Minimal Defect

Consider the system of n^2 equations

$$S = \{(xu_i w_j v_i y, yu_i w_j v_i x) \mid i, j = 1, 2, \dots, n\}\,.$$

Obviously, it has a solution of rank $3n+1$ given by $x = y$. The rank is maximal allowed by the Graph lemma, namely $|\Theta| - 1$. We claim that the system is independent. In order to show this, we have to find words x, y, u_k, v_k, w_k, $k = 1, 2$, such that

$$\begin{aligned} xu_1w_1v_1y &\neq yu_1w_1v_1x, \\ xu_1w_2v_1y &= yu_1w_2v_1x, \\ xu_2w_1v_2y &= yu_2w_1v_2x, \\ xu_2w_2v_2y &= yu_2w_2v_2x\,. \end{aligned}$$

Without loss of generality, we can assume that x is longer than y. We obtain conjugate words $\alpha\beta = y^{-1}x$ and $\beta\alpha = xy^{-1}$ such that

$$\begin{aligned} \alpha\beta u_1w_1v_1 &\neq u_1w_1v_1\beta\alpha, \\ \alpha\beta u_1w_2v_1 &= u_1w_2v_1\beta\alpha, \\ \alpha\beta u_2w_1v_2 &= u_2w_1v_2\beta\alpha, \\ \alpha\beta u_2w_2v_2 &= u_2w_2v_2\beta\alpha\,. \end{aligned}$$

The solution of the conjugation equation implies that $u_1w_2v_1, u_2w_1v_2, u_2w_2v_2 \in \alpha(\beta\alpha)^*$, while $u_1w_1v_1 \notin \alpha(\beta\alpha)^*$. Let $\alpha = a$ and $\beta = bb$, and let

$$u_1w_2v_1 = u_2w_2v_2 = abba\,.$$

Setting $w_2 = b$, we have two possible positions for w_2 in $abba$, yielding its contexts $(u_1, v_1) = (a, ba)$ and $(u_2, v_2) = (ab, a)$. It remains to choose a suitable w_1 which will fit between (u_2, v_2) while not between (u_1, v_1). Putting $u_2w_1u_2 = abbabba$ yields $w_1 = babb$, and indeed, we have $ababbba \notin \alpha(\beta\alpha)^*$. Finally, we have, for example, $y = bb$ and $x = bbabb$.