Test 28. ledna 2019

Write each exercise on separate sheet of paper. All submitted papers must be signed.

Exercise 1 (7 points)**.** Jaromír repeatedly tries to beat computer in chess game. He gets 10 points for winning or loses 2 points loss. Jaromír wins with the probability $1/4$.

(a) Find the average number of games up to Jaromír first won game (including this game).
(b) Suppose Jaromír has now 6 points. Find the probability that anytime during next five games he will have zero or less points.
(c) Determine the distribution of Jaromír points after five games if he starts with 0 points. (negative points are allowed).

(You may use the fact

$$\sum_{k=1}^{\infty} kp^{k-1} = \frac{\mathrm{d}}{\mathrm{d}p} \sum_{k=0}^{\infty} p^k.$$

if $p \in (0,1)$ in (a) part)

Exercise 2 (5 points)**.** (a) Define joint distribution function of random vector (X, Y).

(b) Describe how the distribution function of X is determined if X is an element of random vector náhodného vektoru (X, Y).
(c) Write how is the independence of X and Y characterized by their joint distributon function? Write more conditions for independence of X and Y.

Exercise 3 (7 points)**.** The number of successes in 20 independent trials with probability of success $p \in (0,1)$ in each trial is denoted by X.

(a) Find an estimate of p based on *one realisation* of random variable X. (Recall what X represents in reality).
(b) Find at least two estimates of the veriance $\mathrm{var}X$.
(c) Suppose $X = 9$. Calculate these estimates if possible.

Exercise 4 (6 points)**.** The waiting time between two events may be considered to be continuous random variable X with probability density function

$$f(x) = \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) What is the point estimator of the mean value $\mathsf{E}X$?
(b) Find an asymptotic interval estimate for $\mathsf{E}X$ with confidence level $0,95$.
(c) Find this interval estimate if the average of 100 measurements of the waiting time is 12.

(Use appropriate methods based on limit theorems. The distribution function of standard normal distribution may be found in the table below. The formula $\int_0^\infty x^k \exp(-x)\mathrm{d}x = k!$ for any positive integer k may be useful.)

Exercise 5 (6 points)**.** Define convergence in probability.

(a) Write the law of large numbers and explain what it says.
(b) Consider random sample $X_1, X_2, \ldots, X_n$ from the distribution with mean $\mathsf{E}X$ and finite variance. Write some sufficient condition for f to ensure that $f(\bar{X_n})$ is a consistent estimate of $f(\mathsf{E}X)$?

Exercise 6 (5 points)**.** Toss a 1Kč coin and 2Kč coin independetnly. Let the result of the toss is the value of the coin if the coin shows the head or zero if it shows the tail. Denote (X, Y) the random vector of X the sum of results and Y is the difference of results of 2Kč coin and 1Kč coin. (so, e.g. X may be 0 if both tails appear or 1 if the 1Kč coin shows head and 2Kč coin tail, etc.)

(a) Find the joint and marginal distributions of the vector (X, Y). Are X and Y independent?
(b) Calculate the correlation of X and Y.

Some values of the distribution function of the standard normal distribution:

x	0	0.5	1	1.5	2	2.5	3
$\Phi(x)$	0.5000	0.6915	0.8413	0.9332	0.9772	0.9938	0.9987
x	0.01	0.05	0.10	0.50	0.90	0.95	0.99
$\Phi^{-1}(x)$	-2.3263	-1.6449	-1.2816	0	1.2816	1.6449	2.3263

Remarks: Each exercise has given some number of points. There are 36 point in total. You need at least 20 points to pass the exam. The starred parts are bonus parts and you may get some additional points to get better grade.

The results may not be easily expressed. There may be a series that you cannot add up. Such series may be, however, the correct result.