EE 261 The Fourier Transform and its Applications
Fall 2006
Final Exam Solutions

Notes:

There are 7 questions for a total of 120 points

Write all your answers in your exam booklets

When there are several parts to a problem, in many cases the parts can be done independently, or the result of one part can be used in another part.

Please be neat and indicate clearly the main parts of your solutions

1. (15 points) Let $f(t)$ be a periodic signal of period 1. One says that $f(t)$ has *half-wave symmetry* if

$$f(t - \frac{1}{2}) = -f(t)\,.$$

(a) Sketch an example of a signal that has half-wave symmetry.

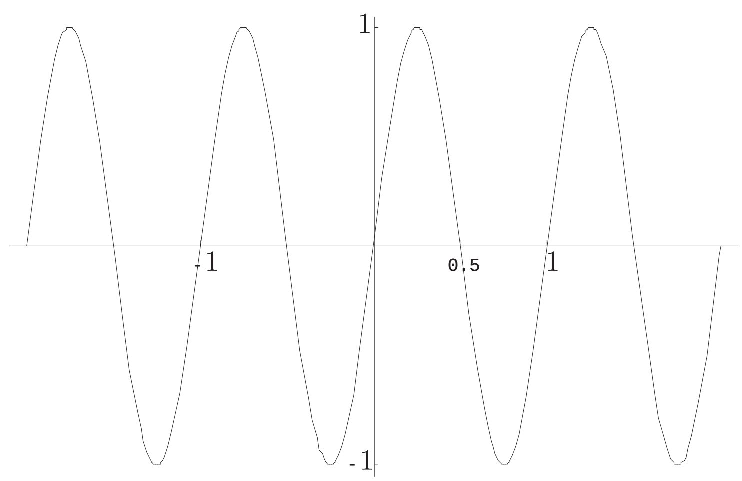
(b) If $f(t)$ has half-wave symmetry and its Fourier series is

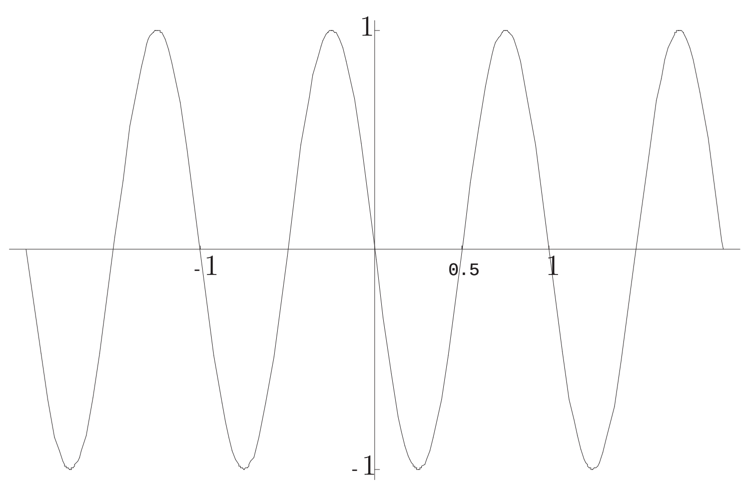
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$$

show that $c_n = 0$ if n is even.
Hint: $-c_n = -\int_0^1 e^{-2\pi i n t} f(t)\, dt = \int_0^1 e^{-2\pi i n t} f(t - \frac{1}{2})\, dt$.

Solution:

(a) A simple example is $f(t) = \sin(2\pi t)$. The graphs of $\sin(2\pi t)$ and $\sin(2\pi(t - \frac{1}{2}))$ are shown.





Algebraically,

$$\sin(2\pi(t - \frac{1}{2})) = \sin(2\pi t - \pi) = -\sin 2\pi t\,.$$

(b) The hint says

$$-c_n = -\int_0^1 e^{-2\pi int} f(t)\,dt = \int_0^1 e^{-2\pi int} f(t-\frac{1}{2})\,dt\,.$$

We make a change of variable $u = t - \frac{1}{2}$ in the second integral:

$$\begin{aligned}\int_0^1 e^{-2\pi int} f(t-\frac{1}{2})\,dt &= \int_{-1/2}^{1/2} e^{-2\pi in(u+\frac{1}{2})} f(u)\,du \\ &= \int_{-1/2}^{1/2} e^{-2\pi inu} e^{-2\pi in\frac{1}{2}} f(u)\,du \\ &= e^{-\pi in} \int_{-1/2}^{1/2} e^{-2\pi inu} f(u)\,du \\ &= e^{-\pi in} c_n, \quad \text{(because we can integrate over any cycle to compute } c_n\text{).}\end{aligned}$$

Thus

$$-c_n = e^{-\pi in} c_n\,.$$

If n is even then $e^{-\pi in} = 1$ and we have

$$-c_n = c_n\,,$$

hence

$$c_n = 0\,.$$

A slightly different route to the same end is as follows. Again it uses the substitution $u = t - \frac{1}{2}$ in an integral.

$$\begin{aligned}c_n &= \int_0^1 e^{-2\pi int} f(t)\,dt \\ &= \int_0^{1/2} e^{-2\pi int} f(t)\,dt + \int_{1/2}^1 e^{-2\pi int} f(t)\,dt \\ &= \int_0^{1/2} e^{-2\pi int} f(t)\,dt - \int_{1/2}^1 e^{-2\pi int} f(t-\frac{1}{2})\,dt \\ &= \int_0^{1/2} e^{-2\pi int} f(t)\,dt - \int_0^{1/2} e^{-2\pi in(u+\frac{1}{2})} f(u)\,du \\ &= \int_0^{1/2} e^{-2\pi int} f(t)\,dt - e^{-\pi in} \int_0^{1/2} e^{-2\pi inu} f(u)\,du\,,\end{aligned}$$

and if n is even the integrals cancel, giving $c_n = 0$.

2. (20 points) *Sampling using the derivative* Suppose that $f(t)$ is a bandlimited signal with $\mathcal{F}f(s) = 0$ for $|s| \geq 1$ (bandwidth 2). According to the sampling theorem, knowing the values $f(n)$ for all integers n (sampling rate of 1) is not sufficient to interpolate the values $f(t)$ for all t. However, if *in addition* one knows the values of the derivative $f'(n)$ at the integers then there is an interpolation formula with a sampling rate of 1. In this problem you will derive that result.

Let $F(s) = \mathcal{F}f(s)$ and let $G(s) = \frac{1}{2\pi i}(\mathcal{F}f')(s) = sF(s)$.

(a) For $0 \leq s \leq 1$ show that

$$(Ш * F)(s) = F(s) + F(s-1)$$
$$(Ш * G)(s) = sF(s) + (s-1)F(s-1)$$

and then show that

$$F(s) = (1-s)(Ш * F)(s) + (Ш * G)(s)\,.$$

(b) For $-1 \leq s \leq 0$ show that

$$(Ш * F)(s) = F(s) + F(s+1)$$
$$(Ш * G)(s) = sF(s) + (s+1)F(s+1)$$

and then show that

$$F(s) = (1+s)(Ш * F)(s) - (Ш * G)(s)\,.$$

(c) Using parts (a) and (b) show that for all s, $-\infty < s < \infty$,

$$F(s) = \Lambda(s)(Ш * F)(s) - \Lambda'(s)(Ш * G)(s)\,,$$

where $\Lambda(s)$ is the triangle function

$$\Lambda(s) = \begin{cases} 1-|s|, & |s| \leq 1\,, \\ 0, & |s| \geq 1\,. \end{cases}$$

(d) From part (c) derive the interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\mathrm{sinc}^2(t-n) + \sum_{n=-\infty}^{\infty} f'(n)(t-n)\mathrm{sinc}^2(t-n)\,.$$

Solutions:

(a) Since $F(s)$ is zero outside of $|s| \leq 1$, if we restrict s to lie between 0 and 1 we only get a few shifts $F(s-n)$ that are nonzero, namely,

$$\begin{aligned}(Ш * F)(s) &= \sum_{n=-\infty}^{\infty} F(s-k) \\ &= \underbrace{\cdots + F(s+2) + F(s+1)}_{=0} + F(s) + F(s-1) + \underbrace{F(s-2) + \cdots}_{=0} \\ &= F(s) + F(s-1)\end{aligned}$$

Since $G(s) = sF(s)$ it's the same thing; only two nonzero terms in $Ш * G$:

$$(Ш * G)(s) = \sum_{n=-\infty}^{\infty} (s-k)F(s-k) = sF(s) + (s-1)F(s-1)\,.$$

Taking these two equations together, we multiply the first by $s-1$,

$$\begin{aligned}(s-1)(Ш * F)(s) &= (s-1)F(s) + (s-1)F(s-1)\\ (Ш * G)(s) &= sF(s) + (s-1)F(s-1)\end{aligned}$$

and subtract the first from the second, giving

$$(Ш * G)(s) - (s-1)(Ш * F)(s) = F(s)\,,$$

or

$$F(s) = (1-s)(Ш * F)(s) + (Ш * G)(s)\,.$$

This holds when $0 \le s \le 1$.

(b) The reasoning is very similar if $-1 \le s \le 0$. In this case

$$(Ш * F)(s) = \sum_{n=-\infty}^{\infty} F(s-k) = F(s+1) + F(s)$$

and

$$(Ш * G)(s) = \sum_{n=-\infty}^{\infty} (s-k)F(s-k) = (s+1)F(s+1) + sF(s)\,.$$

Multiply the first equation by $s+1$:

$$(s+1)(Ш * F)(s) = (s+1)F(s+1) + (s+1)F(s)$$

and subtract the second equation,

$$(s+1)(Ш * F)(s) - (Ш * G)(s) = F(s)\,.$$

(c) We now have

$$\begin{aligned}F(s) &= (1-s)(Ш * F)(s) + (Ш * G)(s)\,, \quad 0 \le s \le 1\\ F(s) &= (1+s)(Ш * F)(s) - (Ш * G)(s)\,, \quad -1 \le s \le 0\,.\end{aligned}$$

Moreover, $F(s) = 0$ outside the interval $-1 \le s \le 1$. From

$$\Lambda(s) = \begin{cases} 1-|s|, & |s| \le 1\,, \\ 0, & |s| \ge 1\,. \end{cases} = \begin{cases} 1-s\,, & 0 \le s \le 1 \\ 1+s\,, & -1 \le s \le 0 \\ 0\,, & |s| \ge 1 \end{cases}$$

and

$$\Lambda'(s) = \begin{cases} -1\,, & 0 \le s \le 1 \\ 1\,, & -1 \le s \le 0 \\ 0\,, & |s| \ge 1 \end{cases}$$

we see that we can write

$$F(s) = \Lambda(s)(Ш * F)(s) - \Lambda'(s)(Ш * G)(s)$$

for all $-\infty < s < \infty$.

(d) To derive the interpolation formula we take the inverse Fourier transform of

$$F(s) = \Lambda(s)(\text{Ш} * F)(s) - \Lambda'(s)(\text{Ш} * G)(s)\,.$$

We work separately with the two terms on the right. For the first,

$$\begin{aligned}
\mathcal{F}^{-1}(\Lambda(\text{Ш} * F))(t) &= \operatorname{sinc}^2 t * (f(t)\text{Ш}(t)) \\
&= \operatorname{sinc}^2 t * \left(\sum_{n=-\infty}^{\infty} f(n)\delta(t-n)\right) \\
&= \sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}^2(t-n)\,.
\end{aligned}$$

For the second, first note that by duality

$$\mathcal{F}^{-1}\Lambda'(t) = -2\pi i t \mathcal{F}^{-1}\Lambda(t) = -2\pi i t \operatorname{sinc}^2 t\,.$$

Then

$$\begin{aligned}
\mathcal{F}^{-1}(\Lambda'(\text{Ш} * G))(t) &= (-2\pi i t \operatorname{sinc}^2 t) * (\frac{1}{2\pi i} f'(t)\text{Ш}(t)) \\
&= -(t \operatorname{sinc}^2 t) * \left(\sum_{n=-\infty}^{\infty} f'(n)\delta(t-n)\right) \\
&= -\sum_{n=-\infty}^{\infty} f'(n)(t-n)\operatorname{sinc}^2(t-n)\,.
\end{aligned}$$

Putting these two results together we obtain the interpolation formula

$$\begin{aligned}
f(t) &= \mathcal{F}^{-1}(\Lambda(\text{Ш} * F))(t) - \mathcal{F}^{-1}(\Lambda'(\text{Ш} * G))(t) \\
&= \sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}^2(t-n) + \sum_{n=-\infty}^{\infty} f'(n)(t-n)\operatorname{sinc}^2(t-n)\,.
\end{aligned}$$

3. (20 points) *The DFT and linear interpolation*

(a) Let $\underline{y}$ be the discrete signal, periodic of order M,

$$\underline{y} = (1\,, \frac{1}{2}\,, 0\,, \dots\,, 0\,, \frac{1}{2})$$

Show that its DFT is

$$\underline{Y}[m] = 1 + \cos(2\pi m/M)\,.$$

(b) Let $\underline{f} = (\underline{f}[0]\,, \underline{f}[1]\,, \dots\,, \underline{f}[N-1])$ be a discrete signal and let $\underline{F} = (\underline{F}[0]\,, \underline{F}[1]\,, \dots\,, \underline{F}[N-1])$ be its DFT. Recall that the upsampled version of $\underline{f}$ is the signal $\underline{h}$ of order $2N$ obtained by inserting zeros between the values of $\underline{f}$, i.e.,

$$\underline{h} = (\underline{f}[0]\,, 0\,, \underline{f}[1]\,, 0\,, \underline{f}[2]\, \dots\,, 0\,, \underline{f}[N-1]\,, 0)\,.$$

Show that $\underline{\tilde{f}} = \underline{h} * \underline{y}$ is the 'linearly interpolated' version of $\underline{f}$:

$$(\underline{f}[0]\,, \frac{\underline{f}[0]+\underline{f}[1]}{2}\,, \underline{f}[1]\,, \frac{\underline{f}[1]+\underline{f}[2]}{2}\,, \underline{f}[2]\,, \dots\,, \underline{f}[N-1]\,, \frac{\underline{f}[N-1]+\underline{f}[0]}{2})\,.$$

Hint: Here we take $M = 2N$ for the period of $\underline{y}$. Note that

$$\underline{y} = \underline{\delta}_0 + \frac{1}{2}\underline{\delta}_1 + \frac{1}{2}\underline{\delta}_{2N-1}$$

and remember the effect of convolving with a shifted discrete δ. Line up the $2N$-tuples.

(c) In a problem set you showed that the DFT of $\underline{h}$ is a replicated form of $\underline{F}$,

$$\underline{H}[m] = \underline{F}[m] \quad m = 0, 1, \dots, 2N-1$$

$$\underline{H} = \overbrace{(\underbrace{\underline{F}[0]\,, \underline{F}[1]\,, \dots\,, \underline{F}[N-1]}_{\underline{F}}\,, \underbrace{\underline{F}[0]\,, \underline{F}[1]\,, \dots\,, \underline{F}[N-1]}_{\underline{F}})}^{\underline{H}}$$

Assuming this, find the DFT of $\underline{\tilde{f}}$.

Solutions:

(a) From the definition of the DFT:

$$\begin{aligned}
\underline{Y}[m] &= \sum_{n-0}^{M-1} \underline{k}[n] e^{-2\pi i m n/M} \\
&= 1 + \frac{1}{2}e^{-2\pi i m/M} + \frac{1}{2}e^{-2\pi i m(M-1)/M} \\
&= 1 + \frac{1}{2}e^{-2\pi i m/M} + \frac{1}{2}e^{-2\pi i m}e^{2\pi i m/M} \\
&= 1 + \frac{1}{2}e^{-2\pi i m/M} + \frac{1}{2}e^{2\pi i m/M} \\
&= 1 + \cos(2\pi m/M)\,.
\end{aligned}$$

(b) We have that

$$\begin{aligned}\underline{\mathrm{h}} * \underline{\mathrm{y}} &= \underline{\mathrm{h}} * (\underline{\delta}_0 + \frac{1}{2}\underline{\delta}_1 + \frac{1}{2}\underline{\delta}_{2N-1}) \\ &= \underline{\mathrm{h}} + \frac{1}{2}(\underline{\mathrm{h}} \text{ shifted by } 1) + \frac{1}{2}(\underline{\mathrm{h}} \text{ shifted by } 2N-1)\end{aligned}$$

With

$$\underline{\mathrm{h}} = (\underline{\mathrm{f}}[0]\,,\,0\,,\,\underline{\mathrm{f}}[1]\,,\,0\,,\,\underline{\mathrm{f}}[2]\;\ldots\,,\,0\,,\,\underline{\mathrm{f}}[N-1]\,,\,0)$$

we have, using periodicity,

$$\frac{1}{2}(\underline{\mathrm{h}} \text{ shifted by } 1) = (0\,,\,\frac{1}{2}\underline{\mathrm{f}}[1]\,,\,0\,,\,\frac{1}{2}\underline{\mathrm{f}}[2]\,,\ldots,\,0\,,\,\frac{1}{2}\underline{\mathrm{f}}[N-1]\,,\,0,\frac{1}{2}\underline{\mathrm{f}}[0])\,.$$

Shifting the components in $\underline{\mathrm{h}}$ to the left by $2N-1$ has, by periodicity, the same effect as shifting them to the right by 1, since $\underline{\mathrm{h}}[k-(2N-1)] = \underline{\mathrm{h}}[k+1-2N] = \underline{\mathrm{h}}[k+1]$. So

$$\frac{1}{2}(\underline{\mathrm{h}} \text{ shifted by } 2N-1) = (0\,,\,\frac{1}{2}\underline{\mathrm{f}}[0]\,,\,0\,,\,\frac{1}{2}\underline{\mathrm{f}}[1]\,,\,\ldots\,,0\,,\frac{1}{2}\underline{\mathrm{f}}[N-1])$$

Adding these up,

$$\begin{aligned}&\underline{\mathrm{h}} + \frac{1}{2}(\underline{\mathrm{h}} \text{ shifted by } 1) + \frac{1}{2}(\underline{\mathrm{h}} \text{ shifted by } 2N-1) = \\ &+ (\underline{\mathrm{f}}[0]\,,\,0\,,\,\underline{\mathrm{f}}[1]\,,\,0\,,\,\underline{\mathrm{f}}[2]\,,\ldots,\,0\,,\,\underline{\mathrm{f}}[N-1]\,,\,0) \\ &+ (0\,,\,\frac{1}{2}\underline{\mathrm{f}}[1]\,,\,0\,,\,\frac{1}{2}\underline{\mathrm{f}}[2]\,,\ldots,\,0\,,\,\frac{1}{2}\underline{\mathrm{f}}[N-1]\,,\,0,\frac{1}{2}\underline{\mathrm{f}}[0]) \\ &+ (0\,,\,\frac{1}{2}\underline{\mathrm{f}}[0]\,,\,0\,,\,\frac{1}{2}\underline{\mathrm{f}}[1]\,,\,\ldots\,,0\,,\frac{1}{2}\underline{\mathrm{f}}[N-1]) \\ &= (\underline{\mathrm{f}}[0]\,,\,\frac{\underline{\mathrm{f}}[0]+\underline{\mathrm{f}}[1]}{2}\,,\,\underline{\mathrm{f}}[1]\,,\,\frac{\underline{\mathrm{f}}[1]+\underline{\mathrm{f}}[2]}{2}\,,\,\underline{\mathrm{f}}[2]\,,\ldots,\,\underline{\mathrm{f}}[N-1]\,,\,\frac{\underline{\mathrm{f}}[N-1]+\underline{\mathrm{f}}[0]}{2})))\,.\end{aligned}$$

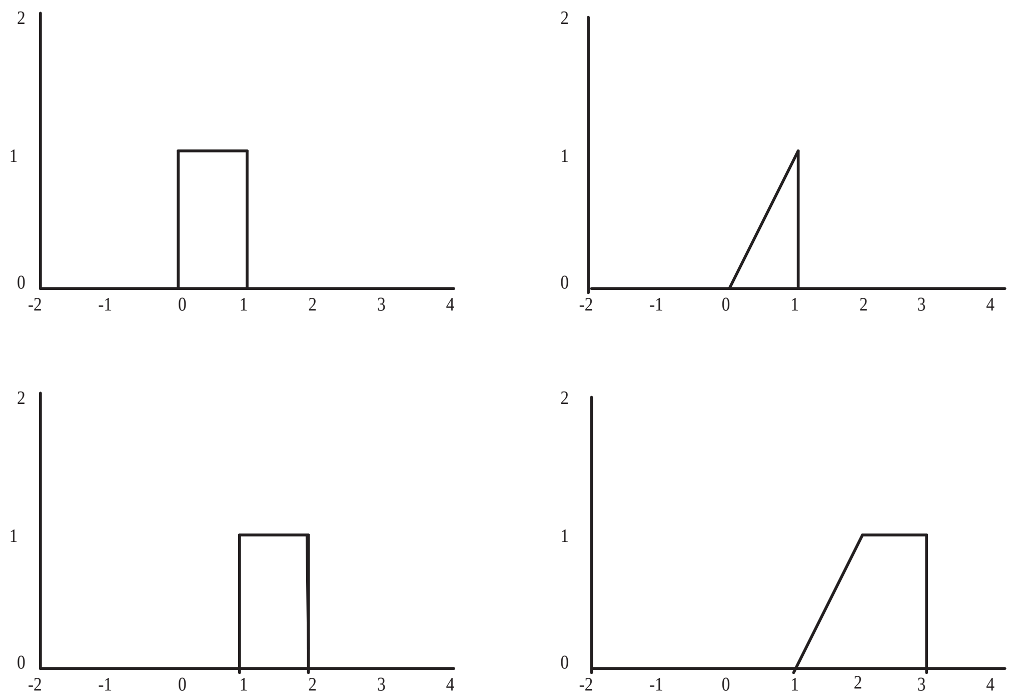
(c) From Part (a) the DFT of $\underline{\mathrm{y}}$ is

$$\underline{Y}[n] = 1 + \cos(2\pi m/M)\,.$$

Thus, by the convolution theorem,

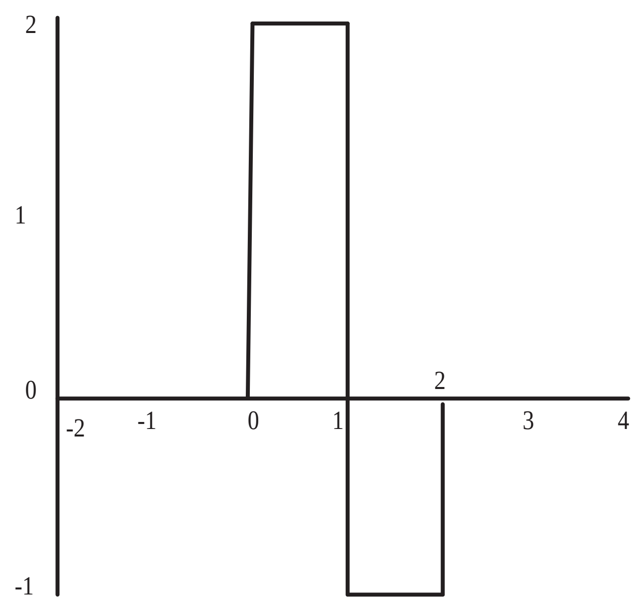
$$\begin{aligned}\underline{\mathcal{F}}\,\widetilde{\underline{\mathrm{f}}}[m] &= (\underline{\mathcal{F}}\,\underline{\mathrm{h}}[m])(\underline{\mathcal{F}}\,\underline{\mathrm{y}}[m]) \\ &= \underline{H}[m]\underline{Y}[m] \\ &= \underline{\mathrm{F}}[m](1+\cos(2\pi m/M))\,.\end{aligned}$$

4. (10 points) A linear system L has the inputs (on the left) and outputs (on the right) shown below.



(a) Is L time-invariant? Justify your answer.

(b) Sketch the output of L given the input below.



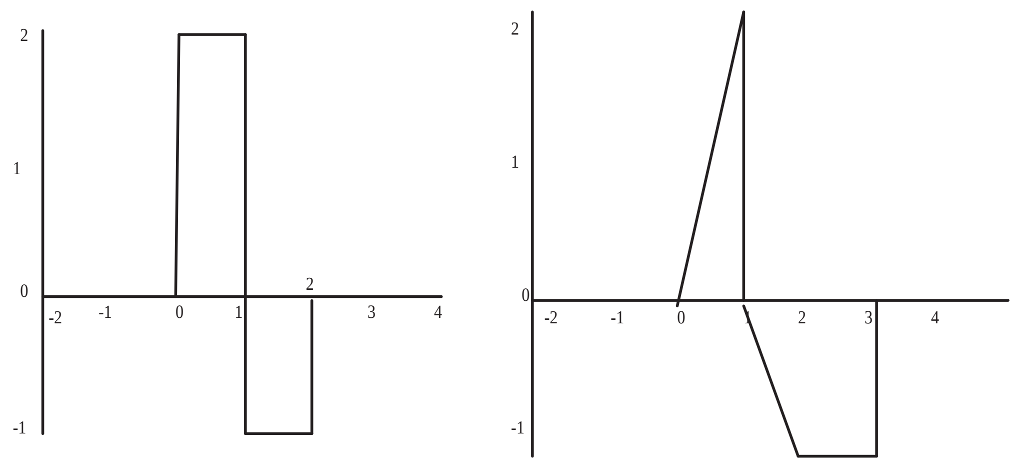
Solutions:

(a) L is not time-invariant. Call the inputs and outputs $v_1(t)$, $w_1(t) = Lv_1(t)$ and $v_2(t)$, $w_2(t) = Lv_2(t)$, respectively. Evidently $v_2(t) = v_1(t-1)$, and if L were time invariant we would have $w_2(t) = w_1(t-1)$. We don't.

(b) The sample input is $2v_t(t) - v_2(t)$. Since L is linear,

$$L(2v_1(t) - v_2(t)) = 2Lv_1(t) - Lv_2(t) = 2w_1(t) - w_2(t)\,,$$

which looks like this:



5. (15 points) Suppose we model the Stanford Clock Tower bells as a system, where the hammer (to hit the bell) is the input, the bell is the system, and the ringing sound is the output.

   (a) Is the system linear (approximately)? Is it time invariant?

   (b) Is this system stable? (Recall that 'stable' means bounded inputs result in bounded outputs.)

   (c) Give an analytic expression that might represent the impulse response, $h(t)$, of the system. Justify your answer.

Solution:

(a) It's reasonable to consider that the system is linear. If we hit the bell with two hammers (adding the inputs) then the sound will mix additively. If we hit the bell twice as hard, for example, the sound will be (approximately) twice as loud. Might be interesting to know how realistic this is!

Likewise, it's reasonable to consider that the system is time-invariant, since if we hit it later we just get the sound later. (Though the shifted sound might be mixing with the fading sound from the earlier hammer blow.)

(b) Stability means that a bounded input will result in a bounded output. This system is stable because the output cannot be unbounded - the ringing sound will never be infinitely loud.

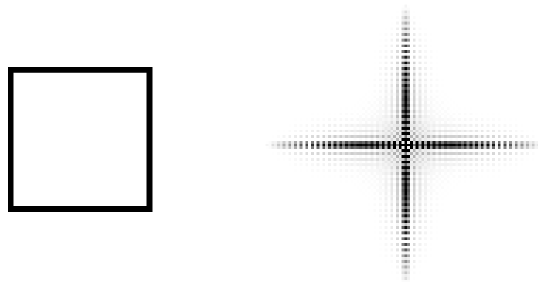
(c) We can consider the impulse response as the output due to an input impulse – imagine hitting the bell with a hammer for a very short period of time. The impulse response must be zero for $t < 0$ because the bell cannot produce sound without some kind of input. This indicates the need to use the unit step function $u(t)$.

Secondly, because the sound is ringing and oscillatory, we know that the impulse response should include a sinusoidal term of some frequency ν_0.

Lastly, the ringing sound is decaying – the bell will not continue to make sound forever and the sound is in fact dying away. Thus, we will need to multiply our impulse response with a decaying exponential term. The impulse response should look something like this:

$$h(t) = u(t)e^{-at}\sin(\nu_0 t)\,, \quad a > 0\,.$$

6. (30 points) Consider the square shown below to be represented by a functions $f(x_1, x_2)$ of x_1 and x_2. The gray level shows the value at a given point, with black being 1 and white being 0. Next to $f(x_1, x_2)$ is a plot of the magnitude of its Fourier transform $|\mathcal{F}f(\xi_1, \xi_2)|$.



Let $a > 1$ be a fixed constant, and let A be the matrix

$$A = \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{pmatrix}$$

The problem involves a number of figures and is stated on the next page.

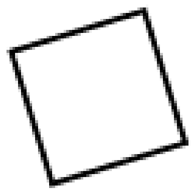
Consider the following modifications of $f(x_1, x_2)$:

1. $f(ax_1, x_2)$ 2. $f(x_1, ax_2)$ 3. $f(x_1 + a, x_2)$

4. $f\left(A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$ 5. $f\left(A^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$ 6. $f(x_1, x_2) * \operatorname{sinc}(ax_1) \operatorname{sinc}(ax_2)$

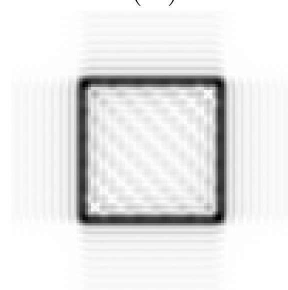
Match the modification 1 – 6 of $f(x_1, x_2)$ with the corresponding plot (i) – (vi) *and* with the plot of the corresponding Fourier transform (A) – (F). Give brief explanations.

(i)

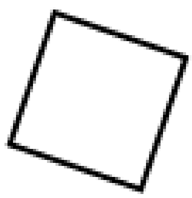


(ii)

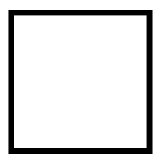
(iii)



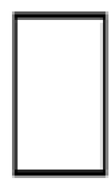
(iv)



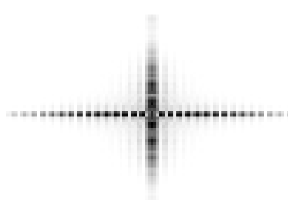
(v)



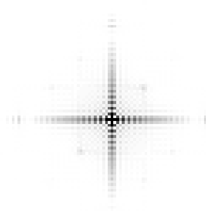
(vi)



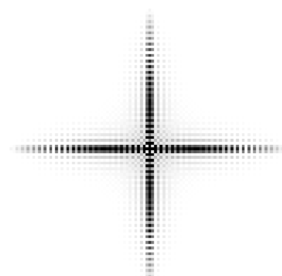
(A)



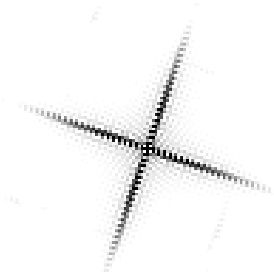
(B)



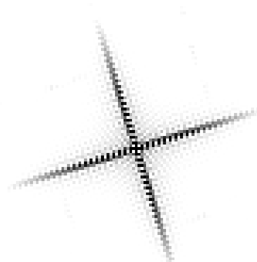
(C)



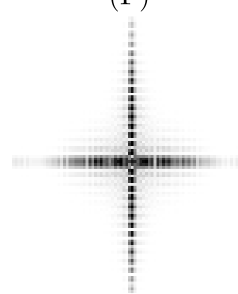
(D)



(E)



(F)



Solutions:

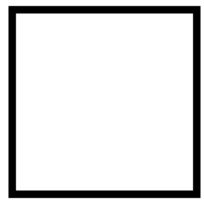
The function $f(x_1, x_2)$ is black over the square shown. To make the matches, you have to ask yourself what set of points will correspond to those points under the given transformation of f. The matches are as follows:

1. $f(ax_1, x_2)$: Since $a > 1$ this shrinks the square in the x_1-direction and so that's figure (vi), i.e., the rectangle in figure (vi) will be stretched to the square via $(x_1, x_2) \mapsto (ax_1, x_2)$ and so $f(ax_1, x_2)$ will be black over the rectangle in (vi). For the Fourier transform, $\mathcal{F}(f(ax_1, x_2) = (1/a)\mathcal{F}f(\xi_1/a, \xi_2)$, so that's *stretched* in the ξ_1 direction and the whole figure is likewise stretched. This matches with figure (A).
2. $f(x_1, ax_2)$. Here the reasoning is the same as in the previous part, but applied in the x_2 and ξ_2 directions. The matches are with (ii) and (F).
3. $f(x_1 + a, x_2)$. This is a shift in the x_1 direction by an amount a to the left, so the square hasn't changed shape, just location relative to where it was before. There is only a phase change in the Fourier transform, which has magnitude 1, thus the plot is the same as for $\mathcal{F}f(\xi_1, \xi_2)$. The matches are with (v) and (C).
4. $f\left(A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$. The matrix A is a counterclockwise rotation through $\pi/6$. As we showed in class, the spectrum is also rotated counterclockwise by $\pi/6$. Since to rotation is counterclockwise by $\pi/6$ the square in (iv) is rotated to the 'straight' square. Similarly with the Fourier transform. The matches are with (iv) and (D).
5. $f\left(A^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$. This is a clockwise rotation through $\pi/6$. The matches are with (i) and (E).
6. $f(x_1, x_2) * \operatorname{sinc}(ax_1)\operatorname{sinc}(ax_2)$. Well, there's only one choice left, so the matches have to be with (iii) and (B). Here's an explanation. Take the Fourier transform to get, by the convolution theorem,

   $$\mathcal{F}(f(x_1, x_2) * \operatorname{sinc}(ax_1)\operatorname{sinc}(ax_2)) = \mathcal{F}f(\xi_1, \xi_2)\Pi_a(\xi_1)\Pi_a(\xi_2)\,.$$

   This cuts off $\mathcal{F}f(\xi_1, \xi_2)$ by a 2D-rect function of width a. That matches with figure (B) (approximately – numerical computations, of course). In the spacial domain the effect is that of a lowpass filter, so the sharp edges of the square are smeared out somewhat. That's figure (iii).

7. (10 points) Again consider the square shown below to be represented by a functions $f(x_1, x_2)$ of x_1 and x_2. The gray level shows the value at a given point, with black being 1 and white being 0. Thus the drawing shows precisely where $f(x_1, x_2) = 1$.



Assume the outside dimension of the square is 1 and the inside dimension of the square is .9. Find the Fourier transform of $f(x_1, x_2)$.

Solution: The 2D rect function (which is separable) $\Pi_{(}x_1, x_2) = \Pi(x_1\Pi(x_2)$ corresponds to a filled-in black square of side length 1. Thus the black rim we see is the result of subtracting from this a 2D rect function of width 0.9. That is,

$$f(x_1, x_2) = \Pi(x_1\Pi(x_2) - \Pi(x_1/.9)\Pi(x_2/.9)\,.$$

Everything in sight is separable, and using the stretch theorem the Fourier transform is

$$\mathcal{F}f(\xi_1, \xi_2) = \operatorname{sinc}(\xi_1)\ \operatorname{sinc}(\xi_2) - .81\operatorname{sinc}(.9\xi_1)\ \operatorname{sinc}(.9\xi_2)\,.$$