EE 261 The Fourier Transform and its Applications
Fall 2007
Solutions to Problem Set One

1. *Some practice with geometric sums and complex exponentials*

   We'll make much use of formulas for the sum of a geometric series, especially in combination with complex exponentials.

   (a) If w is a real or complex number, $w \neq 1$, and p and q are any integers, show that

   $$\sum_{n=p}^{q} w^n = \frac{w^p - w^{q+1}}{1-w}\,.$$

   (Of course if $w = 1$ then the sum is $\sum_{n=p}^{q} 1 = q + 1 - p$.)
   Discuss the cases when $p = -\infty$ or $q = \infty$. What about $p = -\infty$ *and* $q = +\infty$?

   (b) Find the sum

   $$\sum_{n=0}^{N-1} e^{2\pi i n/N}$$

   and explain your answer geometrically.

   (c) Derive the formula

   $$\sum_{k=-N}^{N} e^{2\pi i k t} = \frac{\sin(2\pi t(N+1/2))}{\sin(\pi t)}$$

   Solution

   (a) We're supposed to show

   $$\sum_{n=p}^{q} w^n = \frac{w^p - w^{q+1}}{1-w}\,.$$

   We can derive this from the usual formula, one that you've probably seen – and should certainly know:

   $$\sum_{n=0}^{N} w^n = \frac{1 - w^{N+1}}{1-w}\,.$$

We have

$$\begin{aligned}\sum_{n=p}^{q} w^n &= w^p + w^{p+1} + w^{p+2} + \cdots + w^{q-1} + w^q \\ &= w^p(1 + w + w^2 + \cdots + w^{q-p}) \\ &= w^p \frac{1 - w^{q-p+1}}{1 - w} \\ &= \frac{w^p - w^{q+1}}{1 - w}\end{aligned}$$

Now let's consider the cases when one or the other (or both) limits are infinite. When will the series converge? If $q = \infty$ then to get convergence we have to assume that $|w| < 1$. If so then $w^{q+1} \to 0$ as $q \to \infty$ and

$$\sum_{n=p}^{\infty} w^n = \frac{w^p}{1 - w} .$$

If $p = -\infty$ then, but contrast, we'll get $w^p \to 0$ if $|w| > 1$. Assuming this,

$$\sum_{n=-\infty}^{q} w^n = \frac{-w^{q+1}}{1 - w} .$$

It looks a little funny to me to write it this way, with the minus sign, so I'd prefer

$$\sum_{n=-\infty}^{q} w^n = \frac{w^{q+1}}{w - 1} .$$

Note that if w is real and > 1 then this is positive, as it should be.

If $p = -\infty$ and $q = \infty$ we'd have to have both $|w| < 1$ and $|w| > 1$ to get convergence, an absurd condition. Thus we conclude

$$\sum_{n=-\infty}^{\infty} w^n$$

never converges.

Here's a summary:

$$\sum_{n=p}^{q} w^n = \begin{cases} q - p + 1 \,, & w = 1 \\[2ex] \dfrac{w^p - w^{q+1}}{1 - w} \,, & w \neq 1 \,, -\infty < p \,, q < \infty \\[2ex] \dfrac{w^{q+1}}{w - 1} \,, & |w| > 1 \,, p = -\infty \,, q < \infty \\[2ex] \dfrac{w^p}{1 - w} \,, & |w| < 1 \,, -\infty < p \,, q = \infty \end{cases}$$

(b) From the formula in Part (a) we obtain

$$\sum_{n=0}^{N-1} e^{2\pi ik/N} = \frac{1-e^{2\pi iN/N}}{1-e^{2\pi i/N}} = \frac{1-e^{2\pi i}}{1-e^{2\pi i/N}} = 0$$

This answer also makes complete sense geometrically. The general complex exponential $re^{i\theta}$ can be thought of as a vector in the complex plane, of length r and at an angle θ counterclockwise from the real axis. Thus $w = e^{2\pi i/n}$ is a vector of length 1 at an angle of $2\pi/n$. Similarly, $w^k = e^{2\pi ik/n}$ has length 1 and is at an angle of $2\pi k/n$. The points are distinct and equidistantly spaced $2\pi/n$ radians apart around the unit circle. Consider the n points equally spaced around the unit circle as vertices of a regular n-gon, and the $e^{2\pi ik/n}$ as vectors from 0 to the vertices. The (vector) sum of the points is then the perimeter of the polygon. Viewed as a closed loop, the vector sum is the zero vector.

(c) Again appealing to the formula from Part (a),

$$\sum_{k=-N}^{N} e^{2\pi ikt} = \frac{e^{-2\pi iNt} - e^{2\pi i(N+1)t}}{1-e^{2\pi it}}$$

Now factor out $e^{\pi it}$ from both the numerator and denominator:

$$\frac{e^{-2\pi iNt} - e^{2\pi i(N+1)t}}{1-e^{2\pi it}} = \frac{e^{\pi it}}{e^{\pi it}} \frac{e^{-2\pi i(N+\frac{1}{2})t} - e^{2\pi i(N+\frac{1}{2})t}}{e^{-\pi it} - e^{\pi it}}$$

This is the sort of sleight-of-hand you see a lot in working with combinations and fractions involving complex exponentials. At first, it can be very frustrating not to see what to do, but it becomes a habit to look for such tricks. Now using $e^{ix} - e^{-ix} = 2i\sin x$ we obtain the final formula

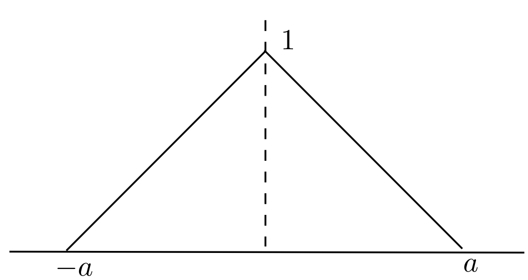
$$\sum_{k=-N}^{N} e^{2\pi ikt} = \frac{\sin(2\pi t(N+1/2))}{\sin(\pi t)} \,.$$

2. *Some practice combining simple signals.* (5 points each)

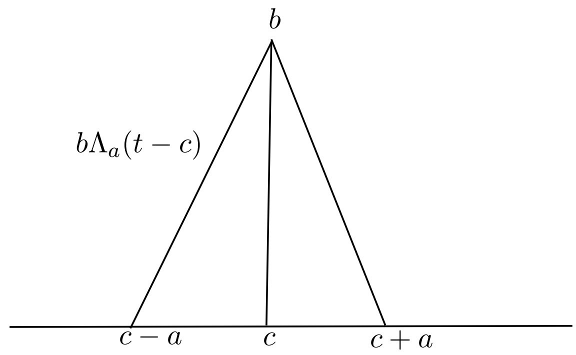
The triangle function with a parameter $a > 0$ is

$$\Lambda_a(t) = \Lambda(t/a) = \begin{cases} 1 - \frac{1}{a}|t| \,, & |t| \le a \\ 0 \,, & |t| > a \end{cases}$$

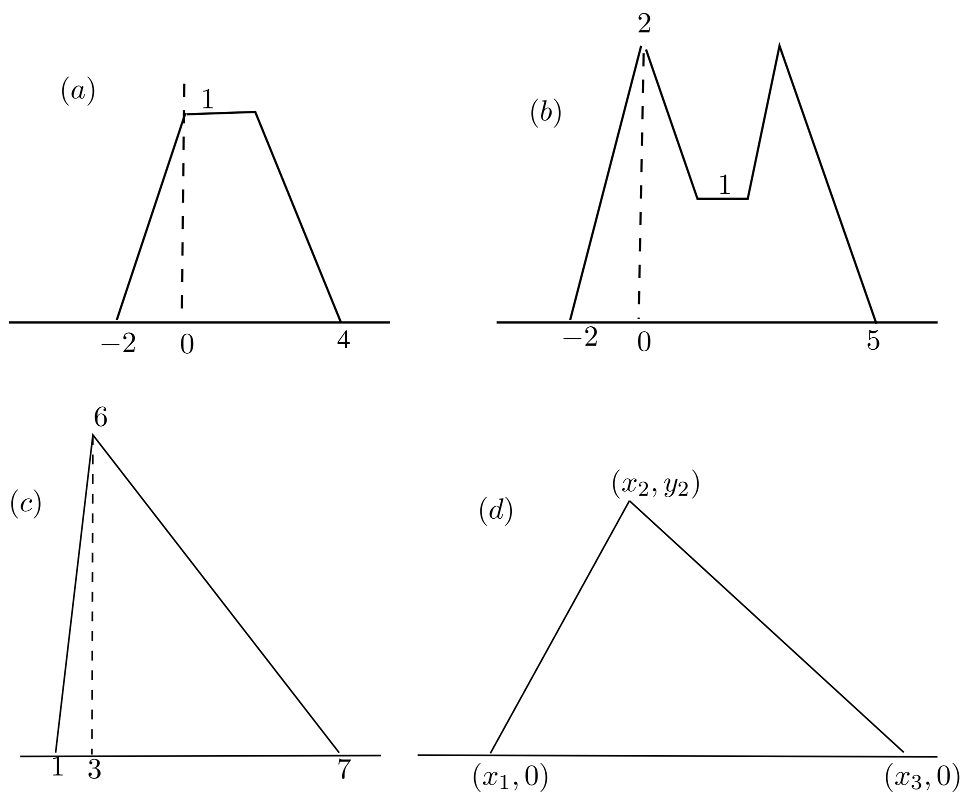
The graph is



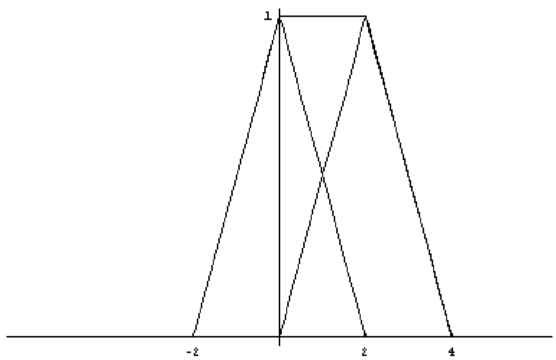
The parameter a specifies the width, namely $2a$. Alternately, a determines the slopes of the sides: the left side has slope $1/a$ and the right side has slope $-1/a$. We can modify Λ_a by scaling the height and shifting horizontally, forming $b\Lambda_a(t-c)$. The slopes of the sides of the scaled function are then $\pm b/a$. The graph is:



Express each of the following as a sum of two shifted, scaled triangle functions $b_1\Lambda_{a_1}(t-c_1) + b_2\Lambda_{a_2}(t-c_2)$. Think of the sum as 'left-triangle' plus a 'right-triangle' ('right' meaning to the right, not having an angle of 90°). For part (d), the values x_1, x_2 and x_3 cannot be arbitrary. Rather, to be able to express the plot as the sum of two Λ's they must satisfy a relationship that you should determine.

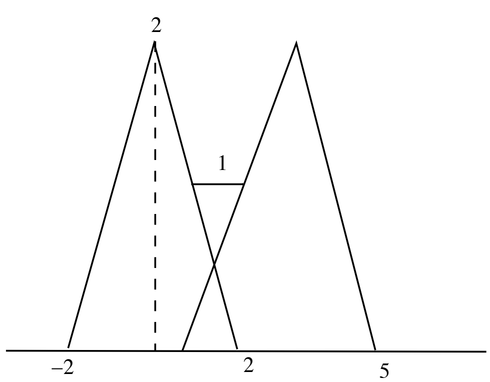


Solutions Parts (a) and (b): To get a flat piece in the sum of two Λ's over a certain interval, as in both parts (a) and (b), we want the slopes to cancel over that interval – where one side is going down the other is going up, and vice versa, so the slopes add to 0. So both Λ_a's should have the same a. For part (a), the max is 1, so it looks like we don't need to scale the heights and can take $b = 1$ for both. The question is how much the Λ's overlap, which is governed by the shift parameter c, more precisely how much one Λ is shifted relative to the other. We see that if half the triangles overlap we get just what we want; bring the left zero-point of the right-hand triangle over to hit at the middle of the base of the left-hand triangle (where the left-hand triangle is at its max). Then as the side of the left-hand triangle decreases from its max, the side of the right-hand triangle increases by the same amount, adding to 1. Here's the picture.



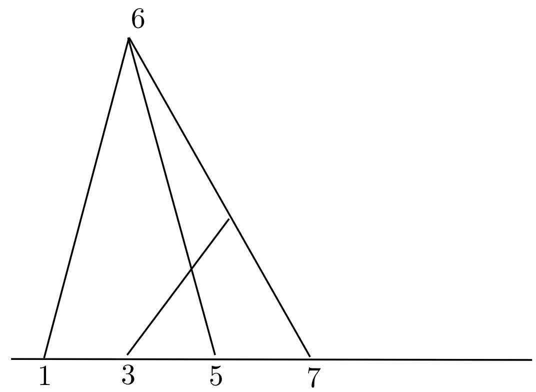
The graph is given by $\Lambda_2(t) + \Lambda_2(t-2)$.

For part(b), again we want overlapping slopes to cancel and we take $a = 2$ for both Λ's. We also scale up both Λ's by multiplying each by 2. To get the flat piece, this time the relative shift of the two Λ's should be such that less than half of the triangles are overlapping. How much less? That's determined by knowing that the height of the flat piece is 1. Bring the left zero-point of the right-hand triangle over to the point where the left-hand triangle has height 1. Here's the picture.



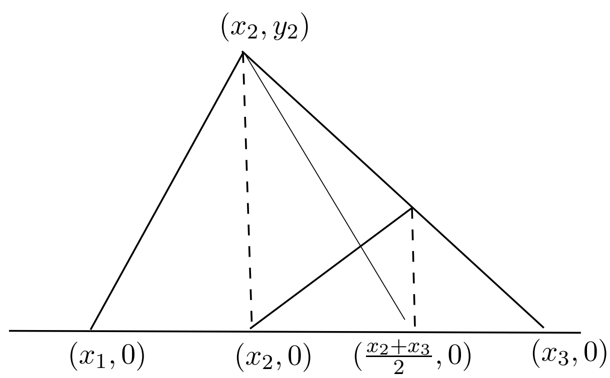
The formula is $2\Lambda_2(t) + 2\Lambda_2(t-3)$.

For part (c), for the left-triangle we take $6\Lambda_2(t-3)$. That's centered at 3, the base goes from 1 to 5, and has sides with slope ± 3. We want to add to that a triangle coming in from the right. So as not to add anything to the maximum height, 6, it's base should go from 3 to 7, so it's a multiple of $\Lambda_2(t-5)$. The slope of the right side of the (big) triangle is $-3/2$, so scale to $3\Lambda_2(t-5)$. Here's a picture



The sum we want is thus $6\Lambda_2(t-3)+3\Lambda_2(t-5)$.

For part (d) we take our cue from the analysis for part (c). We want to express the big triangle as the sum of two Λ's as in the following picture.



The left-triangle is $y_2\Lambda_{x_2-x_1}(t-x_2)$. For the smaller triangle to fit in on the right, the midpoint of its base, $(x_2+x_3)/2$ must be the right endpoint of the base of the left-triangle, which is $x_2+(x_2-x_1)=2x_2-x_1$. Thus we must have

$$2x_2 - x_1 = \frac{x_2+x_3}{2}\,.$$

This can be conveniently rewritten as

$$x_2 - x_1 = \frac{x_3-x_2}{2}\,,$$

which we recognize as saying that the widths of the two component triangle functions are the same, say $2a$, where

$$a = x_2 - x_1 = \frac{x_3-x_2}{2}\,.$$

The scaling factor of the left-triangle is y_2, so that triangle function is

$$y_2\Lambda_a(t-x_2)\,.$$

The slope (which is negative) of the falling line on the right is

$$\frac{-y_2}{x_3-x_2} = \frac{-y_2}{2a}\,.$$

Since the right-triangle also has width $2a$, this implies that the scaling factor of the right-triangle is $y_2/2$, and so we take

$$\frac{y_2}{2}\Lambda_a(t - \frac{x_2 + x_3}{2}).$$

All told, the complete triangle is the sum

$$y_2\Lambda_a(t - x_2) + \frac{y_2}{2}\Lambda_a(t - \frac{x_2 + x_3}{2}).$$

3. *Creating periodic functions.* (5 points each)

   Let $f(t)$ be a function, defined for all t, and let $T > 0$. Define

$$g(t) = \sum_{n=-\infty}^{\infty} f(t - nT).$$

   (a) Provided the sum converges, show that $g(t)$ is periodic with period T. One sometimes says that $g(t)$ is the periodization of $f(t)$.

   (b) Let $f(t) = \Lambda_{1/2}(t)$. Sketch the periodizations $g(t)$ of $f(t)$ for $T = 1/2$, $T = 3/4$, $T = 1$, $T = 2$.

   (c) If a function $f(t)$ is already periodic, is it equal to its own periodization? Explain.

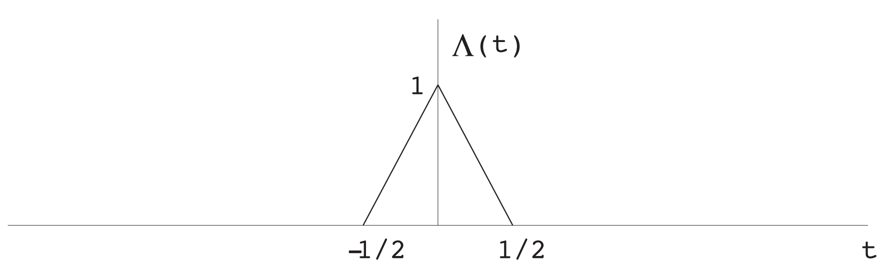
   Solution:

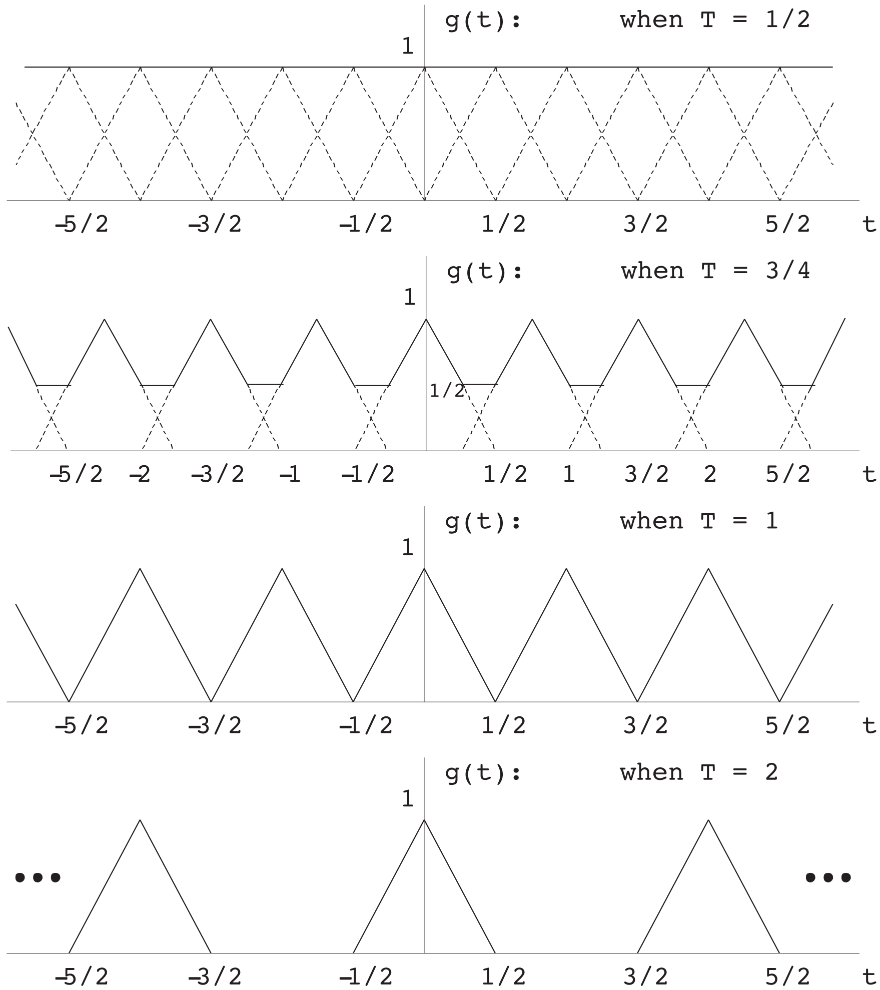
   (a) Using the definition of periodicity, we need to prove that $g(t) = g(t + T)$. So,

$$\begin{aligned} g(t+T) &= \sum_{n=-\infty}^{+\infty} f(t + T - nT) \\ &= \sum_{n=-\infty}^{+\infty} f(t - (n-1)T) \\ &= \sum_{m=-\infty}^{+\infty} f(t - mT) \\ &= g(t), \end{aligned}$$

   where the second to last line was obtained by substituting $m = n - 1$.

   (b) Periodizations of $g(t)$ are sketched below:





(c) If $f(t)$ is periodic the sum

$$g(t) = \sum_{n=-\infty}^{+\infty} f(t - nT)$$

will not converge for any T. As opposed to the situation when we start with a function that is zero outside of some interval, when $f(t)$ is already periodic the values in the sum will just keep piling up. So you can see the problem, take, as an extreme example, the case when T is a period of $f(t)$. In this case we get

$$g(t) = \sum_{n=-\infty}^{+\infty} f(t - nT) = \sum_{n=-\infty}^{+\infty} f(t)\,.$$

Unless $f(t) = 0$ for all t this sum does not converge.

4. *Combining periodic functions.* (5 points each)

(a) Let $f(x) = \sin(2\pi mx) + \sin(2\pi nx)$ where n and m are positive integers. Is $f(x)$ periodic? If so, what is its period?

(b) Let $g(x) = \sin(2\pi px) + \sin(2\pi qx)$ where p and q are positive rational numbers (say $p = m/r$ and $q = n/s$, as fractions in lowest terms). Is $g(x)$ periodic? If so, what is its period?

(c) Show that $f(t) = \cos t + cos\sqrt{2}t$ is not periodic. (Hint: Suppose by way of contradiction that there is some T such that $f(t+T) = f(t)$ for all t. In particular, the maximum value of $f(t)$ repeats. This will lead to a contradiction.)

(d) Consider the voltage $v(t) = 3\cos(2\pi\nu_1 t - 1.3) + 5\cos(2\pi\nu_2 t + 0.5)$. Regardless of the frequencies ν_1, ν_2 the maximum voltage is always less than 8, but it can be much smaller. Use MATLAB (or another program) to find the maximum voltage if $\nu_1 = 2$ Hz and $\nu_2 = 1$ Hz. [From Paul Nahim, *The Science of Radio*]

Solution:

(a) We want to find the smallest number T so that $f(x+T) = f(x)$ for all x, if there is one. The condition is

$$\sin(2\pi mx + 2\pi mT) + \sin(2\pi nx + 2\pi nT) = \sin(2\pi mx) + \sin(2\pi nx).$$

and for this to be true we must have

$$mT = \text{an integer and} \quad nT = \text{an integer, maybe a different integer.}$$

Any T such that n is divisible by T and m is divisible by T will work. In particular $T = 1$ works, so certainly $f(t)$ is periodic. We want the *smallest* T that works. It might be easier if we wrote $\tau = 1/T$; then we're looking for the largest τ which divides both n and m, *i.e.*, the largest τ for which m/τ and n/τ are integers. This is exactly the greatest common divisor (gcd) of m and n. Thus the period is

$$T = \frac{1}{\gcd(m,n)}.$$

Alternately, the frequency is $1/T = \gcd(m,n)$.

(b) As in the previous part, if there is a T such that $g(x+T) = g(x)$ then we must have

$$\frac{m}{r}T = \text{an integer and} \quad \frac{n}{s}T = \text{an integer, maybe a different integer.}$$

Clearly if we take $T = rs$ then this is true, so the function is periodic. What is the *smallest* period? Write $T = a/b$, a and b integers, so the condition looks like

$$\frac{m}{r}\frac{a}{b} = \text{an integer and} \quad \frac{n}{s}\frac{a}{b} = \text{an integer, maybe a different integer.}$$

As a first pass, if we take a to be the least common multiple (lcm) of r and s then r and s will both divide a and the condition will be satisfied for any b. So we take $a = \operatorname{lcm}(r,s)$. To make a/b smallest we then want to take b to be the largest number that divides both m and n (as in the previous part). That is, we take $b = \gcd(m,n)$. The (smallest) period is therefore

$$T = \frac{\operatorname{lcm}(r,s)}{\gcd(m,n)}.$$

Don't believe it? Try a few examples.

(c) Let $f(t) = \cos t + \cos\sqrt{2}t$ and suppose that $f(t)$ is periodic of period T. Note that $f(0) = 2$ and since $\cos t$ and $\cos\sqrt{2}t$ are each ≤ 1 this must be the maximum value of $f(t)$. By assumption $f(T) = f(0) = 2$, so that $\cos T + \cos\sqrt{2}T = 2$ which implies we must also have $\cos T = 1$ and $\cos\sqrt{2}T = 1$. Therefore there are integers m and n such that $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$. But then

$$2\pi m = \frac{1}{\sqrt{2}} 2\pi n,$$

or

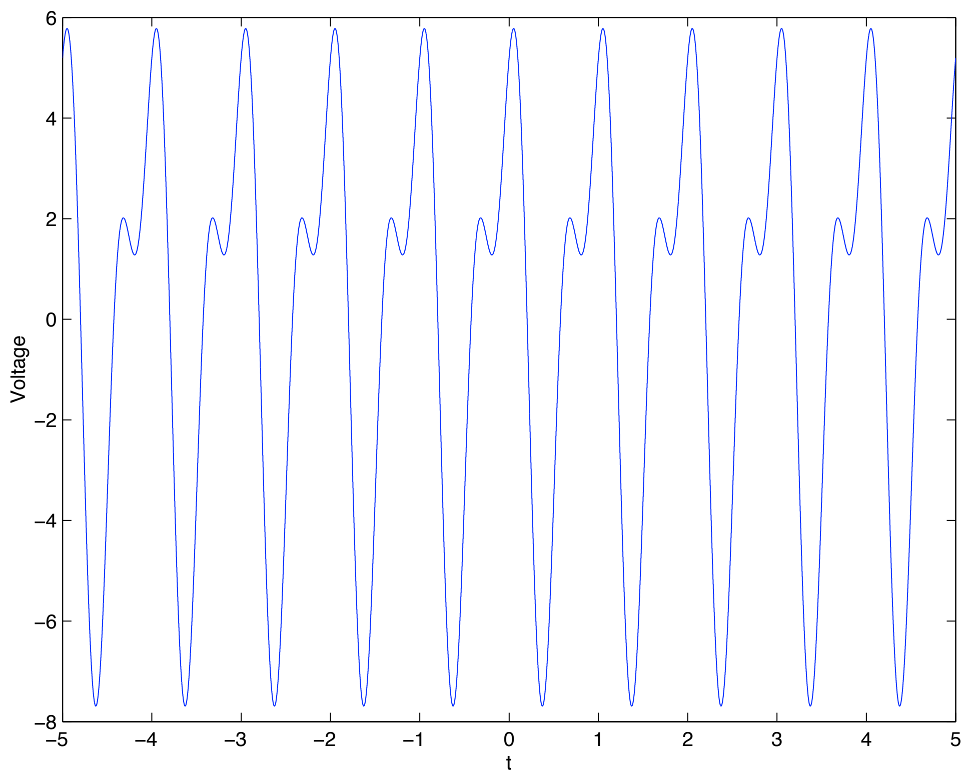
$$\sqrt{2} = \frac{n}{m}.$$

Since $\sqrt{2}$ is irrational this cannot be, and hence $f(t)$ cannot be periodic.

(d) Maximum voltage = 5.7811 (can be a little different according to the sampling interval.) We use the Matlab code:

```
t = -5: 0.0001 :5;   % range of t, where 0.0001 is the sampling interval
% frequency components
v1 = 2;    v2 = 1; V_t = 3*cos(2*pi*v1*t-1.3) +
5*cos(2*pi*v2*t+0.5); plot(t, V_t); disp('Maximum value = ');
disp(max(V_t));
```

The plot is:



5. *Some practice with inner products.* (5 points each)

Let $f(t)$ and $g(t)$ be two signals with inner product

$$(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt\,.$$

Define the *reversed signal* to be

$$f^-(t) = f(-t)\,.$$

Define the *delay operator*, or *shift operator* by

$$\tau_a f(t) = f(t-a)\,.$$

(a) If both $f(t)$ and $g(t)$ are reversed, what happens to their inner product?

(b) If one of $f(t)$ and $g(t)$ is reversed, what happens to their inner product?

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product?

(d) If one of $f(t)$ and $g(t)$ is shifted, what happens to their inner product?

(e) If both $f(t)$ and $g(t)$ are shifted but by different amounts, what happens to their inner product?

(f) What, if anything, changes in these results if f and g are periodic of period 1 and their inner product is

$$(f\,,\,g) = \int_0^1 f(t)\overline{g(t)}\,dt\,.$$

Solutions

(a) The inner product of f^- and g^- is

$$\begin{aligned}
(f^-\,,\,g^-) &= \int_{-\infty}^{\infty} f^-(t)\overline{g^-(t)}\,dt \\
&= \int_{-\infty}^{\infty} f(-t)\overline{g(-t)}\,dt \\
&= \int_{+\infty}^{-\infty} f(u)\overline{g(u)}\,(-du) \quad \text{(making the change of variable } u=-t) \\
&= \int_{-\infty}^{\infty} f(u)\overline{g(u)}\,du \\
&= (f\,,\,g)
\end{aligned}$$

Thus the inner product is unchanged if both signals are reversed, $(f^-\,,\,g^-) = (f\,,\,g)$.

(b) Say f is reversed. Then

$$(f^-\,,\,g) = \int_{-\infty}^{\infty} f^-(t)\overline{g(t)}\,dt = \int_{-\infty}^{\infty} f(-t)\overline{g(t)}\,dt\,.$$

If we again make a change of variables $u = -t$ the integral becomes

$$\int_{-\infty}^{\infty} f(-t)\overline{g(t)}\,dt = \int_{-\infty}^{\infty} f(u)\overline{g(-u)}\,du = (f\,,\,g^-)\,.$$

So the answer to the question 'what happens to their inner product' is, most accurately, 'it's not the same as it was,' but we do find the nice identity

$$(f^-\,,\,g) = (f\,,\,g^-)\,.$$

A similar calculation will show

$$(f\,,\,g^-) = (f^-\,,\,g)\,.$$

(c) Suppose both f and g are shifted by a. Then

$$\begin{aligned}(\tau_a f\,,\,\tau_a g) &= \int_{-\infty}^{\infty} \tau_a f(t)\overline{\tau_a g(t)}\,dt \\ &= \int_{-\infty}^{\infty} f(t-a)\overline{g(t-a)}\,dt \\ &= \int_{-\infty}^{\infty} f(u)\overline{g(u)}\,du \quad \text{(making the change of variable } u = t-a\text{)} \\ &= (f\,,\,g)\end{aligned}$$

The inner product is unchanged if both f and g are shifted by the same amount, $(\tau_a f\,,\,\tau_a g) = (f\,,\,g)$.

(d) Say f is shifted by a. Then

$$\begin{aligned}(\tau_a f\,,\,g) &= \int_{-\infty}^{\infty} \tau_a f(t)\overline{g(t)}\,dt \\ &= \int_{-\infty}^{\infty} f(t-a)\overline{g(t)}\,dt \\ &= \int_{-\infty}^{\infty} f(u)\overline{g(u+a)}\,du \quad \text{(making the change of variable } u = t-a\text{)} \\ &= (f\,,\,\tau_{-a} g)\end{aligned}$$

The inner product changes, unlike when both signals are shifted, but we do see that

$$(\tau_a f\,,\,g) = (f\,,\,\tau_{-a} g)\,.$$

Similarly, if we shifted only g we would get

$$(f\,,\,\tau_a g) = (\tau_{-a} f\,,\,g)\,.$$

(e) Suppose f is shifted by a and g is shifted by b, Using the Part (d),

$$(\tau_a f\,,\,\tau_b g) = (f\,,\,\tau_{-a}\tau_b g) = (f\,,\,\tau_{(b-a)} g)\,.$$

Alternately,

$$(\tau_a f\,,\,\tau_b g) = (\tau_{(a-b)} f\,,\,g)\,.$$

(f) The important thing to note is that the inner product can be calculated on any period, i.e., by integrating over any interval of length 1:

$$(f, g) = \int_{t_0}^{t_0+1} f(t)\overline{g(t)}\,dt\,.$$

Take Part (a), for example, when both f and g are reversed. Then

$$\begin{aligned}(f^-\,,\,g^-) &= \int_0^1 f(-t)\overline{g(-t)}\,dt \\ &= \int_0^{-1} f(u)\overline{g(u)}\,(-du) \quad \text{(using } u = -t\text{)} \\ &= \int_{-1}^{0} f(u)\overline{g(u)}\,du \\ &= (f\,,\,g)\end{aligned}$$

Thus the inner product is unchanged. The rest of the results in the problem are also as they were before.

6. *The Dirichlet Problem, Convolution, and the Poisson Kernel* When modeling physical phenomena by partial differential equations it is frequently necessary to solve a *boundary value problem.* One of the most famous and important of these is associated with Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

where $u(x, y)$ is defined on a region R in the plane. The operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the *Laplacian* and a real-valued function $u(x, y)$ satisfying $\Delta u = 0$ is called *harmonic.* The *Dirichlet problem* for Laplace's equation is this:

> Given a function $f(x, y)$ defined on the boundary of a region R, find a function $u(x, y)$ defined on R that is harmonic and equal to $f(x, y)$ on the boundary.

Fourier series and convolution combine to solve this problem when R is a disk. As with many problems with circular symmetry is helpful to introduce polar coordinates (r, θ), with $x = r\cos\theta$, $y = r\sin\theta$. Writing $U(r, \theta) = u(r\cos\theta, r\sin\theta)$, so regarding u as a function of r and θ, Laplace's equation becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0\,.$$

(You need not derive this.)

(a) Let $\{c_n\}$, $n = 0, 1, \ldots$ be a bounded sequence of complex numbers, let $r < 1$ and define $u(r, \theta)$ by the series

$$u(r, \theta) = \operatorname{Re}\left\{ c_0 + 2\sum_{n=1}^{\infty} c_n r^n e^{in\theta} \right\}.$$

From the assumption that the coefficients are bounded, and comparison with a geometric series, it can be shown that the series converges. Show that $u(r, \theta)$ is a harmonic function.

(b) Suppose that $f(\theta)$ is a real-valued, continuous, periodic function of period 2π and let

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

be its Fourier series. Now form the harmonic function $u(r, \theta)$ as above, with these coefficients c_n. This solves the Dirichlet problem of finding a harmonic function on the unit disk $x^2 + y^2 < 1$ with boundary values $f(\theta)$ on the unit circle $x^2 + y^2 = 1$; precisely,

$$\lim_{r \to 1} u(r, \theta) = f(\theta)\,.$$

You are not asked to show this – it requires a fair amount of work – but assuming that all is well with convergence, explain why one has

$$u(1, \theta) = f(\theta)\,.$$

[This uses the symmetry property of Fourier coefficients.]

(c) The solution can also be written as a convolution: show that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P(r,\theta-\phi)\,d\phi\,,$$

where

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}\,.$$

[Introduce the Fourier coefficients of f. You'll have to sum a geometric series.]

(d) The function $P(r,\theta)$ is called the *Poisson kernel*. Show that it is a harmonic function. [This is a special case of the result you established in Part (a).]

Parts (c) and (d) together exhibit a property of convolution that we'll see repeatedly; The convolution of two functions inherits the nicest properties of each. In this case we convolve a continuous function f (a good property) with a harmonic function P (a nicer property) and the result is a harmonic function.

Solutions

(a) The function is defined by

$$u(r,\theta) = \mathrm{Re}\left\{c_0 + 2\sum_{n=1}^{\infty} c_n r^n e^{in\theta}\right\}\,.$$

We differentiate the series term-by-term. For the r-derivatives we get

$$\begin{aligned}
\frac{\partial}{\partial r}u(r,\theta) &= 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n n r^{n-1} e^{in\theta}\right\}\\
\frac{\partial^2}{\partial r^2}u(r,\theta) &= 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n n(n-1) r^{n-2} e^{in\theta}\right\}
\end{aligned}$$

For the θ-derivative we have

$$\frac{\partial^2}{\partial\theta^2}u(r,\theta) = 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n r^n i^2 n^2 e^{in\theta}\right\} = -2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n r^n n^2 e^{in\theta}\right\}\,.$$

Plug this in to the polar form of the Laplacian:

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} &= 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n n(n-1) r^{n-2} e^{in\theta} + \frac{1}{r}\sum_{n=1}^{\infty} c_n n r^{n-1} e^{in\theta} - \frac{1}{r^2}\sum_{n=1}^{\infty} c_n r^n n^2 e^{in\theta}\right\}\\
&= 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n n(n-1) r^{n-2} e^{in\theta} + \sum_{n=1}^{\infty} c_n n r^{n-2} e^{in\theta} - \sum_{n=1}^{\infty} c_n r^{n-2} n^2 e^{in\theta}\right\}\\
&= 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n n^2 r^{n-2} e^{in\theta} - \sum_{n=1}^{\infty} c_n n r^{n-2} e^{in\theta}\right.\\
&\qquad\left. + \sum_{n=1}^{\infty} c_n n r^{n-2} e^{in\theta} - \sum_{n=1}^{\infty} c_n r^{n-2} n^2 e^{in\theta}\right\}\\
&= 0\,.
\end{aligned}$$

This is probably the way most people did the calculation and that's fine, but realize also that linearity is playing a role here. The essence of the calculation is to show that the individual terms $r^n e^{in\theta}$ are each harmonic. Then so is their linear combination since the Laplacian is a linear operator.

(b) With

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

and

$$u(r,\theta) = \mathrm{Re}\left\{ c_0 + 2\sum_{n=1}^{\infty} c_n r^n e^{in\theta} \right\} .$$

we have, assuming that we can plug in $r = 1$,

$$\begin{aligned}
u(r,1) &= \mathrm{Re}\left\{ c_0 + 2\sum_{n=1}^{\infty} c_n e^{in\theta} \right\} \\
&= \frac{1}{2}(c_0 + \overline{c_0}) + \left(\sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=1}^{\infty} \overline{c_n} e^{-in\theta} \right) \\
&= c_0 + \left(\sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=1}^{\infty} c_{-n} e^{-in\theta} \right) \\
&\qquad \text{(using the symmetry relation } \overline{c_n} = c_{-n} \text{ for the Fourier coefficients)} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=-1}^{-\infty} c_n e^{in\theta} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \\
&= f(\theta)
\end{aligned}$$

(c) We introduce the formula for the Fourier coefficients of $f(\theta)$ into the formula defining $u(r,\theta)$. Recall that

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} f(\phi)\, d\phi \ ;$$

we use ϕ for the variable of integration because we'll be using θ in the definition of u. That is,

$$\begin{aligned}
u(r,\theta) &= \mathrm{Re}\left\{ c_0 + 2\sum_{n=1}^{\infty} c_n r^n e^{in\theta} \right\} \\
&= \mathrm{Re}\left\{ \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi + 2\sum_{n=1}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-in\phi} f(\phi)\, d\phi \right) r^n e^{in\theta} \right\}
\end{aligned}$$

The first term is real

$$\mathrm{Re}\left\{ \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi \right\} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi \, .$$

For the second term. swap the integral and the sum and bring the real part inside:

$$\begin{aligned}
2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi} f(\phi)\,d\phi\right) r^n e^{in\theta}\right\} &= \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} r^n e^{-in\phi} e^{in\theta}\right\} f(\phi)\,d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} r^n e^{in(\theta-\phi)}\right\} f(\phi)\,d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Re}\left\{\sum_{n=1}^{\infty} (re^{i(\theta-\phi)})^n\right\} f(\phi)\,d\phi
\end{aligned}$$

Next we sum the geometric series, which converges since $|re^{-i(\phi-\theta)}| = r < 1$:

$$\sum_{n=1}^{\infty}(re^{i(\theta-\phi)})^n = \frac{re^{i(\theta-\phi)}}{1-re^{i(\theta-\phi)}}\,.$$

And twice the real part of this is

$$\begin{aligned}
\frac{re^{i(\theta-\phi)}}{1-re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{1-re^{-i(\theta-\phi)}} &= \frac{re^{i(\theta-\phi)} + re^{-i(\theta-\phi)} - 2r^2}{|1-re^{i(\theta-\phi)}|^2} \\
&= \frac{2r\cos(\theta-\phi) - 2r^2}{1-2r\cos(\theta-\phi)+r^2}
\end{aligned}$$

Going back to the expression for $u(r,\theta)$ we now get

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \frac{1}{2\pi}\int_0^{2\pi} \frac{2r\cos(\theta-\phi)-2r^2}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\,d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1+\frac{2r\cos(\theta-\phi)-2r^2}{1-2r\cos(\theta-\phi)+r^2}\right) f(\phi)\,d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\,d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta-\phi) f(\phi)\,d\phi\,.
\end{aligned}$$

Wow.

(d) Looking back at our calculations we see that the Poisson kernel is given by

$$P(r,\theta) = \mathrm{Re}\left\{1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}\right\}.$$

This is exactly the form of the sum defining a harmonic function $u(r,\theta)$ as in Part (a) where all the $c_n = 1$.