TheFourierTransformAndItsApplications-Lecture05

Instructor (Brad Osgood):Okay, ready to rock and roll? All right. We've been getting a lot of – as I understand it, we've been getting a lot of email questions about when and where to turn in the homework. You can certainly bring it to class, and that's fine.

But if you find yourself wanting to fill in those last minute comments, I would say that – the policy is the homework should be due by 5:00 on Wednesday, and you can turn it in to the magic filing cabinet. Across from my office – my office is 271 Packard, and there's a little hallway sort of across from there, and there are several gray filing cabinets, one of which has my name and the course number on it.

And if you open that drawer, you will see a wire basket in there with a little sign on it that says something like, "Turn in homework here." So you can just drop your homework in there, okay. So by 5:00 today the first problem assignment is due. All right. Any questions? Anything on anybody's mind?

Okay, all right. Anything else that came up that I should address? No, okay. All right. Today, I wanna finish off our discussion Fourier series. And sometimes we don't really finish the discussion of Fourier series because it will always be a touchstone for reference for some of the other things that we do.

But I wanna finish up the discussion we started last time about using Fourier series to solve the heat equation. And then I wanna talk about the transition from Fourier series to the Fourier transform, and how one gets from the study of periodic phenomenon to the study of non-periodic phenomenon, which is exactly what the Fourier transform is concerned with, by means of limiting process, all right. So that's where I wanna finish up.

But first, let me go back to the discussion we started last time about the heat equation. So this is the use of Fourier. This a classic example, the classic example one might say, of Fourier series, and also shows in a particular case, a very general principle that we will be seeing constantly throughout the course. That's really one of the reasons why I wanna talk about it.

So this is your Fourier series to solve the heat equation, one particular case of a heat equation. And I'll remind you what the setup is. We have a ring, we have a heated ring like that with an initial temperature distribution of – we're calling F of X. So the initial temperature – so I'm thinking of X here as a spacial variable.

I wanted to mention a special variable, although the ring is sitting in two dimensions. So I can think of the ring, if I want, as an interval from zero to one with the end points identified.

At any rate, the fundamental – the important fact here is that since there is periodicity in space, since the ring goes round and round and round, the function F is periodic as a function of the spacial variable, the position on the ring. And we can normalize things to assume the period is one, all right. So that's how Fourier series comes into the picture.

So we take F to be periodic of period one. Then we let the U of XT – now, the temperature is varying, both in position and in time. The temperature change is a function in time, and the temperature is different at different points along the ring.

So I let U of XT be the temperature at a position X at time T. Then U is also a function – it's a periodic function in this spacial variable. U is a periodic function of X. So U XT is periodic in X. That is U of X plus one T at any time T is U of X at the same instant of time. When T is fixed, this periodic is a function of X.

And the physical situation is described by the heat equation, which is also referred to as the defusion equation, and governs many similar processes that involve the diffusion of something through something else [inaudible]. So heat through a region charged through a wire is governed by this sort of equation. I mentioned these last time, holes through a semiconductor. It really has quite a variety of applications.

And some of the techniques that we're talking about here, although they're specialized to this case, can be applied in various forms to many different situations. So you have the heat equation [inaudible] equation, so it relates to the derivatives in time to the derivatives in X in space.

So it says UT is equal to one-half UXX. Here I'm just choosing the constant – there's a constant on the right-hand side of the heat equation. I'm just choosing things so the constant is one-half just to simplify the calculations. So this is – the first derivative with respect to time, and this is the second derivative with respect to X.

All right, now because the function is periodic in X, we can expand it as a Fourier series. So the rigger police are off duty. I'm assuming that everything converges here and there's no question about writing down the sums.

This is sort of a formal operation whose particular – the particular manipulations that I'm gonna do can be justified under reasonable assumptions, but that's not the point. The point is just to see how the techniques can be used to solve the equation.

So I can write the function as a Fourier series. Now remember, if periodic is a function of X, so the [inaudible] on T is on the coefficient. K one from minus infinity to infinity, that's – I'll call it C's of K of T times the periodic term, either the two pi IKX, ano I'm assuming period one. That's the basic assumption. Or rather, that falls from periodicity. You can expand it as a Fourier series. That's how Fourier series get into the picture.

Then you plug into the equation – we did this last time. I'm just reminding you of the setup. Plug into the heat equation and equate coefficients and you get an ordinary differential equation for the C's. The C's are the unknowns there.

So the – you get this equation, you get CK prime of T is equal to minus two pi squared, K squared, times CK of T. That's a simple ordinary differential equation, and we know how to solve it. That's a simple [inaudible].

And solution is CK of T is E to the minus – is the initial condition TK at zero times equal the minus two pi squared, K squared T. That's easy. That is easy. And this is where I think we got to last time.

Now, what is the initial condition? I haven't brought in the initial temperature here, but here's where it comes in. So what is CK of zero? All right, CK depends on time. What is the initial condition?

Well, we can see it, actually. Remember, UK – excuse me, U of XT is the sum from K equals minus infinity to infinity. CK of T times E to the two pi I, KX. So what happens to T equals zero? T equals zero, this is the initial temperature distribution. U at X, any position on the ring, at times zero is F of X. You give it some initial distribution of heat.

So that says that F of X is U of X zero. That's the sum from – if I plug into the series, sum from minus infinity to infinity of CK of zero, E to the two pi IKX.

Now, we have eyes, but if only we but see. What does this say? F is a periodic function. This is an expansion of F in terms of the harmonics, in terms of the building blocks that give you the two pi IKX. What are the coefficients of C, what are the coefficients CK of T in terms of F? What is CK in terms of F, CK of zero in terms of F?

Student:[Inaudible]

Instructor (Brad Osgood):Pardon me? It's a Fourier coefficient of F. So this must be the Fourier series of F. That is to say, i.e. CK of zero is just the K Fourier coefficient of F, F out of K.

Now, as they say, we have eyes, if only we but see. What I meant by that provocative statement is you have to be able to go from the function to the series, but you also have to be able to go from the series to the function.

Here's the function, here's the series. You have to be able to relate and say this is a series for a function, so it must be that these coefficients are the Fourier coefficients for the function.

You're used to having problems like here's the function, compute the Fourier coefficients. All right. Well, in some sense the Fourier coefficient's already computed here, so relate them to the function. Let's write that down.

That is – when I say write that down, what I mean is what does the solution look like. So U of XT, the temperature – this is already very impressive. The temperature at any time T at any position X on the circle is F out of K times E to the minus two pi squared, K squared T, times either the two pi IKX. You can see from this the dependence on time and the dependence on X.

It's very – it's a complicated expression, but it's pretty impressive. You have found that given an initial temperature distribution, this is a formula for how the heat for the temperature at any point X on the ring at any time T. You can already draw some conclusions from this. Namely, as T tends to plus infinity what happens to the temperature? It goes to zero. As T tends to infinity, this term is damping out.

Now see, there's a cute little thing to observe here. This is an exponential, either the minus two pi square, K square T. Now, K is going from minus infinity to infinity, all right. K is both positive and negative here. But K appears here squared, so K squared is always positive. So this, multiply two pi – minus two pi square, K square, this number is always positive, so minus it is always negative. It's either the minus something as T is tending to infinity, all right.

That's a little sort of check on consistency there. So as T tends to zero – this is just an observation. As T tends to infinity, U of XT tends to zero for any value of X. So the ring is cooling off eventually.

Now, this is perfectly fine. Actually, this is a perfectly fine way of writing the solution for reasons which I will explain, and for reasons which we will see often times in the course. I wanna work with this a little bit more, and I wanna write the equation in a different form. This was exactly the sort of thing I had you do in homework. Although, in homework it turned out even a little bit simpler than this. It may not have seemed it, but it did.

I wanna write this sum differently. I mean, again, we've solved the problem in the sense that we have found a formula for the temperature at any point X on the circle at any time T. All I'm saying is there's an alternate way of writing it that itself is very revealing, and in fact, has consequences that go beyond the particular problem that we're studying.

So to do that, I wanna bring back the formula for the actual – for the Fourier coefficient. On the right, F hat of K. I wanna use the explicit formula as an interval. So I wanna use F hat of K is the interval from zero to one – even though I've already used X as the variable here, so the variable for integration I'm gonna – I'll use something else, like I'll call it – I'll just use Y. It doesn't matter what I call it, as long as I'm consistent.

So I'll write this as E to the minus two pi IKY, F of YDY. All right. That's the formula for the K Fourier coefficient. And plug that into the formula for the solution.

So I get U of XT is the sum, and from K – from minus infinity to infinity, this interval is the interval from zero to one, E to the minus two pi IKY, F of YDY. E to the minus two pi squared, K squared, T. E to the two pi IKX. That's why you have to do this on a big blackboard with big chalk.

And I’m gonna combine terms and swap the interval and the sum. Again, the rigger police are off duty, and so interchanging integration and summation is something we can do with impunity.

Mathematicians can do it with impunity also, but it takes them three years of graduate courses before they really feel good about it. We don’t have the time for that.

So this is – I’m gonna swap the interval and the sum, so this is the interval from zero to one of the sum K, going from minus infinity to infinity. So I’m gonna put the terms together like this; E to the minus two pi IKY, E to the two pi IKX, E to the minus – I’m putting all the terms together that get summed. E to the minus two pi squared, K squared T. Those terms all get summed, and then what stays on the outside is F of YDY. I haven’t done anything there, I’ve just rearranged things. And I’ve swapped integration and summation.

And I’ll do one more step, and that’s the interval from zero to one, sum from K going from minus infinity to infinity of – I’ll put these two terms together because they’ve both got I’s in them. E to the to pi IKX minus Y, E to the minus two pi squared, K squared T, F of YDY.

Like I said, I haven’t done anything really, except rearrange terms. I’ve rearranged terms this way because I know what’s gonna happen, or rather, I know how to finish the discussion because I know what years of bitter experience taught the people before me finally.

That is F sum deserves to be singled out for special attention, and deserves to be given its own special name. So I’m gonna write, say, G of XT to be this sum. K going from minus infinity to infinity, E to the two pi IX, E to the minus two pi squared, K squared T.

This is actually G of X minus Y, T. That is the interval. The solution appears in this form. Pardon me?

Student:[Inaudible]

Instructor (Brad Osgood):KX, thank you. Two pi IKX.

So the solution U appears in the following form. I just use that as a shorthand notation. The solution looks like this; U of XT is the interval from zero to one, G of X minus Y, T, F of YDY. And that’s all I wanna – that’s the – I’m not gonna do anything else. I promise you, I won’t do any more rearranging; I won’t do anymore fiddling around. That’s how I’m gonna write the solution.

Now, for those of you who’ve seen this, and you actually saw this – you saw a similar sort of thing on the homework problem. This expresses – so this is an important statement that I’m about to make. Drum roll. This expresses the solution, the general solution that any time X and any time T as the convolution of the initial condition at times zero with this kernel.

This expresses UXT as the convolution of F of X, F of Y, it doesn't matter, F of X with what's called the heat kernel G of XT. So there are a lot of terms there that we haven't heard before, although you may have heard in different context the term convolution, which we are gonna be hearing all the time. And in this special case, you also call this the heat kernel. You also call this the fundamental solution or the Green's function for the heat equation.

So I will just – without saying really anything more, just to introduce you to the terminology, you call G of XT – has a variety of names. That function that I wrote down, G of XT, is called variously the heat kernel. That is it's a kernel for the heat equation. You use the word kernel often when it appears underneath and interval in the context of convolution like this.

Heat kernel is also called the fundamental solution of a heat equation, and it's also called Green's function for the heat equation.

Now, let me just ask, just to take a survey out there. First of all, who's studied this problem before? Who's studied the problem of heat flow sort of this way with Fourier series? All right, so a couple people, but not that many actually. In what class and what context? I'm just curious.

Student:[Inaudible]

Instructor (Brad Osgood):Oh, is that right? Okay. And they did it more or less like this?

Student: Pretty similar.

Instructor (Brad Osgood):Yeah. They probably worried about certain things, like convergence and things like that, right, but screw them.

And how about this terminology? So have you heard this terminology, the heat kernel, the fundamental solution or the Green's function?

Green's – people who have taken many physics courses have often heard the term Green's function for differential equations and things like that. You may have seen that. And we're not gonna – I'm not gonna make a big deal out of it, but again, it's sort of an indication of the kind of techniques and the kind of ideas that come up in this class you see everywhere.

For us, the reason why I went through the solution, and the reason why I wrote the solution this way was to bring up this idea of convolution. Convolution is a very general operation. People in electrical engineering have seen it very early on in their classes on signals and systems. We are gonna see it in all sorts of different context. Not always in terms of differential – not always associated with differential equations, but sometimes associated with differential equations. And it is the kind of thing that comes up.

The fact that you could write a solution of the convolution of two functions, in this case, the initial distribution of heat, and the special functions associated with the equation.

I'm not gonna say anything more about it now. I'm not gonna even give you the general definition of convolution. But all I wanted to do by showing you this was it comes up in a very – in retrospect, in a natural way. It's the sort of thing you should expect to see.

It's one thing – when you're working with any problem almost that has to do with Fourier analysis, it should not surprise you to see convolution coming into the picture somehow. This is an example of that.

On the homework you had another example of it. You had an example of another celebrated problem in mathematical physics, the so-called [inaudible] problem, where again, the solution ultimately could be written as convolutions.

There, the nice thing was that you actually got a closed-form expression for essentially the Green's function, or the fundamental solution for that problem. The plus on kernel is actually a closed-form expression. There's no similar closed-form expression here, or you can't do anything more with this function. It is what it is. It's just this infinite sum.

For the problem you had in homework, again, you get an infinite sum that comes in, but it collapses because it's a geometric series. And you get a nice closed-form expression for the plus on kernel.

But in principle, the principles that are operating here are very similar for the two problems. And again, you'll – it's sort of a fact that took a long time to sort out, that when you have solution – when you have partial differential equations, which govern many physical phenomenon, the solutions often appear in the form of convolution with a special solution, so-called fundamental solution, with the initial conditions, with the initial data. It's something you should expect to see.

This is a pretty big major secret of the universe, all right. So take that to heart, you'll see this. Be afraid, be very afraid. No, no, show no fear. It's what you should expect.

And with that – pretty impressive. And as I say, this is also part of your intellectual heritage, all right. You should know this solution, know this approach to the problem. It's a very famous problem. It had all sorts of far reaching consequences. It should be part of your soul.

I believe with that we bid adieu to Fourier series, although as I say we'll come back to it from time to time. And actually, maybe if not right this day where we bid adieu to it because right now what I wanna do is talk about the transition from Fourier series to Fourier transforms.

And that is the transition from periodic phenomenon to non-periodic phenomenon. So I wanna make a transition from Fourier series to Fourier transforms. And this is the transition from periodic phenomenon to non-periodic phenomenon.

Now, I've said before, and you'll hear me say it again, we have other than make a lot of choices in this classes, and choices for how to cover the material. This is not the only way of doing it, all right.

In many treatments of the Fourier transform, you don't make this – you don't do it this way. That is, you don't start with Fourier series and then try to make the transition to Fourier transforms. It's just the Fourier transform is presented as sort of a [inaudible]. God and the machine, or whatever. It's just there, and it's justified by its many uses and its important applications. That's fine, and that's quite justifiable.

I didn't wanna do that because I wanted to show you the basic phenomenon associated with periodicity. It's an important enough topic, and I wanted you to sort of have some of those things – I wanted to try to cultivate your intuition a little bit for those sorts of ideas. But you don't have to do it this way.

If you'll look at Bracewell's book, which is a very common and popular book that's been used for this course – Ron Bracewell just passed away, actually, was a professor in the electrical engineering department here for many, many years. His book starts out with the Fourier transform, with no mention of Fourier series. And then later on, it actually recovers some of the ideas of Fourier series based on the Fourier transform. And you can do that, but that's a choice, and I have made a different choice. That's all that's involved here.

Now, the transition from periodic to non-periodic phenomena and where we're gonna accomplish that is to view a non-periodic phenomena at the limiting case of a periodic phenomenon as the period tends to infinity, all right.

So we'll do this by viewing non-periodic function, and we'll say phenomena. So non-period function as sort of a limiting case of a periodic phenomenon – periodic function as the period tends to infinity, period tends to infinity.

It's a little tricky to do this, actually. It's not completely – it doesn't happen completely automatically. It takes a little work. It's not a completely automatic process.

Now, the other thing to realize is there are actually two – I'll do it over here – there are actually two aspects to this. So once again, let me remind you of the Fourier case – Fourier series case. There are two aspects to the Fourier transform, really. Two aspects, there's analysis and synthesis.

So the analysis is the Fourier series is forming – so again, if F of T is periodic, the you have the Fourier coefficients, the interval from zero to one, E to the minus two pi IKX, F of RKT, F of TDT. That's analysis. That's analyzing the function, the signal into its constituent components. Figuring out how much each complex exponential contributes to the whole by this much, this amount.

Then there's synthesis. The synthesis is writing the series. We're covering the function from its constituent components. Sum from minus infinity to infinity, F out of K, E to the two pi IKT. So that's synthesis. And both of those things generalize to the Fourier transform.

The Fourier transform is the generalization of the Fourier coefficient. The inverse Fourier transform is the generalization of the Fourier series.

For a transform is a generalization – or the limiting case if you want to think about it that way. Generalization that is limiting case in the sense that I'm talking about here is the period tends to infinity case of the Fourier coefficient. So that's the analysis decomposing a signal into its constituent parts.

The inverse Fourier transform is a generalization, or the limited case in the Fourier series. It is a limiting case of a Fourier series. That's the synthesis part of the equation, so this is part of the discussion.

Now, so how do I set this up in order to take a limit? So I keep saying that it's the limiting case is the period tends to infinity, so that means I have to give you the setup for Fourier series, for Fourier coefficients and for Fourier series, when the period is not one, but the period is some other number, capital T, that I can let tend to infinity.

So we need a setup when say F of T is periodic of period T. Then also I want to let T tend to infinity. I want to let T tend to infinity. How do I do this?

Well, you read about this, I hope, I expect. Let me tell you what the formulas are. It's not hard. The building blocks for a signal of period T are complex exponentials of period T, so that's E to the two pi IKT over T.

Or I'm actually gonna write it a little bit differently. I'm gonna write it as E to the two pi ITK over T. It doesn't matter. I just switched the parenthesis there. Same thing.

These are periodic of period capital T. And the corresponding Fourier series – the Fourier series are the form CKK going from minus infinity to infinity, C, sum K, E to the two pi IKX – KT – K – [inaudible] write like this. K over TT, or K little T over capital T. I'm gonna write it like that for reasons you'll see in just a second. That's what the Fourier series looks like.

What are the coefficients? Well, again, one can repeat all the arguments that we did when we were working with functions of period one, where actually you can use the results for the period one case to derive a general result.

At any rate, what you find is C sub K is given by one over T, times the interval from zero to T of either the minus two pi IK over T, little T, F of T, DT. Did I say here anywhere that F is little, F is periodic of period T? I guess I didn't say that, so sorry, let me say it now. F of T here is a given signal, so I'm assuming it is periodic of period capital T. So F of T, periodic of period capital T.

That's the formula the K Fourier coefficient when the function has period capital T. And actually, you use this again in the homework problem.

Okay. Now, one other thing, I'm gonna write this a little bit differently again for reasons which you will see in a moment because I have in mind – taking a limit here as capital T tends to infinity. I can also write this the function's periodic of period T, doesn't matter what interval you integrate over.

If I know the function on an interval of length T I know it everywhere, so I can also write this formula – and again, this is discussed a little bit more detailed in the notes – I can also write the formula as instead of integrating from zero to T, I can integrate from minus T over two, to T over two as symmetric interval from negative to positive. Same thing, either the minus two pi IK over capital T, T, F of TDT. Okay, fine.

Now, this is really no different than what I've done before, except I've written things a little bit more generally. Instead of a function of period one, I have a function of period one, I have a function of period capital T.

But the formula for the Fourier coefficient is perfectly [inaudible] to what I had before, and the formula for the Fourier series is perfectly [inaudible] to what I had before. So again, that's analysis, that's synthesis. You analysis the function into its constituent parts, and then you synthesis it by forming the corresponding sum.

Now, I wanna point something out here that's very interesting. How would you draw a picture of the spectrum of these cases? What's a picture of these things? What's a picture of the spectrum, picture in the frequency side, picture of the spectrum of frequencies?

Well, in the case of period one – let's take period one. Then the Fourier coefficients are given by the usual formula we had before, and you might draw the spectrum like this. There's a coefficient of zero, that's the zero coefficient, C zero. Then there's the – well, let's see.

Zero, one, two. I have frequencies at all the integers. Minus one, minus two, and so on and so on. Three, minus three. And there's space, so to speak, the harmonics, the frequencies of space are one apart. Here is like absolute value of C zero. Here is absolute value of C one. Here's – I say absolute value because they're actually complex numbers. The coefficients are complex numbers, so I can't actually plot them.

But I can plot – I can get a picture of the spectrum by plotting the absolute value. Here's the absolute value of C two, and here's the absolute value of C three, whatever it is.

And what will I get on the negative side, what will I get on the negative side? On the negative side I get the same absolute values because of the symmetry relation, C minus K is equal to CK bar. So the absolute value of C minus K is – the magnitude of C minus K is the magnitude of CK bar, and the magnitude of a conjugate of a complex number is the same thing as the magnitude of the number. So this is also the magnitude of CK.

The picture's the same on the left, so this is the magnitude of C minus one, the magnitude of C minus two, the magnitude of C three, whatever. Magnitude of C minus three, whatever it looks like. That's the picture.

And in fact, if you've ever worked with a spectrum analyzer, and if I have the chance I'm gonna bring one into class, you see pictures like this. You see a signal and you see these bars that are at the different frequencies, okay.

Now, the important thing here is – the reason I mentioned this is they're spaced one apart. Because a period one – the frequencies are also one apart, one, two, three, and so on and so on.

Now, what about for a function of period T? So the spacing here – the spacing of the frequencies is one. The spacing of the frequencies is one. They're one apart if you were gonna draw the pictures.

Now, what about if you had period T? What about if I had period T? Well, what is the picture there? The picture is like this, the Fourier series looks like this; the Fourier series for a function of period T looks like the sum from minus infinity to infinity, C sub K, E to the two pi I, K over T times T.

So the harmonics, they're indexed by K, but they're periodic of period T, so really, the harmonic you're interested in is in some sense tagged by K over capital T. So zero, one over T, two over T, three over T, minus one over T, minus two over T, minus three over T, and so on.

So if you were to draw a picture of the spectrum that would correspond to a series that looked like this, you would draw a picture that went something like this; you would draw a zero, so this is now period – so this is for period one.

For period T, you would draw the first harmonic sort of at one over T, the second harmonic is two over T, the third harmonic is three over T, and so on and so on.

Then this minus one over T, minus two over T and so on, the spacing is one over T apart. And here's the zero Fourier coefficient, here's the first Fourier coefficient, here's the second Fourier coefficient, here's the third Fourier coefficient. The ones on the negative – corresponding the negative frequencies have the same – when I say here's the first, here's the second, here's the third, I mean the magnitude because again, you can't plot complex numbers. You're just plotting the magnitude.

And on the left, I have the same picture because it's symmetric. So it's like this, like this, like that and so on. Minus three over T and so on. They're spaced one over T apart. But the spectrum has spacing one over T.

Now, this is another example – remember early on when I first talked about frequency and wavelength, and I talked about the inverse relationship, or the reciprocal relationship between frequency and wavelength, very first day of class. That's something you saw a long time ago.

This is our second related example of an inverse or reciprocal relationship between, in this case, I'm gonna say the two domains. How the function appears in the time domain, and how the function appears in the frequency domain. How this is a function of time, and how it is in terms of its constituent frequency parts.

If the period is T, then the spacing and the frequencies is one over T. There's a reciprocal relationship here between the period and the frequencies. Between the period – that's often – and what's happening to the function in time, or what's sometimes referred to as the time domain of the function, and the frequencies between the period and the frequencies.

Or viewing the function in the frequency domain, viewing it in terms of its constituent parts. Reciprocal X of a period T means frequency of spacing on the frequency is a one over T. Frequency spaced one over T apart.

All right. I didn't say anything about the size of T here, but that's a general reciprocal relationship between the two domains. And again, it's something you're gonna see throughout this course. So a reciprocal relationship between the two view of the function. And that reciprocity is exactly mediated by Fourier techniques, Fourier series in this case, or very soon the Fourier transform.

But again, it's the kinda thing – this is the sort of intuition you have to start to develop. You're viewing the function one way, you expect certain things. You view the function in the other domain you expect the reciprocal phenomenon.

Now, if T is less than one, and one over T is bigger than one, so T less than one. So a function which repeats more frequently than once a second implies larger or smaller spacing than one, implies one over T is bigger than one, so the spacing is larger. Spacing – one over T is bigger than one, spacing bigger than zero, bigger than one.

The spectrum is spread out. If T is bigger than one, if you have a long period, the spectrum is compressed. The spacing is one over T. One over T is less than one, so the spacing in the spectrum is squeezed, compressed.

In particular, as T is going – as capital T is going to infinity, which is the case ultimately I wanna deal with, the spectrum is getting more and more – the spacing of the spectrum is smaller and smaller. The frequencies are getting closer and closer together. One over T, two over T, three over T, four over T. They're getting closer and closer. One over T is getting smaller and smaller, and they're getting closer together. As T tends to infinity, the spectrum becomes sort of continuous.

I'm gonna make this more precise, or rather, I'm gonna make this more explicit if not more precise in just a minute. But the idea is the spectrum's getting closer, the frequencies are getting spaced closer and closer together because the spacing is one over T. The spacing is one over T, and if T is tending to infinity, the spacing is tending to zero. That's what I mean by the fact that the spectrum is getting continuous, so the spectrum is getting squeezed. Now, that's the formula here for the coefficient here once again.

Now, let me just start now – no, one thing at a time. So let me – once again, let me write down the formula for the coefficient. CK is one over T. This is the definition for the Fourier coefficient when the function has period T. So I'm going to integrate from minus T over two to T over two, E to the minus two pi I, K over T with a T, F of TDT.

Now, I wanna let T tend to infinity here, and use this as a way of passing from periodic to non-periodic phenomenon, and use this to pass from periodic to non-periodic.

But as I say, it's not quite straightforward. And let me tell you why. I can't just take the limit as T tends to infinity there and get – and the Fourier transform pop out of that. It doesn't work. You can't just let T tend to infinity and get the Fourier transform. You have to tickle it a little bit.

And actually, let me leave this picture up on the board and show over here, all right. Let me tell you what the setup is gonna be and what I wanna do.

So imagine I have some function. The picture's not periodic. But suppose it's fine, I can extend. So suppose F of T looks like this. Sum [inaudible] sum, sum interval going from A to B, and it's zero beyond – less than A and bigger than B, all right. That's my function.

So I take some big number, T, and I periodize this, say bigger than – so the minus T over two is less than A, and plus T over two is bigger than B. So here's my function; it's zero, less than A, it's zero, bigger than B. I take some – and I wanna approximate this thing, but I wanna imagine this is a periodic function. So I take some big period beyond where the function is zero, and I periodize it to be period T, all right. Take a big T and periodize to have period T.

Okay, fine. Now, write down the formula for the Fourier coefficient. So imagine, if the function here were fixed, if that's all I worried about, I let T go to infinity, then I'm sort of approximating that non-periodic phenomenon by a periodic phenomenon with a very big period, all right. That's gonna be my goal.

But the problem is – and I wanna see what happens to the Fourier coefficient. So write down C sub K. The Fourier coefficient looks like this, C sub K is one over T, the interval from minus T over two to T over two, E to the minus two pi I, K over T, little T, F of T, DT, okay. That's the K Fourier coefficient.

But now, F is zero, less than A and bigger than B, and those numbers are fixed. Those are sort of given to us. So this is equal to one over T times the interval from A to B. E to the minus two pi I, K over T, little T, F of T, BT. Because the function F, I'm assuming, is zero, less than A and bigger than B, okay.

Now, this interval in absolute value is gonna be bounded. The interval from A to B, E to the minus – an absolute value – E to the minus two pi I, K over T, little T, F of T, DT, the absolute value is less than the interval and the absolute value. This is less than or equal to the interval from A to B, the interval of the complex – the absolute value of the complex exponential, which is one, times the function DT. The absolute value of a complex exponential is one. I'm almost there, almost there.

So this is just equal to the interval, this absolute value. You'll see why I'm doing this in just a second, really, really, really. This is equal to the interval from A to B, F of T, DT because the absolute value of the complex exponential is just one. That's a fixed number. It's like M.

Okay. So what does that say about the Fourier coefficient? Watch what I did here. I just wrote down the formula for the K Fourier coefficient. What I wanna see – I wanna convince you there's a little bit of problem with directly letting T tend to infinity. If I directly let T tend to infinity, everything is gonna die.

Here's the K Fourier coefficient. It's one over T times this interval. What about that interval? That interval is that interval, and this interval in absolute value is bounded – say goodbye to this drawing now.

So this says that in absolute value, C sub K is less than equal to one over T times M for all K, all right. So as T tends to infinity, C sub K tends to zero. The Fourier coefficients die.

So I had this wonderful idea. I said I'm gonna approximate a non-periodic phenomenon by a periodic phenomena, a very large period, and I'm gonna let the period tend to infinity. Sounds great. Sounds like a very natural thing to do.

I write down the formula for the Fourier series, great. I got these coefficients for the Fourier series, great. I'm gonna let T tend to infinity, great. The coefficients are gonna tend to zero, not great. I'm not gonna get a formula that's gonna help me as T tends to infinity. Not great.

I think I have to quit right now. Tomorrow, on Friday, I will tell you how to save this in a very nice, easy way that's gonna lead to everything.

[End of Audio]

Duration: 53 minutes