EE 261 The Fourier Transform and its Applications
Fall 2007
Problem Set One Due Wednesday, October 3

1. *Some practice with geometric sums and complex exponentials*

   We'll make much use of formulas for the sum of a geometric series, especially in combination with complex exponentials.

   (a) If w is a real or complex number, $w \neq 1$, and p and q are any integers, show that

   $$\sum_{n=p}^{q} w^n = \frac{w^p - w^{q+1}}{1-w}.$$

   (Of course if $w = 1$ then the sum is $\sum_{n=p}^{q} 1 = q + 1 - p$.)
   Discuss the cases when $p = -\infty$ or $q = \infty$. What about $p = -\infty$ *and* $q = +\infty$?

   (b) Find the sum

   $$\sum_{n=0}^{N-1} e^{2\pi i n/N}$$

   and explain your answer geometrically.

   (c) Derive the formula

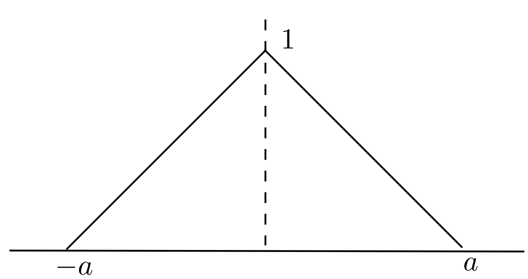
   $$\sum_{k=-N}^{N} e^{2\pi i k t} = \frac{\sin(2\pi t(N + 1/2))}{\sin(\pi t)}$$

2. *Some practice combining simple signals.* (5 points each)

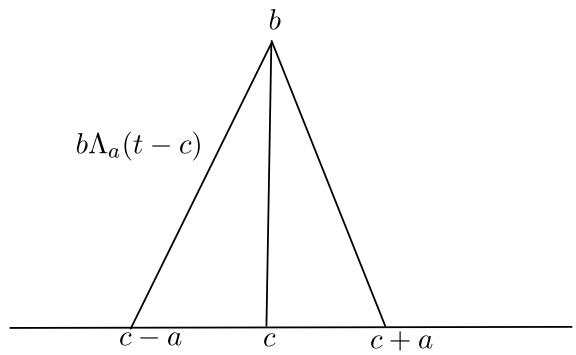
   The triangle function with a parameter $a > 0$ is

   $$\Lambda_a(t) = \Lambda(t/a) = \begin{cases} 1 - \frac{1}{a}|t|\,, & |t| \leq a \\ 0\,, & |t| > a \end{cases}$$

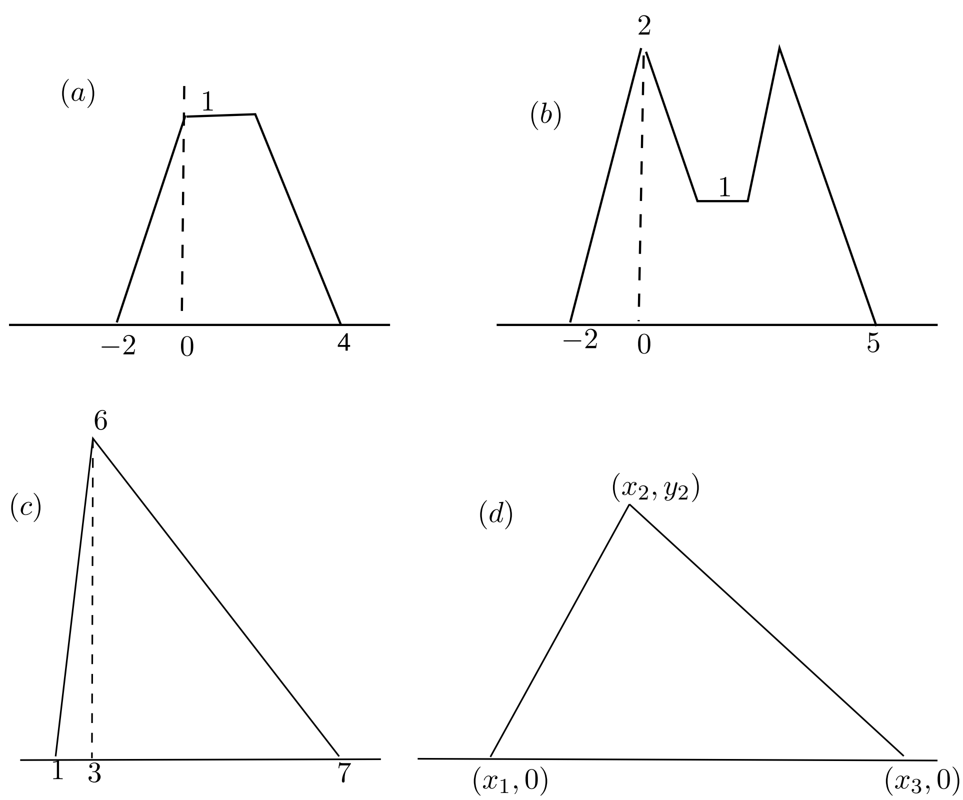
   The graph is



The parameter a specifies the width, namely $2a$. Alternately, a determines the slopes of the sides: the left side has slope $1/a$ and the right side has slope $-1/a$. We can modify Λ_a by scaling the height and shifting horizontally, forming $b\Lambda_a(t-c)$. The slopes of the sides of the scaled function are then $\pm b/a$. The graph is:



Express each of the following as a sum of two shifted, scaled triangle functions $b_1\Lambda_{a_1}(t-c_1)+b_2\Lambda_{a_2}(t-c_2)$. Think of the sum as 'left-triangle' plus a 'right-triangle' ('right' meaning to the right, not having an angle of 90°). For part (d), the values x_1, x_2 and x_3 cannot be arbitrary. Rather, to be able to express the plot as the sum of two Λ's they must satisfy a relationship that you should determine.



3. *Creating periodic functions.* (5 points each)

Let $f(t)$ be a function, defined for all t, and let $T > 0$. Define

$$g(t) = \sum_{n=-\infty}^{\infty} f(t - nT).$$

(a) Provided the sum converges, show that $g(t)$ is periodic with period T. One sometimes says that $g(t)$ is the periodization of $f(t)$.

(b) Let $f(t) = \Lambda_{1/2}(t)$. Sketch the periodizations $g(t)$ of $f(t)$ for $T = 1/2$, $T = 3/4$, $T = 1$, $T = 2$.

(c) If a function $f(t)$ is already periodic, is it equal to its own periodization? Explain.

4. *Combining periodic functions.* (5 points each)

(a) Let $f(x) = \sin(2\pi mx) + \sin(2\pi nx)$ where n and m are positive integers. Is $f(x)$ periodic? If so, what is its period?

(b) Let $g(x) = \sin(2\pi px) + \sin(2\pi qx)$ where p and q are positive rational numbers (say $p = m/r$ and $q = n/s$, as fractions in lowest terms). Is $g(x)$ periodic? If so, what is its period?

(c) Show that $f(t) = \cos t + cos\sqrt{2}t$ is not periodic. (Hint: Suppose by way of contradiction that there is some T such that $f(t + T) = f(t)$ for all t. In particular, the maximum value of $f(t)$ repeats. This will lead to a contradiction.)

(d) Consider the voltage $v(t) = 3\cos(2\pi\nu_1 t - 1.3) + 5\cos(2\pi\nu_2 t + 0.5)$. Regardless of the frequencies ν_1, ν_2 the maximum voltage is always less than 8, but it can be much smaller. Use MATLAB (or another program) to find the maximum voltage if $\nu_1 = 2$ Hz and $\nu_2 = 1$ Hz. [From Paul Nahim, *The Science of Radio*]

5. *Some practice with inner products.* (5 points each)

Let $f(t)$ and $g(t)$ be two signals with inner product

$$(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt\,.$$

Define the *reversed signal* to be

$$f^{-}(t) = f(-t)\,.$$

Define the *delay operator*, or *shift operator* by

$$\tau_a f(t) = f(t - a)\,.$$

(a) If both $f(t)$ and $g(t)$ are reversed, what happens to their inner product?

(b) If one of $f(t)$ and $g(t)$ is reversed, what happens to their inner product?

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product?

(d) If one of $f(t)$ and $g(t)$ is shifted, what happens to their inner product?

(e) If both $f(t)$ and $g(t)$ are shifted but by different amounts, what happens to their inner product?

(f) What, if anything, changes in these results if f and g are periodic of period 1 and their inner product is

$$(f\,,\,g) = \int_0^1 f(t)\overline{g(t)}\,dt\,.$$

6. *The Dirichlet Problem, Convolution, and the Poisson Kernel* When modeling physical phenomena by partial differential equations it is frequently necessary to solve a *boundary value problem.* One of the most famous and important of these is associated with Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

where $u(x, y)$ is defined on a region R in the plane. The operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the *Laplacian* and a real-valued function $u(x, y)$ satisfying $\Delta u = 0$ is called *harmonic.* The *Dirichlet problem* for Laplace's equation is this:

> Given a function $f(x, y)$ defined on the boundary of a region R, find a function $u(x, y)$ defined on R that is harmonic and equal to $f(x, y)$ on the boundary.

Fourier series and convolution combine to solve this problem when R is a disk. As with many problems with circular symmetry is helpful to introduce polar coordinates (r, θ), with $x = r\cos\theta$, $y = r\sin\theta$. Writing $U(r, \theta) = u(r\cos\theta, r\sin\theta)$, so regarding u as a function of r and θ, Laplace's equation becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0\,.$$

(You need not derive this.)

(a) Let $\{c_n\}$, $n = 0, 1, \ldots$ be a bounded sequence of complex numbers, let $r < 1$ and define $u(r, \theta)$ by the series

$$u(r, \theta) = \mathrm{Re}\left\{ c_0 + 2\sum_{n=1}^{\infty} c_n r^n e^{in\theta} \right\}\,.$$

From the assumption that the coefficients are bounded, and comparison with a geometric series, it can be shown that the series converges. Show that $u(r, \theta)$ is a harmonic function.

(b) Suppose that $f(\theta)$ is a real-valued, continuous, periodic function of period 2π and let

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

be its Fourier series. Now form the harmonic function $u(r, \theta)$ as above, with these coefficients c_n. This solves the Dirichlet problem of finding a harmonic function on the unit disk $x^2 + y^2 < 1$ with boundary values $f(\theta)$ on the unit circle $x^2 + y^2 = 1$; precisely,

$$\lim_{r \to 1} u(r, \theta) = f(\theta)\,.$$

You are not asked to show this – it requires a fair amount of work – but assuming that all is well with convergence, explain why one has

$$u(1,\theta) = f(\theta)\,.$$

[This uses the symmetry property of Fourier coefficients.]

(c) The solution can also be written as a convolution: show that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P(r,\theta-\phi)\,d\phi\,,$$

where

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}\,.$$

[Introduce the Fourier coefficients of f. You'll have to sum a geometric series.]

(d) The function $P(r,\theta)$ is called the *Poisson kernel*. Show that it is a harmonic function. [This is a special case of the result you established in Part (a).]

Parts (c) and (d) together exhibit a property of convolution that we'll see repeatedly; The convolution of two functions inherits the nicest properties of each. In this case we convolve a continuous function f (a good property) with a harmonic function P (a nicer property) and the result is a harmonic function.