is a $\wedge$-stable function cone, since $\mathcal{P} \subset S(\mathcal{T}) \subset C_{\mathcal{P}}(X)$ (see Remark 9.3). Moreover, $\widehat{S(\mathcal{T})} = S(\mathcal{T})$ and therefore

$$H(S(\mathcal{T})) = S(\mathcal{T}) \cap (-S(\mathcal{T})) = H(\mathcal{T}).$$

The following simple lemma is the key to the simpliciality of $S(\mathcal{T})$ (it shows that $\mathcal{P} \subset \widetilde{S(\mathcal{T})}$).

LEMMA 13.3. *Let $p \in \mathcal{P}$ and $t \in -S(\mathcal{T})$ such that $t \le p$. Then there exists $q \in \mathcal{P} \cap H(\mathcal{T})$ such that $t \le q \le p$.*

Proof. Let $q := R_t$. Then $q \in \mathcal{P}$ by definition of a potential cone, and obviously $t \le q \le p$. For every $T \in \mathcal{T}$, $t \le Tt \le Tq$, where $Tq \in \mathcal{W}$, and therefore $q = R_t \le Tq \le q$. Thus $q \in H(\mathcal{T})$. □

THEOREM 13.4. *$S(\mathcal{T})$ is a simplicial cone. The minimal representing measures form an idempotent $\mathcal{P}$-dilation $D^{\mathcal{T}}$ and, for every $p \in \mathcal{P}$,*

$$D^{\mathcal{T}} p = \sup\{q \in H(\mathcal{T}) \cap \mathcal{P} \colon q \le p\} = R_p^{Ch_{S(\mathcal{T})} X}.$$

Proof. By Lemma 13.3, $\mathcal{P} \subset \widetilde{S(\mathcal{T})}$. Hence, by Theorem 11.2, $S(\mathcal{T})$ is a simplicial cone and the minimal measures $\mu_x \in \mathcal{M}_x(S(\mathcal{T}))$, $x \in X$, are determined by

$$\mu_x(p) = \sup\{q(x) \colon q \in H(\mathcal{T}) \cap \mathcal{P}, q \le p\} \qquad (p \in \mathcal{P}).$$

Let $p \in \mathcal{P}$ and $q_1, q_2 \in H(\mathcal{T}) \cap \mathcal{P}$ such that $q_1 \le p$ and $q_2 \le p$. Then $q_1 \vee q_2 \in -S(\mathcal{T})$ and $q_1 \vee q_2 \le p$. So, by Lemma 13.3, there exists $q \in H(\mathcal{T}) \cap \mathcal{P}$ such that $q_1 \vee q_2 \le q \le p$. By (2) in Lemma 12.3, we obtain that the function $x \mapsto \mu_x(p)$ is contained in $\mathcal{W}$, that is the measures μ_x, $x \in X$, define a $\mathcal{P}$-dilation which we shall denote by $D^{\mathcal{T}}$ or D for short.

By Theorem 11.6, D is idempotent and hence, by Proposition 13.2, for every $p \in \mathcal{P}$,

$$Dp = R_p^{b(D)} = R_p^{Ch_{S(\mathcal{T})} X}.$$

□

COROLLARY 13.5. *Suppose that $H(\mathcal{T})$ is a function space. Then $H(\mathcal{T})$ is simplicial, $\mathrm{Ch}_{H(\mathcal{T})} X = \mathrm{Ch}_{S(\mathcal{T})}$, and the minimal representing measures with respect to $H(\mathcal{T})$ and $S(\mathcal{T})$ coincide.*

Finally, let $\mathfrak{T}$ be a decreasingly filtered family of sets of $\mathcal{P}$-dilations on X and

$$S(\mathfrak{T}) := \bigcup_{\mathcal{T}\in\mathfrak{T}} S(\mathcal{T}), \qquad H(\mathfrak{T}) := \bigcup_{\mathcal{T}\in\mathfrak{T}} H(\mathcal{T}).$$

Obviously,

$$W(S(\mathfrak{T})) = \bigcup_{\mathcal{T}\in\mathfrak{T}} W(S(\mathcal{T})) = S(\mathfrak{T})$$

and

$$H(\mathfrak{T}) = S(\mathfrak{T}) \cap (-S(\mathfrak{T})) = H(S(\mathfrak{T})).$$

THEOREM 13.6. *$S(\mathfrak{T})$ is a simplicial cone. The minimal representing measures form an idempotent $\mathcal{P}$-dilation $D^{\mathfrak{T}}$ and, for every $p \in \mathcal{P}$,*

$$D^{\mathfrak{T}}p = \sup\{q \in H(\mathfrak{T}) \cap \mathcal{P} \colon q \le p\} = R_p^{Ch_{S(\mathfrak{T})}X}.$$

Proof. Reasoning as before we conclude that $S(\mathfrak{T})$ is simplicial and that $D^{\mathfrak{T}}$ is a $\mathcal{P}$-dilation on X such that, for every $p \in \mathcal{P}$,

$$D^{\mathfrak{T}} = \sup\Big\{\bigcup_{\mathcal{T}\in\mathfrak{T}} \{q \in \mathcal{P} \colon q \le p, Tq = q \text{ for every } T \in \mathcal{T}\}\Big\} = \sup_{\mathcal{T}\in\mathfrak{T}} D^{\mathcal{T}}p$$

and $D^{\mathfrak{T}}p = R_p^{Ch_{S(\mathfrak{T})}X}$. □

COROLLARY 13.7. *Suppose that $H(\mathfrak{T})$ is a function space. Then $H(\mathfrak{T})$ is simplicial, $Ch_{H(\mathfrak{T})}X = Ch_{S(\mathfrak{T})}X$, and the minimal representing measures with respect to $H(\mathfrak{T})$ and $S(\mathfrak{T})$ coincide.*

14 A general minimum principle

In Section 18 we shall need a minimum principle which involves lower semicontinuous functions (Proposition 14.1 and Theorem 14.2). It is stronger than Corollary 10.5 and will lead to a generalization of Proposition 2.1 (see Proposition 15.2).

Let us assume that $\mathcal{F}$ is a convex cone of lower semicontinuous functions $> -\infty$ on X. Of course, we define the corresponding Choquet boundary $Ch_{\mathcal{F}}X$ by

$$Ch_{\mathcal{F}}X := \big\{x \in X \colon \mathcal{M}_x(\mathcal{F}) = \{\varepsilon_x\}\big\},$$

where $\mathcal{M}_x(\mathcal{F})$ denotes the set of all measures μ on X such that $-\infty < \mu(f) \le f(x)$ for every $f \in \mathcal{F}$.

PROPOSITION 14.1 (Bauer). *Let us suppose that X is compact, that $\mathcal{F}$ is linearly separating, and that there exists a strictly positive function $f_0 \in \mathcal{F} \cap \mathcal{C}(X)$. Then every function $f \in \mathcal{F}$, which is positive on $Ch_{\mathcal{F}}X$, is positive on X.*

Proof. (a) Let us first assume that $1 \in \mathcal{F}$ and let $\mathcal{A}$ be the set of all non-empty compact sets A in X such that

$$\mu(X \setminus A) = 0 \qquad \text{for all } x \in A \text{ and } \mu \in \mathcal{M}_x(\mathcal{F}).$$

Of course, $X \in \mathcal{A}$. Moreover, the set $\mathcal{A}$ is inductively ordered by the converse inclusion relation. So, by Zorn's lemma, every set $A \in \mathcal{A}$ contains a minimal set $A' \in \mathcal{A}$.

We now fix $f \in \mathcal{F}$, assume that $\alpha := \inf f(X) < 0$, and consider the non-empty compact set $A := \{f = \alpha\}$. Let $x \in A$ and $\mu \in \mathcal{M}_x(\mathcal{F})$. Since $\mu(1) \leq 1$ and $\alpha < 0$, we obtain that

$$\alpha \leq \int \alpha \, d\mu \leq \int f \, d\mu \leq \alpha,$$

hence $\mu(X) = 1$ and $\int (f - \alpha) \, d\mu = 0$, that is, $\mu(X \setminus A) = 0$. Thus $A \in \mathcal{A}$.

Let $A' \in \mathcal{A}$ be minimal such that $A' \subset A$. We intend to show that A' consists of one point only. Let us suppose the contrary. Then there exists a function $g \in \mathcal{F}$ such that $g|_{A'}$ is non-constant. Let $\beta := \inf g(A')$ and

$$A'' := A' \cap \{g = \beta\}.$$

Then A'' is a non-empty compact set, $A'' \neq A'$. If $x \in A''$ and $\mu \in \mathcal{M}_x(\mathcal{F})$, then

$$\beta = \int_{A'} \beta \, d\mu \leq \int_{A'} g \, d\mu = \int g \, d\mu \leq g(x) = \beta,$$

hence $\int_{A'} (g - \beta) \, d\mu = 0$ and therefore

$$\mu(X \setminus A'') = \mu(X \setminus A') + \mu(A' \setminus A'') = 0.$$

Thus $A'' \in \mathcal{A}$, contradicting the minimality of A'. So A' reduces to a singleton $\{x\}$ and the preceding statements on A' imply that $\mathcal{M}_x(\mathcal{F}) = \{\varepsilon_x\}$, that is, $x \in Ch_{\mathcal{F}}X$. Since $f(x) = \alpha < 0$, the statement of the Proposition follows provided $1 \in \mathcal{F}$.

(b) Let us now consider the general case. Let $f_0 \in \mathcal{F} \cap \mathcal{C}(X)$, $f_0 > 0$, and

$$\mathcal{F}_0 = \{\frac{f}{f_0} : f \in \mathcal{F}\}.$$

Then $\mathcal{F}_0$ satisfies the assumptions of (a) and, for every $x \in X$,

$$\mathcal{M}_x(\mathcal{F}_0) = \big\{\frac{f_0 \mu}{f_0(x)} : \mu \in \mathcal{M}_x(\mathcal{F})\big\},$$

hence $Ch_{\mathcal{F}_0}X = Ch_{\mathcal{F}}X$. Finally, let $f \in \mathcal{F}$, $f \geq 0$ on $Ch_{\mathcal{F}}X$. Then $f/f_0 \in \mathcal{F}_0$ and $f/f_0 \geq 0$ on $Ch_{\mathcal{F}_0}X$. By (a), $f/f_0 \geq 0$ on X, that is, $f \geq 0$ on X. □

THEOREM 14.2. *Let us assume that $\mathcal{F}$ contains a function cone $\mathcal{P}$ such that all functions in $\mathcal{F}$ are lower $\mathcal{P}$-bounded. Then every function $f \in \mathcal{F}$, which is positive on $Ch_{\mathcal{F}}X$, is positive on X.*

Proof. Let us suppose that there exists a function $f \in \mathcal{F}$ such that f is not positive, but $f \geq 0$ on $Ch_{\mathcal{F}}X$. Then there are $p, q \in \mathcal{P}$ such that $-p \leq f$, $q > 0$, $p \in o(q)$, and $g := f + q$ is not positive. We define

$$K := \{g \leq 0\} \quad \text{and} \quad \mathcal{G} := (\mathcal{P} + \mathbb{R}_+ q)|_K.$$

Then K is a non-empty compact subset of $X \setminus Ch_{\mathcal{F}}X$. So, fixing $x \in K$, there exists a measure $\mu \in \mathcal{M}_x(\mathcal{F})$ such that $\mu \neq \varepsilon_x$.

Let $\nu := 1_K \mu$. Of course, for every $u \in \mathcal{P}$, $\int u\, d\nu \leq \int u\, d\mu \leq u(x)$. In addition, since $g > 0$ outside K,

$$\int g\, d\nu \leq \int g\, d\nu + \int_{X \setminus K} g\, d\mu = \int g\, d\mu \leq g(x). \tag{14.1}$$

So $\nu \in \mathcal{M}_x(\mathcal{G})$. If $\mu(X \setminus K) > 0$, then the first inequality in (14.1) is strict, hence $\nu \neq \varepsilon_x$. If, however, $\mu(X \setminus K) = 0$, then $\nu = \mu \neq \varepsilon_x$.

Thus $x \notin Ch_{\mathcal{G}}K$, that is, $Ch_{\mathcal{G}}K = \emptyset$. By Proposition 14.1, this implies that $g \geq 0$ on K and hence $g \geq 0$ on X, a contradiction.

□

15 Minimum principle and sheaf properties

From now we fix a family $(H_U)_{U \in \mathcal{U}}$ of harmonic kernels on X (see Definition 7.1; as before, X is a locally compact space with countable base).

The following Proposition 15.2 will show, in particular, that the convex cones ${}^*\mathcal{H}(W)$ and the linear spaces $\mathcal{H}(W)$, W open in X, remain the same, if $(H_U)_{U \in \mathcal{U}}$ is replaced by a family $(H_U)_{U \in \tilde{\mathcal{U}}}$ of harmonic kernels, where $\tilde{\mathcal{U}} \subset \mathcal{U}$ or $\mathcal{U} \subset \tilde{\mathcal{U}}$. First a lemma which shows how Evans functions are used.

LEMMA 15.1. *Let $U \in \mathcal{U}$ and let $g \colon \partial U \to (-\infty, \infty]$ be lower semicontinuous. Then $\liminf_{x \to z}(H_U g + w)(x) \geq g(z)$ for every $z \in \partial U$ and for every Evans function w on U.*

Proof. Given w and z, there exists an ultrafilter $\mathcal{F}$ on U such that $\lim \mathcal{F} = z$ and

$$\liminf_{x \to z}(H_U g + w)(x) = \lim\nolimits_{\mathcal{F}}(H_U g + w).$$

If $\mathcal{F}$ is regular, then $\lim_{\mathcal{F}}(H_U g + w) \geq \lim_{\mathcal{F}} H_U g \geq g(z)$. If $\mathcal{F}$ is not regular, then $\lim_{\mathcal{F}} w = \infty$ and hence $\lim_{\mathcal{F}}(H_U g + w) = \infty$. □

PROPOSITION 15.2. *Let W be open in X and let $v \colon W \to (-\infty, \infty]$ be lower semicontinuous such that, for every $x \in W$ and every neighborhood U of x in W, there exists $V \in \mathcal{U}(U)$ with $H_V v(x) \leq v(x)$. Then $v \in {}^*\mathcal{H}(W)$.*

If, in addition, W is relatively compact and $\liminf_{x \to z} v(x) \geq 0$ for every $z \in \partial W$, then $v \geq 0$.

Proof. 1. Let us postpone the proof of $v \in {}^*\mathcal{H}(W)$ and suppose first that W is relatively compact and $\liminf_{x\to z} v(x) \ge 0$ for every $z \in \partial W$. We extend v to a lower semicontinuous function on the compact set $\overline{W}$ taking the value 0 on ∂W. Let

$$\mathcal{F} := {}^*\mathcal{H}^+(X)|_{\overline{W}} + \mathbb{R}_+ v.$$

If $x \in V \in \mathcal{U}(W)$ such that $H_V v(x) \le v(x)$, then $H_V w(x) \le w(x)$ for every $w \in \mathcal{F}$, but $H_V(x,\cdot) \ne \varepsilon_x$, since $H_V(x,\cdot)$ is supported by ∂V. So $Ch_{\mathcal{F}}\overline{W} \subset \partial W$. By (H_5) and Proposition 14.1, we hence see that $v \ge 0$.

2. Now we shall prove that $v \in {}^*\mathcal{H}(W)$, W being any open subset of X. Let $U \in \mathcal{U}(W)$ and $f \in \mathcal{C}(\overline{W})$ such that $f \le v$. Since v is the limit of an increasing sequence of such functions f, we only have to show that $H_U f \le v$ on U. Let w be an Evans function on U. By Lemma 15.1, for every $z \in \partial U$,

$$\begin{aligned} \liminf_{x\to z, x\in U} (v - H_U f + w)(x) &\ge v(z) + \liminf_{x\to z, x\in U} (H_U(-f) + w)(x) \\ &\ge v(z) + (-f)(z) \ge 0. \end{aligned}$$

By the first part of our proof, replacing W and v by U and $v - H_U f + w$, respectively, we see that $v - H_U f + w \ge 0$ on U. By (H_4), we finally conclude that $v - H_U f \ge 0$. □

COROLLARY 15.3. *Let V, W be open sets in X, $V \subset W$. Let $v \in {}^*\mathcal{H}^+(V)$ and $w \in {}^*\mathcal{H}^+(W)$ such that $\liminf_{x\to z, x\in V} v(x) \ge w(x)$ for every $z \in \partial V \cap W$. Finally, let $u\colon W \to [0,\infty]$ be the function $v \wedge w$ on V and the function w on $W \setminus V$. Then $u \in {}^*\mathcal{H}^+(W)$.*

Proof. Clearly, u is lower semicontinuous on W. If $x \in V$ and $U \in \mathcal{U}(V)$, then $H_U v(x) \le v(x)$ and $H_U w(x) \le w(x)$, hence $(H_U v \wedge w)(x) \le (v \wedge w)(x)$. If $x \in W \setminus U$ and $U \in \mathcal{U}(W)$, then $H_U u(x) \le H_U w(x) \le w(x) = u(x)$. Thus $u \in {}^*\mathcal{H}^+(W)$ by Proposition 15.2. □

COROLLARY 15.4. *Let W be the union of open sets W_i, $i \in I$, in X and let $v\colon W \to (-\infty,\infty]$. Then v is hyperharmonic (harmonic, resp.) if and only if, for every $i \in I$, the restriction of v on W_i is hyperharmonic (harmonic, resp.).*

Proof. The proof (which is straightforward) will be left to the reader. □

16 Convergence properties of harmonic functions

In this short section we shall establish some convergence properties of harmonic functions which are of independent interest. It should be clear that these results hold as well for arbitrary open sets W in X, since obviously the restrictions of the harmonic kernels H_U, $U \in \mathcal{U}(W)$, on W form a family of harmonic kernels on W.

PROPOSITION 16.1. *Let (h_n) be a decreasing sequence in $\mathcal{H}^+(X)$. Then $\inf h_n$ is harmonic on X.*

Proof. Let $h = \inf h_n$ and $U \in \mathcal{U}$. Then $h \in \mathcal{B}^+(X)$, $H_U h$ is continuous on U by (H_3), and

$$H_U h = \inf H_U h_n = \inf h_n = h \quad \text{on } U.$$

□

COROLLARY 16.2. *Let $\mathcal{F}$ be a decreasingly filtered subset of $\mathcal{H}^+(X)$. Then there exists a decreasing sequence (h_n) in $\mathcal{F}$ such that* $\inf h_n = \inf \mathcal{F}$. *In particular,* $\inf \mathcal{F} \in \mathcal{H}^+(X)$.

Proof. By Lemma 9.8, there exists a decreasing sequence (h_n) in $\mathcal{F}$ such that $\inf h_n = \inf \mathcal{F}$. An application of Proposition 16.1 finishes the proof. □

COROLLARY 16.3. *Let $U \in \mathcal{U}$ and let $f \geq 0$ be a bounded function on ∂U. Then the function $H_U^*\colon x \mapsto \int^* f(y)\, H_U(x, dy)$ is harmonic on U.*

Proof. It suffices to note that $\mathcal{F} := \{H_U g|_U \colon g \in \mathcal{B}_b^+(X), g \geq f\}$ is a decreasingly filtered subset of $\mathcal{H}^+(U)$ and that $H_U^* f = \inf \mathcal{F}$. □

17 Potentials

A function $s \in \mathcal{S}^+(X)$ will be called *potential* (on X), if the constant 0 is the greatest harmonic minorant of s. Let $\mathcal{P}(X)$ denote the set of all potentials on X.

REMARK 17.1. It is not hard to show that, for example, in the classical case, $\mathcal{P}(X)$ is the set of all G^μ, μ measure on $X = \mathbb{R}^d$, $d \geq 3$, such that $G^\mu \not\equiv \infty$.

To be able to apply the results of Section 13 we intend to show that $\mathcal{P}(X) \cap \mathcal{C}(X)$ is a cone of potentials in the sense of Section 12 and that

$$\mathcal{W} := \{\sup p_n \colon p_n \in \mathcal{P},\ p_n \leq p_{n+1} \text{ for every } n \in \mathbb{N}\} = {}^*\mathcal{H}^+(X)\}. \tag{17.1}$$

An important step will be the following result, where

$$R_f := \inf\{v \in {}^*\mathcal{H}^+(X) \colon v \geq f\}$$

(provided (17.1) is proven, this will be in agreement with our definition at the beginning of Section 12).

PROPOSITION 17.2. *Let f be a numerical function on X which is $\mathcal{S}^+(X)$-bounded. Then $\hat{R}_f \in \mathcal{S}^+(X)$ and, outside the support of f, the function R_f is harmonic* (*and coincides with* $\hat{R}_f$).

Moreover, if f is continuous, then $R_f = \hat{R}_f \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ and $H_U R_f = R_f$ for every $U \in \mathcal{U}$ such that $H_U f \geq f$.

For the proof we need some preparations.

PROPOSITION 17.3. *Let $\mathcal{V}$ be a subset of ${}^*\mathcal{H}^+(X)$, $v := \inf \mathcal{V}$. Then $\hat{v} \in {}^*\mathcal{H}^+(X)$ and, for every $x \in X$,*

$$\hat{v}(x) = \sup_{x \in U \in \mathcal{U}(W)} H_U^* v(x) = \lim_{U \in \mathcal{U}, U \downarrow x} H_U^* v(x). \tag{17.2}$$

In particular, $\hat{R}_f \in {}^*\mathcal{H}^+(X)$ for every $f\colon X \to \overline{\mathbb{R}}$. Moreover, $\hat{R}_f = R_f$, if f is lower semicontinuous.

Proof. Let $U \in \mathcal{U}$. Then $H_U^* \hat{v} \le H_U^* v \le \inf_{w \in \mathcal{V}} H_U w \le \inf_{w \in \mathcal{V}} w = v$, where $H_U \hat{v}$ is lower semicontinuous on U by (H_3). Hence $H_U \hat{v} \le \hat{v}$. Thus $\hat{v} \in {}^*\mathcal{H}^+(X)$.

Finally, let $f\colon X \to \overline{\mathbb{R}}$. Then $\hat{R}_f \in {}^*\mathcal{H}^+(X)$ by the previous considerations. If f is lower semicontinuous, then $R_f \ge f$ implies that $\hat{R}_f \ge f$ and hence $\hat{R}_f \ge R_f$. The converse inequality is trivial. □

An immediate consequence is the following.

COROLLARY 17.4. *Let $\mathcal{V}_n \subset {}^*\mathcal{H}^+(X)$ and $v_n := \inf \mathcal{V}_n$, $n \in \mathbb{N}$. Then $\widehat{v_1 + v_2} = \hat{v}_1 + \hat{v}_2$. If $v_n \uparrow v$, then $\hat{v}_n \uparrow \hat{v}$.*

Moreover, let $f_n\colon X \to [0, \infty]$, $n \in \mathbb{N}$, Then $\hat{R}_{f_1+f_2} \le \hat{R}_{f_1} + \hat{R}_{f_2}$, where equality holds if $R_{f_1+f_2} = R_{f_1} + R_{f_2}$. If $R_{f_n} \uparrow R_f$, then $\hat{R}_{f_n} \uparrow \hat{R}_f$.

Proof. If $w_1, w_2 \in {}^*\mathcal{H}^+(X)$ such that $w_j \ge f_j$, $j = 1, 2$, then $w_1 + w_2 \in {}^*\mathcal{H}^+(X)$ and $w_1 + w_2 \ge f_1 + f_2$. Therefore $R_{f_1} + R_{f_2} \ge R_{f_1+f_2}$. The proof is finished applying Proposition 17.3. □

More important for us is the following.

COROLLARY 17.5. *Let $v \in {}^*\mathcal{H}^+(X)$, $U \in \mathcal{U}$ and $u \in {}^*\mathcal{H}^+(U)$ such that $H_U v \le u \le v$. Then $\hat{u} \in {}^*\mathcal{H}^+(X)$, $\hat{u}(z) = \liminf_{x \to z, x \in U} u(x)$ for every $z \in \partial U$, and $\hat{u} = u$ on the complement of ∂U.*

In particular, $\widehat{H_U v} \in {}^*\mathcal{H}^+(X)$. Moreover, the mapping $v \mapsto \widehat{H_U v}$ is positively homogeneous, additive, and σ-continuous.

Proof. Let w be an Evans function on U. By Lemma 15.1, $u + w \ge v$ at ∂U. Let $\tilde{u}\colon X \to [0, \infty]$ be equal to $(u+w) \wedge v$ on U and equal to v on $X \setminus U$. By Corollary 15.3, $\tilde{u} \in {}^*\mathcal{H}^+(X)$. Let $\mathcal{V}$ be the set of all functions obtained this way, varying the Evans function w. By (H_4), $\inf \mathcal{V} = u$. Thus $\hat{u} \in {}^*\mathcal{H}^+(X)$ by Proposition 17.3. The additional statements about $\hat{u}$ are easily verified.

The properties of $v \mapsto \widehat{H_U v}$ follow easily from the corresponding properties of the mapping $v \mapsto H_U v$, Corollary 17.4 (for the additivity), and Proposition 17.3 (for the σ-continuity). □

Proof of Proposition 17.2. By Proposition 17.3, $s := \hat{R}_f \in \mathcal{S}^+(X)$. Let $U \in \mathcal{U}$ such that $f = 0$ on $\overline{U}$ and let $v \in \mathcal{S}^+(X)$ such that $v \ge f$. By Corollary 17.5, $\widehat{H_U v} \in \mathcal{S}^+(X)$. Moreover, $v \ge \widehat{H_U v}$ and $\widehat{H_U v} \ge f$, since $\widehat{H_U v} = v$ outside $\overline{U}$. Therefore

$$R_f = \inf\{v \in \mathcal{S}^+(X)\colon v \ge f\} = \inf\{\widehat{H_U v}\colon v \in \mathcal{S}^+(X)\colon v \ge f\}.$$

Since each function $\widehat{H_U v}$ is harmonic on U, we conclude, by Proposition 16.2, that R_f is harmonic on U (and hence $\hat{R}_f = R_f$ on U).

Let us now suppose that f is continuous. Then $s := R_f \in \mathcal{S}^+(X)$ (and hence $\hat{R}_f = R_f$). Let $U \in \mathcal{U}$ such that $H_U f \geq f$. Then $H_U s \geq f$ and hence $\widehat{H_U s} \geq f$. Since $\widehat{H_U s} \in \mathcal{S}^+(X)$, we see that $\widehat{H_U s} \geq R_f = s$. Trivially $\widehat{H_U s} \leq H_U s \leq s$. Thus $H_U s = s$. It remains to show that s is upper semicontinuous and real.

Let us fix $x \in X$, $\varepsilon > 0$, and $a, b \in \mathbb{R}$ such that $a \leq b$, $a \leq \sup_{U \in \mathcal{U}_x} H_U s(x)$, and $f(x) \leq b \leq s(x)$. By (H_5), there exists $t \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that

$$b - a < t(x) < b - a + \varepsilon.$$

Then $f(x) - t(x) \leq b - t(x) < a$, hence there exists $U \in \mathcal{U}$ such that $x \in U$ and

$$f(x) - t(x) < H_U s(x).$$

So there exists $V \in \mathcal{U}(U)$ such that $x \in V$ and $f - t < H_U s$ on V, and hence

$$f \leq t + \widehat{H_V s},$$

since $H_U s = H_V H_U s \leq H_V s$ and $H_V s = s \geq f$ on V^c. Therefore $s = R_f \leq t + \widehat{H_V s}$. In particular, $s(x) < \infty$. Choosing $a := \sup_{U \in \mathcal{U}_x} H_U s(x)$, $b := s(x)$ and using the continuity of $t + \widehat{H_V s}$ on V, we finally conclude that

$$\limsup_{y \to x} s(y) \leq t(x) + \widehat{H_V s}(x) < s(x) - a + \varepsilon + \widehat{H_V s}(x) \leq s(x) + \varepsilon.$$

Thus s is upper semicontinuous at x. □

COROLLARY 17.6. *For every $v \in {}^*\mathcal{H}^+(X)$, there exists an increasing sequence (s_n) in $\mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that $v = \sup s_n$.* [6]

Proof. Let (φ_n) be an increasing sequence in $\mathcal{K}^+(X)$, $\varphi_n \uparrow v$. Let $s_n := R_{\varphi_n}$, $n \in \mathbb{N}$. By (H_5) and Proposition 17.2, (s_n) is an increasing sequence in $\mathcal{S}^+(X) \cap \mathcal{C}(X)$. Since $\varphi_n \leq s_n \leq v$ and $\sup \varphi_n = v$, we finally see that $\sup s_n = v$. □

PROPOSITION 17.7. *For every $s \in \mathcal{S}^+(X)$, there exist unique $h \in \mathcal{H}^+(X)$ and $p \in \mathcal{P}(X)$ such that $s = h + p$. Moreover,*

$$h = \inf\{R_p^{K^c} \colon K \text{ compact in } X\}. \tag{17.3}$$

The decomposition of s in Proposition 17.7 is called *Riesz decomposition*, where the functions h and p are called the *harmonic part* and the *potential part* of s, respectively.

Proof of Proposition 17.7. Let $s \in \mathcal{S}^+(X)$ and let g denote the right side of (17.3). By Proposition 17.2, each function $R_p^{K^c}$, K compact in X, is harmonic on the interior of K. Considering an exhaustion of X by a sequence (K_n) it follows from Proposition 16.1 that $g \in \mathcal{H}^+(X)$.

[6] By (H_3), ${}^*\mathcal{H}^+(X) \cap \mathcal{C}(X) = \mathcal{S}^+(X) \cap \mathcal{C}(X)$.

Now let $\tilde{g}$ be any harmonic minorant of s, let K be a compact set in X and $u \in {}^*\mathcal{H}^+(X)$ such that $u \geq s$ on K^c and hence $u \geq \tilde{g}$ on K^c. If W is a relatively compact open neighborhood of K, then $u - \tilde{g} \geq 0$ on W by Proposition 15.2. Therefore $u \geq \tilde{g}$ on X, which, taking the infimum on the functions u and the compact sets K, leads to the inequality $g - \tilde{g} \geq 0$.

Thus g is the greatest harmonic minorant of s, the greatest harmonic minorant of $s - g$ is 0, that is, $s - g$ is a potential, and we have the decomposition $s = g + (s - g)$.

If $h \in \mathcal{H}^+(X)$ and $p \in \mathcal{P}(X)$ such that $s = p + h$, then obviously $h \leq g$ and $g - h$ is a harmonic minorant of p, hence $g - h \leq 0$. Thus $h = g$ and $p = s - g$ finishing the proof. □

REMARK 17.8. In particular,

$$\mathcal{P}(X) = \big\{s \in \mathcal{S}^+(X)\colon \inf\{R_p^{K^c}\colon K \text{ compact in } X\} = 0\big\} \tag{17.4}$$

which immediately implies that $\mathcal{P}(X)$ is a convex cone. Clearly, $\mathcal{P}(X)$ is a hereditary subcone of $\mathcal{S}^+(X)$, that is, if $q \in \mathcal{S}^+(X)$ and $q \leq p$ for some $p \in \mathcal{P}(X)$, then $q \in \mathcal{P}(X)$.

PROPOSITION 17.9. *Let $U \in \mathcal{U}$ and $x \in U$. Then there exists $p \in \mathcal{P}(X) \cap \mathcal{C}(X)$ such that $H_U p(x) < p(x)$.*

Proof. By Proposition 17.7, it suffices to find a function $s \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that $H_U s(x) < s(x)$. By (H_5), there exists a strictly positive function $s_0 \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$. If $H_U(x, \cdot) = 0$, then $H_U s_0(x) = 0 < s_0(x)$. So let us suppose that $H_U(x, \cdot) \neq 0$ and let y be a point in the support of $H_U(x, \cdot)$. By (H_5) and Proposition 17.6, there exist $s_1, s_2 \in \mathcal{S}^+(X) \cap \mathcal{C}(X)$ such that $s_1(x)s_2(y) < s_1(y)s_2(x)$. Defining

$$s := \inf\{s_1(x)s_2, s_2(x)s_1\} \quad \text{and} \quad t := s_2(x)s_1$$

we have $s, t \in \mathcal{S}^+(X)$, $s \leq t$, $s(y) < t(y)$, and $s(x) = t(x)$. Thus

$$H_U s(x) < H_U t(x) \leq t(x) = s(x).$$

□

As desired at the beginning of this section we shall now prove the following.

PROPOSITION 17.10. *$\mathcal{P}(X) \cap \mathcal{C}(X)$ is a function cone and ${}^*\mathcal{H}^+(X)$ is the set of all limits of increasing sequences in $\mathcal{P}(X) \cap \mathcal{C}(X)$.*

Moreover, $(\mathcal{P}(X) \cap \mathcal{C}(X))_\sigma = \mathcal{P}(X) \cap \mathcal{C}(X)$[7] and $R_f \in \mathcal{P}(X) \cap \mathcal{C}(X)$, whenever $f \in \mathcal{C}(X)$ is $\mathcal{P}(X)$-bounded.

Proof. We already know that $\mathcal{F} := \mathcal{P}(X) \cap \mathcal{C}(X)$ is a convex cone. If $f \in \mathcal{C}(X)$ is $\mathcal{P}(X)$-bounded, then $R_f \in \mathcal{P}(X)$ by Proposition 17.2 and Remark 17.8. By Proposition 17.9, every function in $\mathcal{K}(X)$ is $\mathcal{F}$-bounded. Therefore the proof of Proposition 17.6 now shows that ${}^*\mathcal{H}^+(X)$ is the set of all limits of increasing sequences in $\mathcal{F}$. In particular, $\mathcal{F}$ is linearly separating by (H_5).

[7] A closer analysis would reveal that the sum of any sequence of potentials which is superharmonic is a potential.

Next we claim that $\mathcal{F}_\sigma = \mathcal{F}$. So let (p_n) be a sequence in $\mathcal{F}$ such that the sum $\sum_{n=1}^\infty p_n$ is continuous and real. We have to show that p is a potential. Obviously, for all compact subsets K of X and $N \in \mathbb{N}$,

$$R_p^{K^c} \le \sum_{n=1}^N R_{p_n}^{K^c} + \sum_{n=N+1}^\infty p_n. \tag{17.5}$$

Given $x \in X$ and $\varepsilon > 0$, we first may first choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^\infty p_n(x) < \varepsilon$ and then a compact set K in X such that $R_{p_n}^{K^c}(x) < \varepsilon/N$ for every $1 \le n \le N$. Hence, by (17.4), (17.5) implies that $p \in \mathcal{F}$. Using Proposition 17.9, we now easily see that there exists a strictly positive $p_0 \in \mathcal{F}$.

It remains to prove that $\mathcal{F}$ is adapted, that is, $\mathcal{F} \subset o(\mathcal{F})$. Given $p \in \mathcal{F}$, we may choose a decreasing sequence (f_n) in $\mathcal{C}(X)$ such that, for every $n \in \mathbb{N}$, $0 \le f_n \le p$ on X, $f_n = p$ outside a compact set K_n, and the interiors of the sets $\{f_n = 0\}$ cover X. Then (R_{f_n}) is a decreasing sequence in $\mathcal{F}$ such that $\inf R_{f_n} = 0$ by (17.4) and hence $R_{f_n} \downarrow 0$ locally uniformly. So there exists a subsequence (p_n) of (R_{f_n}) such that $q := \sum_{n=1}^\infty p_n \in \mathcal{C}(X)$ and hence $q \in \mathcal{F}$. Since $R_{f_m} = p$ outside K_m, $m \in \mathbb{N}$, we know that, given $N \in \mathbb{N}$, there exists a compact set K in X such that, for every $1 \le n \le N$, $q_n = p$ outside K and hence $q \ge Np$ outside K. Thus $p \in o(q)$. $\square$

18 The simplicial cones $\mathbf{S(W)}$ and $\mathbf{S_0(W)}$

We shall now apply the results of Section 13 to the structure given by the family $(H_U)_{U\in\mathcal{U}}$ of harmonic kernels on X. To that end let us simply write $\mathcal{P}$ instead of $\mathcal{P}(X) \cap \mathcal{C}(X)$. We recall that $\mathcal{P}$ is a function cone and that the set $\mathcal{W}$ of all limits of increasing sequences in $\mathcal{P}$ is ${}^*\mathcal{H}^+(X)$ (Proposition 17.10).

Let W be an arbitrary open subset of X (for example, $W = X$). We define

$$\begin{aligned} S(W) &:= \{s \in \mathcal{C}_\mathcal{P}(X) \colon s \text{ is superharmonic on } W\}, \\ S_0(W) &:= \bigcup \{S(V) \colon \overline{W} \subset V,\ V \text{ open}\}. \end{aligned}$$

Of course, $S_0(W) \subset S(W)$ and hence $Ch_{S_0(W)}X \subset Ch_{S(W)}X$. (It is interesting to study the question under which conditions on W the cone $S_0(W)$ is dense in $S(W)$ and hence $Ch_{S_0(W)}X = Ch_{S(W)}X$, see [6].)

Moreover, let

$$\begin{aligned} H(W) &:= \{h \in \mathcal{C}_\mathcal{P}(X) \colon h \text{ is harmonic on } W\} = S(W) \cap (-S(W)), \\ H_0(W) &:= \bigcup \{H(V) \colon \overline{W} \subset V,\ V \text{ open}\} = S_0(W) \cap (-S_0(W)). \end{aligned}$$

By Proposition 17.5, the mappings $v \mapsto \widehat{H_U v}$ define $\mathcal{P}$-dilations T_U, $U \in \mathcal{U}$, such that $T_U(x,\cdot) = H_U(x,\cdot)$ for every $x \in U$ and $T_U(x,\cdot) = \varepsilon_x$ for every $x \in X \setminus \overline{U}$. We claim that

$$S(W) = S(\mathcal{T}), \quad \text{where } \mathcal{T} = \{T_U \colon U \in \mathcal{U}(W)\}).$$

Of course, $S(\mathcal{T}) \subset S(W)$, since a function $s \in \mathcal{C}_{\mathcal{P}}(X)$ is superharmonic on W if and only if, for every $U \in \mathcal{U}(W)$, $H_U s \le s$ on U. The converse follows immediately by the following lemma (cf. Proposition 3.3).

LEMMA 18.1. *Let $U \in \mathcal{U}$ and $z \in \partial U$. Then the measure $T_U(z,\cdot)$ is supported by ∂U and there exists a sequence (x_n) in U such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} H_U(x_n,\cdot) = T_U(z,\cdot)$, that is, $\lim_{n\to\infty} H_U f(x_n) = T_U f(z)$ for every $f \in \mathcal{C}(\partial U)$.*

Proof. By Corollary 17.5, $T_U q(z) = \liminf_{x\to z, x\in U} H_U q(x)$ for every $q \in \mathcal{P}$. Fixing a strict $p \in \mathcal{P}$ we may hence choose a sequence (x_n) in U such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} H_U p(x_n) = T_U p(x)$. Of course, for every $q \in \mathcal{P}$, $\liminf_{n\to\infty} H_U q(x_n) \ge T_U q(x)$. This implies that $\lim_{n\to\infty} H_U(x_n,\cdot) = T_U(z,\cdot)$ and hence $T_U(z,\cdot)$ is supported by ∂U. $\square$

Thus the results of Section 13 imply the following (which can be applied to the classical case and to the heat equation).

THEOREM 18.2. *$S(W)$ is a simplicial cone. The minimal representing measures form an idempotent $\mathcal{P}$-dilation D and, for every $p \in \mathcal{P}$,*

$$Dp = \sup\{q \in H(W) \cap \mathcal{P} \colon q \le p\} = R_p^{Ch_{S(W)}X}.$$

COROLLARY 18.3. *Suppose that $H(W)$ is a function space. Then $H(W)$ is simplicial, $\mathrm{Ch}_{H(W)}X = \mathrm{Ch}_{S(W)}X$, and the minimal representing measures with respect to $H(W)$ and $S(W)$ coincide.*

Of course, we know that $Ch_{S(W)}X \subset X \setminus W$. It is not so easy to really determine the Choquet boundary in potential-theoretic terms (see [2] and [4]), as we did in the classical case (see Theorem 5.1).

THEOREM 18.4. *$S_0(W)$ is a simplicial cone. The minimal representing measures form an idempotent $\mathcal{P}$-dilation D and, for every $p \in \mathcal{P}$,*

$$Dp = \sup\{q \in H_0(W) \cap \mathcal{P} \colon q \le p\} = R_p^{Ch_{S_0(W)}X}.$$

COROLLARY 18.5. *Suppose that $H_0(W)$ is a function space. Then $H_0(W)$ is simplicial, $\mathrm{Ch}_{H_0(W)}X = \mathrm{Ch}_{S_0(W)}X$, and the minimal representing measures with respect to $H_0(W)$ and $S_0(W)$ coincide.*

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