

EE 261 The Fourier Transform and its Applications
Fall 2007
Problem Set Nine Solutions

1. (10 points) *2D Convolution* Find and sketch the function defined by the following convolution:

$$g(x, y) = \Pi(x)\Pi(y) * \Pi(x)\Pi(y)$$

Solution:

Let $h(x, y) = \Pi(x)\Pi(y)$. Then

$$\mathcal{F}h(\xi, \eta) = \text{sinc } \xi \text{ sinc } \eta$$

Now by the convolution theorem

$$\mathcal{F}g(x, y) = \mathcal{F}h(\xi, \eta)\mathcal{F}h(\xi, \eta)$$

so

$$\mathcal{F}g(x, y) = \text{sinc}^2 \xi \text{ sinc}^2 \eta$$

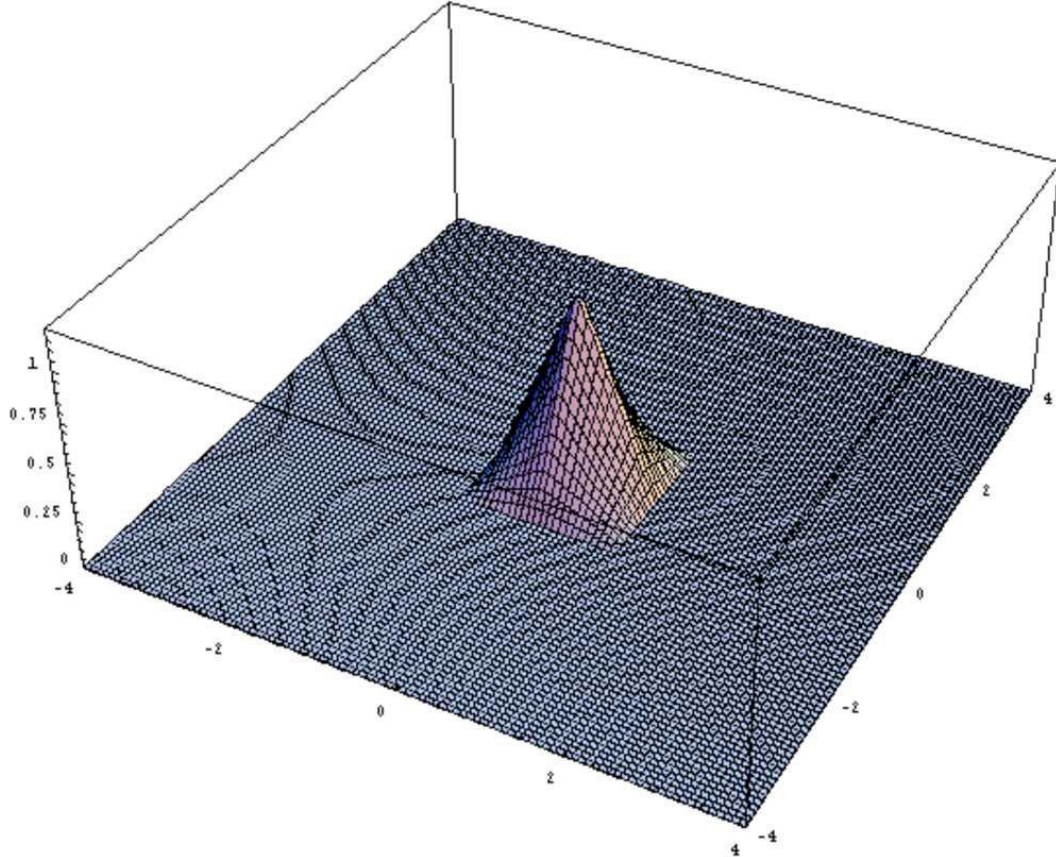
In turn, this is separable, so

$$g(x, y) = (\mathcal{F}^{-1}\text{sinc}^2)(x) (\mathcal{F}^{-1}\text{sinc}^2)(y)$$

or

$$g(x, y) = \Lambda(x)\Lambda(y)$$

Here's a plot – it looks like a pyramid with a square base:



2. (10 points) *2D Radial Stretch Theorem* If $g(r)$ is a circularly symmetric function with Hankel transform $G(\rho)$, show that the Hankel transform of $g(ar)$ is $\frac{1}{|a|^2}G\left(\frac{\rho}{a}\right)$.

Solution:

Recall that

$$G(\rho) = 2\pi \int_0^\infty g(r) J_0(2\pi\rho r) r dr$$

Let $u = ar$ so that $r = \frac{u}{a}$ and $dr = \frac{du}{a}$. Since $g(-r) = g(r)$ (a property of radial functions) we can assume $a > 0$ without loss of generality. Then

$$\begin{aligned} \mathcal{F}\{g(ar)\} &= 2\pi \int_0^\infty g(ar) r J_0(2\pi\rho r) dr \\ &= \frac{1}{a^2} 2\pi \int_0^\infty g(u) u J_0\left(2\pi\rho \frac{u}{a}\right) du \\ &= \frac{1}{a^2} G\left(\frac{\rho}{a}\right) \\ &= \frac{1}{|a|^2} G\left(\frac{\rho}{a}\right) \end{aligned}$$

3. (20 points) *2D Fourier Transforms* Find the 2D Fourier Transforms of:

(a) $\sin 2\pi ax_1 \sin 2\pi bx_2$

Solution:

Because the function is separable we have

$$\begin{aligned}
 \mathcal{F}(\sin 2\pi ax_1 \sin 2\pi bx_2) &= \mathcal{F} \sin 2\pi ax_1 \mathcal{F} \sin 2\pi bx_2 \\
 &= \frac{1}{2i}(\delta(\xi_1 - a) - \delta(\xi_1 + a)) \frac{1}{2i}(\delta(\xi_2 - b) - \delta(\xi_2 + b)) \\
 &= -\frac{1}{4}(\delta(\xi_1 - a)\delta(\xi_2 - b) - \delta(\xi_1 - a)\delta(\xi_2 + b) \\
 &\quad -\delta(\xi_1 + a)\delta(\xi_2 - b) + \delta(\xi_1 + a)\delta(\xi_2 + b)) \\
 &= -\frac{1}{4}(\delta(\xi_1 - a, \xi_2 - b) - \delta(\xi_1 - a, \xi_2 + b) \\
 &\quad -\delta(\xi_1 + a, \xi_2 - b) + \delta(\xi_1 + a, \xi_2 + b))
 \end{aligned}$$

The plot of this would be four spikes, two up, two down, at the four vertices of the rectangle (a, b) , $(-a, b)$, $(a, -b)$, $(-a, -b)$.

(b) e^{-ax^2}

Solution:

You can use the radial stretch theorem, above, on this, knowing the Fourier transform of a Gaussian. That gives, in polar coordinates,

$$\mathcal{F}f(\rho) = \frac{\pi}{a} e^{-\frac{\pi^2}{a}\rho^2}$$

You can also do it directly using that $f(x, y)$ is separable,

$$f(x, y) = e^{-ax^2} e^{-ay^2},$$

and appealing to the formulas for the Fourier transform of a one-dimensional Gaussian.

(c) $e^{-2\pi i(ax+by)} \cos(2\pi cx)$

Solution:

Again, this is separable

$$\begin{aligned}
 \mathcal{F} \left\{ e^{-2\pi i(ax+by)} \cos(2\pi cx) \right\} &= \mathcal{F} \left\{ e^{-2\pi iax} \cos(2\pi cx) e^{-2\pi iby} \right\} \\
 &= \mathcal{F} \left\{ e^{-2\pi iax} \cos(2\pi cx) \right\} \mathcal{F} \left\{ e^{-2\pi iby} \right\} \\
 &= \left(\delta(\xi + a) * \frac{1}{2}[\delta(\xi + c) + \delta(\xi - c)] \right) \delta(\eta + b) \\
 &= \frac{1}{2}[\delta(\xi + a + c) + \delta(\xi + a - c)]\delta(\eta + b)
 \end{aligned}$$

This can also be expressed using the multi-dimensional notation for the product of delta functions of different variables

$$\mathcal{F}f(\xi, \eta) = \frac{1}{2}[\delta(\xi + a + c, \eta + b) + \delta(\xi + a - c, \eta + b)]$$

(d) $\cos(2\pi(ax + by))$ Hint: Use the addition formula for the cosine.

Solution:

One way to do this is to write

$$\cos(2\pi(ax + by)) = \cos 2\pi ax \cos 2\pi by - \sin 2\pi ax \sin 2\pi by,$$

i.e., as a separable function. Then (much like part (a))

$$\begin{aligned} \mathcal{F}(\cos(2\pi(ax + by))) &= \mathcal{F}(\cos 2\pi ax \cos 2\pi by) - \mathcal{F}(\sin 2\pi ax \sin 2\pi by) \\ &= \mathcal{F}(\cos 2\pi ax)\mathcal{F}(\cos 2\pi by) - \mathcal{F}(\sin 2\pi ax)\mathcal{F}(\sin 2\pi by) \\ &= \frac{1}{4}(\delta(\xi - a) + \delta(\xi + a))(\delta(\eta - b) + \delta(\eta + b)) \\ &\quad + \frac{1}{4}(\delta(\xi - a) - \delta(\xi + a))(\delta(\eta - b) - \delta(\eta + b)) \\ &= \frac{1}{4}(\delta(\xi - a)\delta(\eta - b) + \delta(\xi - a)\delta(\eta + b) \\ &\quad + \delta(\xi + a)\delta(\eta - b) + \delta(\xi + a)\delta(\eta + b)) \\ &\quad + \frac{1}{4}(\delta(\xi - a)\delta(\eta - b) - \delta(\xi - a)\delta(\eta + b) \\ &\quad - \delta(\xi + a)\delta(\eta - b) + \delta(\xi + a)\delta(\eta + b)) \end{aligned}$$

These are some combinations and cancelations here, leading to

$$\begin{aligned} \mathcal{F}(\cos(2\pi(ax + by))) &= \frac{1}{2}\delta(\xi - a)\delta(\eta - b) + \frac{1}{2}\delta(\xi + a)\delta(\eta + b) \\ &= \frac{1}{2}(\delta(\xi - a, \eta - b) + \delta(\xi + a, \eta + b)) \end{aligned}$$

The plot is two spikes at opposite corners of the rectangle, (a, b) and $(-a, -b)$.

4. (20 points) Given a function $f(x_1, x_2)$ define

$$\begin{aligned} g(x_1, x_2) &= f(-x_1, x_2), & h(x_1, x_2) &= f(x_1, -x_2) \\ k(x_1, x_2) &= f(x_2, x_1) & m(x_1, x_2) &= f(-x_2, -x_1) \end{aligned}$$

Find $\mathcal{F}g$, $\mathcal{F}h$, $\mathcal{F}k$, and $\mathcal{F}m$ in terms of $\mathcal{F}f$. Show your work. In each case interpret your result geometrically in terms of an image and its spectrum.

Solution:

(a) For $g(x_1, x_2) = f(-x_1, x_2)$ we find

$$\begin{aligned} \mathcal{F}g(\xi_2, \xi_1) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(x_1 \xi_1 + x_2 \xi_2)} f(-x_1, x_2) dx_1 dx_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i((-u) \xi_1 + x_2 \xi_2)} f(u, x_2) du dx_2 \\ &\quad \text{(making the (single) change of variable } u = -x_1) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(u(-\xi_1) + x_2 \xi_2)} f(u, x_2) du dx_2 \\ &= \mathcal{F}f(-\xi_1, \xi_2) \end{aligned}$$

Geometrically, $f(-x_1, x_2)$ is a reflection across the x_2 -axis. The result says that the spectrum is likewise reflected across the ξ_2 -axis.

- (b) The calculation for $h(x_1, x_2) = f(x_1, -x_2)$ is virtually the same, resulting in $\mathcal{F}h(\xi_1, \xi_2) = \mathcal{F}f(\xi_2, -\xi_2)$. This time the result says that a reflection across the x_1 -axis corresponds to a reflection across the ξ_1 -axis.
- (c) For $k(x_1, x_2) = f(x_2, x_1)$ if you write the definition

$$\mathcal{F}k(\xi_1, \xi_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(x_1 \xi_1 + x_2 \xi_2)} f(x_2, x_1) dx_1 dx_2$$

you might be tempted just to say that this is a switch of x_1 and x_2 so the answer is obviously $\mathcal{F}f(\xi_2, \xi_1)$. That's the correct answer, but it's not immediately obvious. Indeed, write the integral for $\mathcal{F}f(\xi_2, \xi_1)$. That's

$$\mathcal{F}f(\xi_2, \xi_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(x_1 \xi_2 + x_2 \xi_1)} f(x_1, x_2) dx_1 dx_2.$$

Is it so obvious that this is the preceding integral? It works, but you have to make a two-dimensional change of variables.

Another approach, however, is to use the general stretch theorem and write

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and write} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that $k(\underline{x}) = f(A\underline{x})$. Note that $\det A = -1$, so $|\det A| = 1$. You can easily see that $AA = I$ (applying A once reverses the coordinates and applying it again reverses them back) so that $A^{-1} = A$ and then, since A is symmetric, $A^{-T} = A$. Taking all this into account, according to the general stretch theorem we have

$$\mathcal{F}g(\underline{\xi}) = \mathcal{F}f(A\underline{\xi}) \quad \text{or} \quad \mathcal{F}g(\xi_1, \xi_2) = \mathcal{F}f(\xi_2, \xi_1).$$

The flip $k(x_1, x_2) = f(x_2, x_1)$ represents a reflection across the line $x_1 = x_2$. The result says that the spectrum is reflected across the line $\xi_1 = \xi_2$.

- (d) The final case $m(x_1, x_2) = f(-x_2, -x_1)$ is handled in a way similar to the preceding case, writing

$$\begin{pmatrix} -x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{so using} \quad -A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Arguing just as above then leads to

$$\mathcal{F}g(\xi_1, \xi_2) = \mathcal{F}f(-\xi_2, -\xi_1).$$

Geometrically, $m(x_1, x_2) = f(-x_2, -x_1)$ is a reflection across the line $x_2 = -x_1$. The result says that this produces a reflection across the line $\xi_2 = -\xi_1$

5. (15 points) *2D DFT* Let f be a $M \times N$ matrix (you can think of f as an $M \times N$ image). The 2D DFT of f is given by the following formula:

$$\mathcal{F}f[k, l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n] \omega_N^{-ln} \omega_M^{-km}$$

where

$$\omega_N = e^{2\pi i/N}, \quad \omega_M = e^{-2\pi i/M}.$$

- (a) *Separability:* Is the 2D DFT separable? If yes, how can you take advantage of this property to compute the 2D DFT of the matrix f ?

Solution

Yes, the 2D DFT is separable. To see this, let f_m be the m^{th} row of the matrix f : it's a vector of length N . Let $\mathcal{F}f_m$ be the 1D DFT of f_m then:

$$\mathcal{F}f_m[l] = \sum_{n=0}^{N-1} f_m[n]\omega_N^{-ln} = \sum_{n=0}^{N-1} f[m, n]\omega_N^{-ln}$$

Then, from the given formula for the 2D DFT, we can easily see that:

$$\mathcal{F}f[k, l] = \sum_{m=0}^{M-1} \mathcal{F}f_m[l]\omega_M^{-km}$$

In other words, this is a 1D DFT of the vector of length M which consists of the l^{th} entry in each vector $\mathcal{F}f_m$.

So we can take advantage of the separability by first computing the 1D DFT of the rows of f and then doing another 1D DFT to find the $[k, l]$ entry in $\mathcal{F}f$.

- (b) *Modulation:* Let g be an $M \times N$ image obtained from f in the following way: $g[m, n] = f[m, n]\omega_M^{mk_0}\omega_N^{nl_0}$ where k_0 and l_0 are integers. What is $\mathcal{F}g$ in terms of $\mathcal{F}f$?

Solution:

$$\begin{aligned} \mathcal{F}g[k, l] &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} g[m, n]\omega_N^{-ln}\omega_M^{-km} \\ &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n]\omega_M^{k_0n}\omega_N^{l_0n}\omega_N^{-ln}\omega_M^{-km} \\ &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n]\omega_M^{-(k-k_0)n}\omega_N^{-(l-l_0)n} \\ &= \mathcal{F}f[k - k_0, l - l_0] \end{aligned}$$

- (c) *2D convolution* Given 2 matrices f and g of size $M \times N$, their 2D convolution is given by:

$$(f * g)[m, n] = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f[u, v]g[m - u, n - v]$$

Show that $\mathcal{F}(f * g) = \mathcal{F}f \times \mathcal{F}g$, where on the right we mean the $M \times N$ matrix whose elements are the products of the corresponding elements of $\mathcal{F}f$ and $\mathcal{F}g$.

Hint: Write out the expression for $\mathcal{F}(f * g)$. You'll get a nasty looking expression with four nested sums. But swap the order of summation, change the variable of summation (analogous to changing the variable of integration) and use the fact that the 1D DFT can be computed by summing over any N (or M) consecutive indices.

Solution:

Let $h = f * g$. Then

$$\begin{aligned}
 \mathcal{F}h[k, l] &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} h[m, n] \omega_N^{-ln} \omega_M^{-km} \\
 &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f[u, v] g[m-u, n-v] \omega_N^{-ln} \omega_M^{-km} \\
 &= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f[u, v] \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} g[m-u, n-v] \omega_N^{-ln} \omega_M^{-km} \\
 &= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f[u, v] \sum_{s=-u}^{M-1-u} \sum_{t=-v}^{N-1-v} g[s, t] \omega_N^{-l(t+v)} \omega_M^{-k(s+u)} \\
 &= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f[u, v] \mathcal{F}g[k, l] \omega_N^{-lv} \omega_M^{-ku} \\
 &= \mathcal{F}g[k, l] \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} f[u, v] \omega_N^{-lv} \omega_M^{-ku} \\
 &= \mathcal{F}g[k, l] \times \mathcal{F}f[k, l]
 \end{aligned}$$

6. (15 points) *Image Segmentation*

Obtain the image `dog.jpg` from

<http://see.stanford.edu/materials/Isottaee261/dog.jpg>

- (a) Load the gray-scale image in Matlab using the `imread` command. Convert the matrix image from type `uint8` to `double`. Plot the initial image.
- (b) Now, consider the matrices:

$$\mathbf{B}_x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the Matlab `conv2` command (with the option `'same'` - run `help` for details), filter the initial image by convolving it with the matrix \mathbf{B}_x . Filter the initial image once more by convolving it with the matrix \mathbf{B}_y . Find all the entries of the two filtered images that are greater than 10 and make the corresponding entries of the initial image equal to 255. Plot the resulting image. What do you observe?

- (c) Repeat the steps in part b) only this time instead of filtering the initial image, use a 'smoothed' version of it. Generate the smoothed version by passing the initial image through an ideal Low Pass 2-D filter. Now, increase / decrease the LP filter's width. In each case, plot the 'smoothed image' and the final resulting image. What do you observe?

For filter: http://see.stanford.edu/materials/Isottaee261/LP_filter.txt

Solution:

- (a) Please see attached Matlab code and plots.
- (b) Please see attached Matlab code and plots. Essentially, this process results in over-segmentation of the picture due to local irregularities in the gradient images. Smoothing of the initial picture can help eliminate some of these irregularities.
- (c) Please see attached Matlab code and plots. After sufficient smoothing of the initial image, a satisfactory segmentation can be achieved.

```
close all;
clear all;
gap_thr=10;

%===== Part a =====
img=imread('dog.jpg');
img=double(img);
fig=figure(); set(fig, 'PaperPosition',[0.25 0.25 6 6]);
imagesc(img);colormap('gray'); title('Initial Image');
print('-depsc','InitImg.eps');

%===== Part b =====
i=0;
for f=0.5:-0.06:0.08;
    filt_img=img;
    if f<0.5
        h=zeros(size(img));
        Xmax=size(img,1);
        Ymax=size(img,2);
        xo=ceil(Xmax*f);
        yo=ceil(Ymax*f);
        h(1:xo,1:yo)=1;
        h(Xmax-xo+1:Xmax,1:yo)=1;
        h(1:xo,Ymax-yo+1:Ymax)=1;
        h(Xmax-xo+1:Xmax,Ymax-yo+1:Ymax)=1;
        filt_img=real(iff2(fft2(img).*h));
        fig=figure(); set(fig, 'PaperPosition',[0.25 0.25 6 6]);
        imagesc(filt_img);colormap('gray');
        title(sprintf('Smoothed Image (f=%g)',f));
        print('-depsc',sprintf('SmoothedImg%d.eps',i));
    end

    boundaries=zeros(size(img));
    By=[ 0 1 0 ; 0 -1 0 ; 0 0 0 ];
    img_conv2_y=conv2(filt_img,By,'same');
    Bx=[ 0 0 0 ; 1 -1 0 ; 0 0 0 ];
    img_conv2_x=conv2(filt_img,Bx,'same');
```



```
boundaries(find(abs(img_conv2_y)>gap_thr))=255;
boundaries(find(abs(img_conv2_x)>gap_thr))=255;
img_with_bound=img;
img_with_bound(find(boundaries))=255;

fig=figure(); set(fig, 'PaperPosition',[0.25 0.25 6 6]);
imagesc(boundaries); colormap('gray');
title(sprintf('Object Boundaries (f=%g)',f));
print('-depsc',sprintf('Boundaries%d.eps',i));
fig=figure(); set(fig, 'PaperPosition',[0.25 0.25 6 6]);
imagesc(img_with_bound);colormap('gray');
title(sprintf('Initial Image with Object Boundaries (f=%g)',f));
print('-depsc',sprintf('Init_Img_with_Bound%d.eps',i));
i=i+1;
end
```

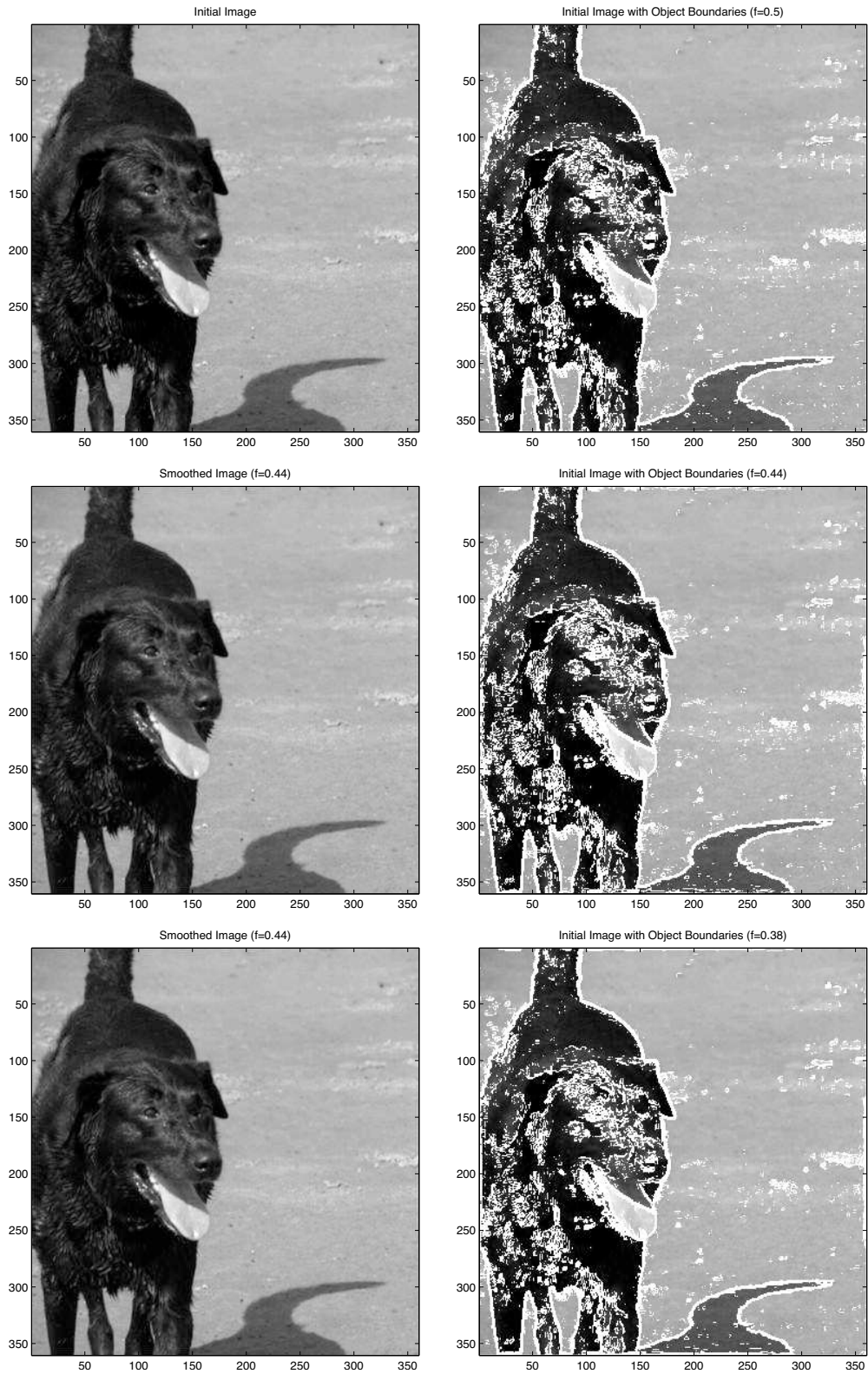


Figure 1: Matlab Plots(1/3).

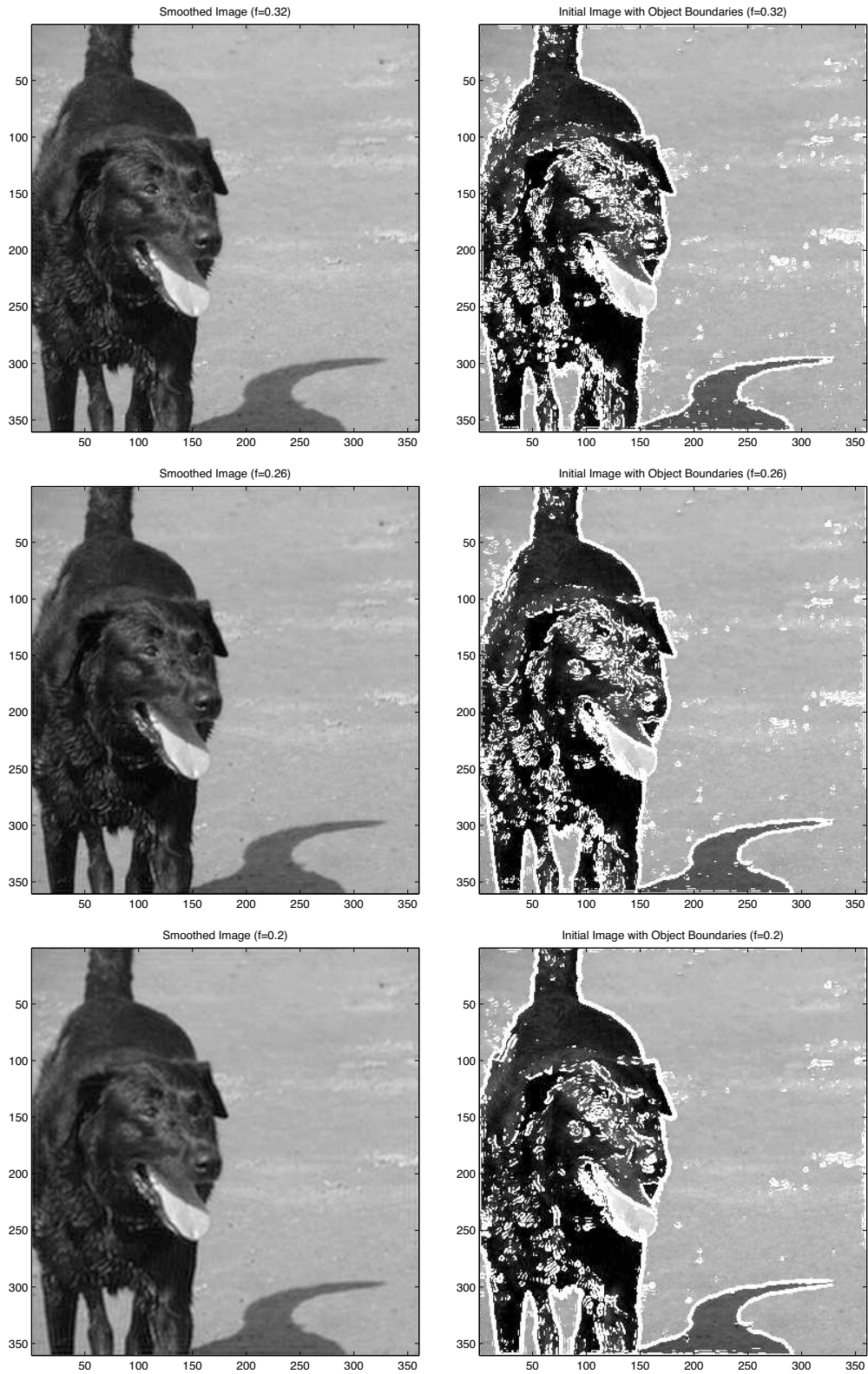


Figure 2: Matlab Plots(2/3).

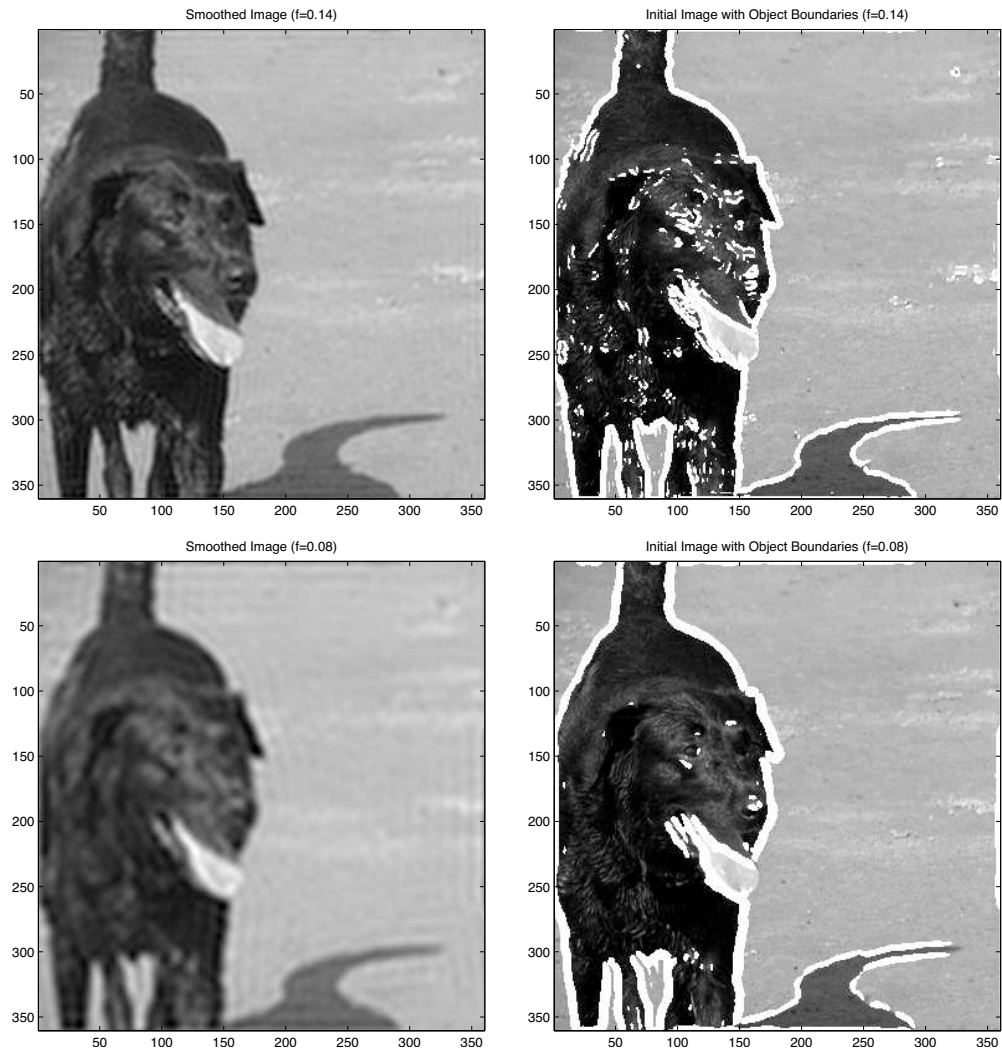


Figure 3: Matlab Plots(3/3).