TheFourierTransformAndItsApplications-Lecture07

Instructor (Brad Osgood):I'm gonna upload the solutions to Problem Set 1; I'm gonna do that shortly. There were a couple of people who joined the class late, so I gave them a couple of extra days. Put that Daily away. I know it's exciting about that great Cardinal team. Anyway, so I'll put it up either today – sometime over the next couple days, all right? And the latest problem set, Problem Set 3, is now posted on the website. All right. Any questions about anything; anything on anybody's mind? How do we like our Fourier Transform so far, class?

Student: Whoo.

Instructor (Brad Osgood): Yeah, all right. That's the spirit. All right. So let me remind you – as a matter of fact, let me reintroduce the star of the show. So let's recall the Fourier Transform and its inverse, and I want to make a couple of general remarks before plunging back into specific properties, specific transforms and some properties, and its inverse.

So, again, f of t is a signal and the Fourier Transform or function, same thing, the Fourier Transform, I use this notation. I want to comment about that, again, in just a second. Integral from my infinity – infinity of either the -2p I ST, F of T, VT and the inverse Fourier Transform looks very similar except for a change in sign in the exponential. So the inverse Fourier Transform of – I use a different function, although it doesn't matter. We're gonna go from -8 to 8 of either the +2p I ST, G of S, DS, okay?

All right. Now, I want to make a few general comments here. This isn't one I have to write down, but it's actually quite a complicated operation, all right? Integration is not such a simple operation. Integrating a function against the complex exponential is gonna make the thing oscillate, and computing an infinite integral from -8 to 8 brings with it all sorts of peril, all right?

Now, the rigor police are off duty, for the most part, all right? So we are not gonna be so concerned about the existence of those integrals. That is an issue, all right, and it is something we're gonna have to deal with but not right away. Right now, I just want to get a little practical, hands-on know how with using the formulas and being comfortable with the formulas.

So I'm not gonna worry about the convergence right now. We will talk a little bit more about later, although, it's never gonna be a big issue for us, or rather, we're gonna – well, we're gonna talk about it in a number of different ways, all right, but it is something to worry about.

So when I write down the definition of the Fourier Transform, what I really should have said was the Fourier Transform is defined by that integral whenever the integral exists, all right? So the question is the existence of the integral on the existence of the integral, that

is to say convergence of the integral – of the integral. All right, so more on this, to some extent, later.

The set of points but just to introduce a bit of terminology here, something that I'm sure you've said, and I actually have used, I think, somewhat informally. The set of points for which the integral does exist, that is to say, for which the Fourier Transform exists, is called the spectrum, all right? So the set of S of S in R for which the Fourier Transform is defined, that is to say for which the integral exists, is called the spectrum.

So when I made the bold statement last time that every signal has a spectrum, and the signal is determined by a spectrum, I was thinking of exactly this – that is that, in many cases, the integral is not an issue; that is it will exist, all right? And the cases when it doesn't exist, of course, that poses an additional problem; you have to do some further analysis.

All right. Now, that's one comment that I wanted to make. As I said, the rigor police are off duty. I'll tell you when we have to worry about those kinds of questions. Now, that thing I wanted to say is that our definition of the Fourier Transform and the notation I'm using for the Fourier Transform is not universal. As a matter of fact, there is no universal definition or universal standard, universal notation for the Fourier Transform.

So definitions and notations vary, all right? You should be aware of this, and it's not that – and no single notation is perfect, all right? Different notations are useful in different context. So you see – you will see, for example, the notation F Hat – and really, sort of, corresponding to what we did for Fourier Coefficiency, we'll see the notation F Hat for F Hat of S, or what I'm calling, for the Fourier Transform.

You will also see – this is a pretty cute notation. You'll see an upside down Hat for the inverse Fourier Transform. I'll call it T, all right? You'll see that notation. It's not uncommon at all, all right? You also see – that's probably the cutest one, but it, sort of, gets lost in typesetting. You also see – a very common notation in engineering especially is to use a lowercase letter for the function – for the signal, and an uppercase letter for the Fourier Transform.

So you also see f of t for the signal, and F of S for the Transform, all right? The problem with this notation is that it's not so good when you talk about duality, which we're gonna talk about today, actually, because it really makes more of a distinction somehow then there ought to be made.

Anyway, but it's useful – all these notations are useful in some context, but no notation is useful in every context. There's always some ambiguity; there's always some clash, all right? And you have to be flexible, and you also have to know which notation the particular author, the reference that you're looking at, is employing because people will use different notations.

There's also a notation that people use for the, sort of, sibling relationship, which might mean more to you in just a second when I talk about duality, between a function and its Fourier Transform. So some people write things like F of T corresponds to F of S. I, sort of, use that notation, but there's this other notation, which I think is just idiotic, which is like a subset notation or something like that, right?

Like, does it go this way, or the other way, or something like that? I hate that notation, but it's used. Braceval uses this, God rest his soul, but it's a stupid notation. I'm sorry, you know? He's dead; it's gone. All right. Finally – and it may have something like that. Anyway, say no notation is perfect, but some notations are worse than others.

Finally, it's not universally agreed on how the Fourier Transform itself should be defined, all right? There are different definitions – I think, actually, let me just go back. I think the script F notation that I'm using and a lot of other people use also is the least ambiguous notation. It's sometimes awkward in certain context, but they say it's the least ambiguous, and somehow I tend to gravitate toward that more than others, but, again, you can choose what you want, and you'll see them all in use.

Anyway, I was gonna say there are also different definitions that are in use. For example, one sometimes defines the Fourier Transform of – because where do you put the 2p and where do you put the minus sign, all right? So you will see this definition. You'll see the Fourier Transform of F at S is the integral for -8 to 8 of E to the I ST, F of T, DT, with no 2p up there, or you sometimes what you see is I omega T instead of S, all right?

And you'll see the inverse Fourier Transform – excuse me, a minus, without the 2p. You'll see this notation. You'll see the Fourier Transform of F at S is 1 over 2p × the integral for -8 to 8 of either the -I ST, F of T, DT, sometimes putting the factor of 1 over 2p out front. You will also sometimes see the Fourier Transform with a plus sign and the inverse Fourier Transform with a minus sign, all right?

You'll also see the Fourier Transform F at S is integral from – or sometimes with a 1 over 2p out in front. All combinations are in use, all right? E to the +I ST, F of T, DT, and the inverse Fourier Transform is the integral for -8 to 8 be to the -I ST, F of S, DS, all right? You'll see all these notations in use, and you just have to know – somebody has to tell you or somehow otherwise you have to figure out which particular convention is in use, all right?

This one is actually especially popular in physics, all right? It's quite often in problems in optics, for example, where the Fourier Transform comes in. You often see this is the definition of the Fourier Transform, sometimes with a 2p in there. You can stick it here; you can stick it there. If you don't like it, you can stick it someplace else, and this is the definition for the inverse Fourier Transform, all right?

All I can say is be careful out there.

In the notes, I moved this in various places; I'm not sure the best place to put this. I think it's now at the end of the chapter called – the chapter on convolution, which we'll get to very shortly, there's a section called Chase the Constant or something like that, which I stole from a book by Tom Kerner.

And that tells you how the different formulas change when you, you know, we change the plus sign to a minus sign, when you change where you put the 2p and so on, all right? And so it's, sort of, it's meant to serve as a dictionary to help you translate one case from another case, or one convention from another convention when you're likely to run across them, and you will be likely to run across them. Okay. So I felt like I ought to tell you that.

Now, I think I may print that out separately and actually just give it to you, sort of, as a dictionary so you can make that translation because, as I say, it's just a pain in the neck; what can I say? But it's somehow in the nature of the subject that there's not a universal definition. I think it's fair to say that the one that I've given there is probably the most commonly – even then I'm a little bit hesitant to say that, but I think it's safe to say it's the most commonly accepted convention, but by no means, is it universal.

Okay. All right. And we also had last time – a reminder we did last time, we had two basic examples of the Fourier Transform, calculating the Fourier Transform, and, again, at this stage, the only way we have to calculate the Fourier Transform is by recourse of the definition. You have to carry out the integration if you can, all right? There's no way of getting around it. We'll learn, then, today and then going on other techniques that will allow us to calculate new Fourier Transforms from old Fourier Transforms, but, for now, right at the beginning, there's nothing to do except plug into the formula.

So the basic examples we had, which come up very often in applications, are the rectangle function, p of T, and that's the function that looks like this. It's 1 from -½ to ½ and then 0 outside the interval, and I don't care how it's defined at the end points because it doesn't make any difference in the calculation, and the Fourier Transform that is the sync function, Fp of S is sync of S, which is, in my convention, sign of pS over pS, and that's, by the way, of course, another thing that's not universally agreed upon. Some people define the sync function without the p in here. Some people just say the sign of S over S, or sign of X over X without the p. What are you gonna do? What are you gonna do?

The other example that we had, I didn't carry out the calculation completely, but, again, it was based on just calculating the integral. In this case, you had to use integration by parts. It's the triangle function has Slope 1 going up from -1 to 1, and then down from 1 to 1, 0, that's lambda of T, and the Fourier Transform there is the sync², nice result, sync² of S, okay?

All right. I want to do one more particular case that's really cool, actually, and also comes up quite a bit in applications, and then we are going to take the different path, the second path that I talked about, that is talk about some general properties of the Fourier

Transform, and how you'd use the Fourier Transform to find – how you can formulate some general properties that will allow you to find Fourier Transforms of combinations of functions or modifications of functions.

But let me do one more explicit example for you, and that is our friend the bell curve, the Gaussian. That's a very important function in many applications, and it has a remarkable property with regard to the Fourier Transform. So, as the third example, let me do the basic Gaussian F of T = E to -p T².

Now, again, there's a question of normalization here, all right? The shape is the familiar bell-shaped curve, which comes up in probability distributions, and I'll talk about that probably a little bit next time or the time after that when I talk about the central limit there.

I mean, it has a height of one, and with this normalization, that is putting the p in the exponent, that normalizes the area under the curve to be one. It's not the only way of doing it, but it's the way of doing it that's, somehow, most convenient – most natural when you're working with Fourier Transforms.

So the result when you make this normalization is the integral from -8 to 8, either the -p T², DT is 1. Now, as tempted as I am, I'm not gonna show you why that's true. I'm not gonna do the calculation. That's one of the most famous tricks in all of mathematics to get that result going back to Oilier.

You cannot do this by direct integration because the function, E to the -p X² has no elementary anti-derivative. So you can't do this with the fundamental theorem of calculus. It has to be done by a very devious other means, and it's done in the notes. You should look it over. If you have any questions, ask me about it because I never get tired of talking about it because it's such a famous and elegant trick, but, in the interest of time, I think I won't go through the derivation, all right, but that's an important result, and a surprising result. I mean, why Ep 1, why should they all come together in such a simple formula? But they do.

So here's what I want to show you, and to let the cat out of the bag, I'll show you where it comes from is that the Gaussian is itself – no, the Gaussian is itself. That's – I am myself too, and you are yourselves, but that's not so interesting. What's interesting is the Fourier Transform of the Gaussian is itself. That is if F of T is equal to, for this normalization, either the -p T², and what I want to show you is the Fourier Transform is the same function, either the -p S². I'm using a different variable there, but the basic fact is that the Fourier Transform of the Gaussian is the Gaussian.

Now, I mean, it's a very striking result because computing the Fourier Transform is a complicated operation, as I'm gonna go through this to show you how it works. So why a function should turn out to be itself under Fourier Transforms is really quite striking.

We'll also see, soon, that – you remember, you've heard me talk about many reciprocal relationships between the time domain and the frequency domain, and we're gonna see various examples of that, and, somehow, what this tends to say is – for reasons which you'll understand a little bit better later – is that somehow the Gaussian is equally spread out in the time and the frequency domain; there's no difference. Because, often, what happens is if a function is spread out in one domain, it's stretched out and it's squeezed together in the other domain or vice versa, all right?

We'll see that, actually, as an example of – well, we'll see that as a basic property of the Fourier Transform, and what this says, the fact that the Gaussian – the Fourier Transform of the Gaussian is itself, means, somehow, it is equally spread out in both time and frequency, all right? Which is something you wouldn't occur to you to say or would occur to you to think about unless you knew this result, all right?

So why is something like this true, all right? We have, again, no recourse other than to the definition. So let me write – in this case, I think it's a little bit easier actually to use the capital letter notation because of I'm going be differentiating and performing some other unnatural acts on the Fourier Transform, so watch.

So let me call, again, if F of T is = to either the -p T², let me call capital F of S its Fourier Transform. So that's the integral for -8 to 8, E to the -2p I ST, E to the -p T², DT, okay? And I am actually gonna evaluate this integral, all right? Not by any, sort of, appeal to the fundamental theorem of calculus or anything like that. That won't work, but there is a clever trick that will actually allow us to carry out the integration after an initial, little slight of hand.

That is to say I'm gonna differentiate with respect to S, all right? That is F prime of S, and this can be justified – the rigor police are off duty, but even if they're on duty, I would have no trouble in asserting that I can find the derivative of this function by differentiating under the integral sign. So the integral for -8 to 8, the derivative of this thing is DDS. The only thing that depends on S here is this complex exponential, either the 2p I ST, and then either the -p T² stays the same; it doesn't get hit by the derivative. So the derivative of this with respect to S is equal to the integral for -8 to 8 -2p I T, E to the -2p I ST, then E to the -p T², DT, okay?

All right. Now, watch this. You factor out the I, I'm gonna write this out as I × the integral for -8 to 8. I want to group terms in a very suggestive way. So it's gonna be E to the -2p I ST, and then 2p T - 2p T × E to the -p T², DT, and I group the terms this way because it cries out to be integrated by parts, all right? Differentiating with respect to S brings down this factor of 2p T or -2p T here, all right? This absolutely cries out to be integrated by parts. This is the U. This is the DV, okay?

And if I do that, what happens? Well, if DV, once again, is -2p T × E to the -p T², VT, then V is E to the -p T², all right? Derivative of this E to the -p T² brings me down with respect to T, brings down to the -2p T, all right? And if U is equal to E to the -2p I ST,

then DU = -2p I S, E to the -2p I ST, 2p I ST, DT, all right? I'm differentiating there with respect to T, or D-ing with respect to T, so to speak, okay?

All right. So, again, integration by parts tells us that the integral from A to B of UDV is UV, evaluated between AB - the integral from A to B of VDU, all right? So how does that work out in this case? In this case, there's an I times the whole thing, right? So there's an I × U × V, which is E to the -p T², that's a V, -p T² × E to the minus – where's U? U is – yeah, -2p I ST, and evaluated between -8 to 8, minus the integral from -8 to 8 of VDU. V is either the -p T², DU is -2p I S, E to the -2p I ST, DT, all right? Closed braces because there's an I in front of the whole thing. Cool? Way cool.

All right. Now, what about these boundary terms here? What about the terms U × V between -8 to 8. What happens to that? Gone, all right, because this thing has absolutely value one. E to the -p T² is going to 0 at both +8 and -8, all right? So this is killing this off. It is gone.

What remains? What remains is -2p I S, a minus sign here, and then there's an I in front, all right? So if I got all that right, it's gonna be an I × an I, and I × I is a -1, right? So this is gonna be – right. There's a minus, a minus, and then an I × an I gives you an extra -1. It's gonna be minus – the integral for -8 to 8 of 2p I S – 2p S × E to -2p I ST, E to the -p T², DT, okay?

Now, I'm integrating with respect to T. This comes out. This depends on S. So this is -2p S integral for -8 to 8 of E to the -2p ST, E to the -p T², DT. Brilliant. I've gotten back to where I started. All those years of education, all that work, what did it get me? Back to where I started, except there's an extra factor out front. That is this is equal to -2p S × F of S, the original Fourier Transform. That integral is the Fourier Transform of the Gaussian. Integral for -8 to 8 , E to the -2p I ST × E to the -p T², all right?

So what have I shown here? I have F prime of S = -2p S × F of S. Oh, but that's just a simple differential equation for F, kids, right? So that says that F of S = the initial value F of 0, E to the -p S². That's the only solution to that baby, okay?

And what is F of 0; what is capital F of 0? Capital F of 0, that's the value 0 of the Fourier Transform actually, right? When S = 0 this is -8 to 8, E to the -2p I, 0 × T, E to the -p T², DT. That's the integral for -8 to 8, E to the -p T², DT, which is 1, okay? So what is the actual retail value of the answer? That says that F of S, capital F of S is E to the -p S². This says that capital F of S is E to the -p S². Done. Fantastic, fantastic, all right?

So again, I take the Fourier Transform of the function E to the -p T², that corresponds to E if I want to use that notation for p S², all right? The Fourier Transform of the Gaussian, when it's normalized this way is itself. If you change the normalization, you're gonna get a different answer, although, we'll figure out how to do that. We'll figure out how to make such changes, but for this Gaussian the way it's normalized, E to the -p T², its Fourier Transform is itself – quite remarkable, and quite important, all right, quite important.

Okay. All right. Now, that's about all I want to do at this stage for specific transforms. We only have three, actually: the rectangle function, the triangle function, and the Gaussian. There are other examples in the book, all right? There are other examples on those. There's a one-sided exponential decay, the two-sided exponential decay. I'm not gonna go through the calculations, all right? If this were a regular calculus class, I'd go through all those calculations, but you're beyond that. You can read the calculations; you can certainly use the results, all right?

They all have to rely on using the definition of the Fourier Transform. There's no other recourse. That's all you have to work with is the definition, all right? So that means all you have at your disposal is – well, there are tricks like this, and then but, typically, they're just integration techniques, and the integration techniques are usually integration by substitution or integration by part; that's all that's use in any of those derivations, all right? So I'll let you go through those things, and certainly use the formulas. There's no reason to memorize them especially. I mean, you can always look them up if you need them; they're there, and so on.

Instead, what I want to do now is I want to talk about some general properties of the Fourier Transform. I mentioned there are two paths that we want to follow. One is finding specific transforms, and then one is finding general properties of the Fourier Transform that will allow us to find the Fourier Transform of different combinations of the functions. So now I want to pursue that second path and look for general properties.

And the first one I want to look at often goes under the heading of duality. So I want to explain that to you. It's actually very simple, but – Fourier Transform Duality, all right? The formulas are extremely simple, but for some reason – well, for reasons which I'll explain to you, many times students really agonize over these formulas, all right? So I want to go through it in a little bit of detail.

Here, what I want to do is I want to exploit the similarity in the formulas for the Fourier Transform and its inverse. That's what this is about, between the formulas for the Fourier Transform and the inverse Fourier Transform, okay? The Fourier Transform of – let me write down the definition again. The Fourier Transform of F of S is the integral from -8 to 8, E to the -2p I ST, F of T, DT, okay? Now, remember the Fourier Transform – say this to yourself early and often – the Fourier Transform is an operation that turns one function into another function, all right? To evaluate the transform, I have to evaluate it at a variable, and in this case, I'm calling the variable S. Fine, that means I plugged S into the formula, and that tells me how to compute the Fourier Transform, fine.

What if I plug in -S into the formula, all right? The Fourier Transform of F evaluated at -S. The Fourier Transform is an operation that turns a function into another function, and to evaluate it, I have to plug in a point. I plug in the point -S. What do I get? I get the integral for -8 to 8 of E to the -2p I × -ST, F of T, DT, which is the integral from -8 to 8 of either the +2p I ST, F of T, DT, which is, if according to my formula for the inverse Fourier Transform, the inverse Fourier Transform of F at S.

I'm gonna write that down one more time. Okay. The Fourier Transform of F at -S is the inverse Fourier Transform of F at S. Fine, that's exactly what that formula says. Now, people, this gives some people just fits, all right? Why? Because this is an example of being wedded to an interpretation, all right? This is an example of – if it gives you fits, it's because you're too wedded to an interpretation and wedded to your variables, all right?

Because they say how can it be S on both sides of the equation? If I take the Fourier Transform, I get into the frequency domain. If I take the inverse Fourier Transform, I get back into the time domain, but you're calling it S in both cases, man. How can you do that? Are you allowed to do that? What kind of fool are you, man?

It doesn't make sense, all right? That's because you always think of – or people cling to the idea that they cling to one particular interpretation. You have the signal and the time domain. This Fourier Transform is something in the frequency domain. You cannot mix the two in your head or anywhere, ever, all right?

And so this formula bothers people. You take the Fourier Transform evaluator minus, it's the same thing as taking the inverse Fourier Transform, but I say to you, as I said before when I was doing it, and that's why I was making the point, the Fourier Transform is a formula, is an operation, that turns one function into another function.

To write down the formula, you have to evaluate the operation at a variable. It doesn't matter what I call the variable. I can it S. I can call it T. I can call it zippity do da. I can call it anything, all right, as long as I am consistent, all right? And there is no inconsistency; there is no error in this at all, all right? So don't think in terms of time, frequency, or whatever. Think in terms of the mathematical operation. The Fourier Transform evaluate a -S is the inverse Fourier Transform.

Now, there is a neater way of writing this. This statement actually and related statements, which I'll show you now, are – and this is exactly sometimes called Duality, or the Principle of Duality, or Duality for Fourier Transforms, or whatever. You would have a tough time – this is an example, actually, where you would have a tough time with other notations, all right? If you used the capital letter notation, all right, that would give you fits because the capital letter notation somehow forces you to distinguish between the two domains, little f of t, capital F of S, you know? And that would make it hard to write down that formula in a way that really made sense, all right? But if you use the operational notation here or the operator notation, then it's easy to write down. It's unambiguous, and there's no problem with it, all right?

Now, maybe, like I said, maybe you're looking at me like, "What is the problem? I don't have any problem with this, but, trust me, I have seen many times many people have a lot of trouble with that formula because they're trying to interpret it in ways that it can't be interpreted. It's just a mathematical statement, all right? It's a very useful mathematical statement as it turns out, but it is a mathematical statement. It's not a statement about time and frequency – not to my knowledge, all right?

Now, I'm gonna write this another way, write more neatly. I think actually it's also a good idea, I think, to try to write things in as variable a free way as you can, all right? Or try to introduce – even if you have introduce a little extra notation to try to not write your variables because in this subject writing variables and naming variables just can get things pretty muddled, all right? So I want to write this a little bit more neatly in a way that will actually allow me to write the duality formulas without any variables, where there's, sort of, no agonizing, or perhaps, no agonizing.

So I want to introduce the reverse signal. As a matter of fact, I think this was even on the first homework, first or second homework, all right? That is if F of T is a signal, then I define F - T just to be F of -T. All right. Not a big deal, all right. So you reverse – if you want to think of T as time, you're reversing time; that's why I call it the reverse signal, okay?

Then, first of all, it's a nice way of expressing evenness and oddness, actually, then in terms of the reverse signal. F is even if the reverse signal is equal to itself because to say that F is even is to say that F of -T is equal to F of T. So say if you reverse a signal, you get the same signal back again. F is odd, if you reverse the signal, you get back minus the signal. Because F is odd, if F of -T is -F of T, all right? So it's a nice simple, let's say, variable free way of writing it, if you want.

And what does this formula say; what does the duality formula say, the first duality formula say? Once again, with variables, it says the Fourier Transform evaluated at -S is equal to the inverse Fourier Transform of F evaluated at S, all right? I want to write this in a variable-free way using the reversed signal. What is the Fourier Transform of F evaluated at -S? It is the reverse of the Fourier Transform. Now, watch very carefully where I put the parenthesis. This is equal to this because the Fourier Transform of F reversed, evaluated at S is the Fourier Transform of F at -S, okay?

So this statement in a variable-free way says the Fourier Transform of F reversed is the inverse Fourier Transform of F – kind of, nice. I mean, the reverse of the Fourier Transform is the inverse Fourier Transform. That's why, in fact, some people talk about the Forward Fourier Transform and the Reverse Fourier Transform.

Actually, they don't even use the term transform and inverse. They sometimes talk about the forward transform and the backwards transform or the reverse transform, and I think what they're often thinking about is this formula, although there's this extra sign change in there. So it says the Fourier Transform of F reversed is the inverse Fourier Transform of F.

Now, what about – so that's one form – I didn't do anything new there. I just rewrote the formula in a variable-free way. It's a way that I can tend to remember a little bit more easily, to tell you the truth.

Let's look at something else. Let's look at the Fourier Transform of F-, the reverse signal, all right? Notice the parentheses there, all right? I'm not reversing the Fourier Transform

of F; I'm taking the Fourier Transform of the reverse of F. What is this? Well, in order to evaluate this, I have to evaluate it at a point.

So the Fourier Transform of F- at S is, by definition, the integral for -8 to 8 – be careful here – of E to the -2p I ST, F reversed FT, DT. That's my definition of the Fourier Transform. The Fourier Transform of whatever I plug in here is what I plug into the formula. In this case it's F-, F reversed. This is the integral for -8 to 8 of E to the -2p I, ST, F of -T, DT, by definition, right?

Now, we're gonna work with that a little bit. We're gonna work with that last expression. We're gonna make a change in variable. All right. This is really an exercise in logic and notation, all right? And I'm only doing it because I can tell you that logic and notation can be a problem in this subject, all right? Finding your way through the notation, and making sure you're saying the right thing, and not saying the wrong thing can be a challenge. I have seen generations of students, very smart students, you know, get exactly balled up over these points. So that's why I'm doing them, and let's hope I can get through correctly.

I'm gonna make a change of variable. I'm gonna let U = -T, all right? That's, sort of, the obvious thing to do here. I want to get rid of this F of -T there, so I'm gonna make a change of variable in the integral, U = -T. So DU = -DT, okay? So what happens to the integral? The integral from -8 to 8 of E to the -2p I ST, F of -T, DT, becomes – let's do one step at a time, all right?

This is the integral of E to the -2p I ST, so T = -U. So it's -2p I S -U, F of U, and DT -DU, all right? Then you have to change the limitave integration also, all right? So if I let U = -T, and if T goes from -8 to 8, then U goes from 8 to -8, all right? So U goes from +8 to -8. Once again, if T goes from -8 to +8, then -T goes from +8 to -8.

But so if that's the integral, either the 2p I SU, F of U, DU, and there was an extra minus sign out front, so it's minus the integral from 8 to -8. That switches the limits of integration. That becomes the integral for -8 to 8, okay? What remains, dear friends? What is that integral? You recognize that as the inverse Fourier Transform of F evaluated at S, okay?

What is the conclusion? The Fourier Transform of the reverse signal is also the inverse Fourier Transform. If I don't write the variable in there, all right? The Fourier Transform of F reversed is the inverse Fourier Transform, all right? That's another duality theorem. Now, I want to combine this with the earlier one that I had – where was the earlier one I had?

Was the inverse Fourier Transform of F – what inverse Fourier Transform of F was the Fourier Transform of F reversed. That's this one over here, all right? That's the first duality statement that I had. If I combine this one with this one, then I get a very nice statement. I get that the Fourier Transform of F reversed is the inverse Fourier Transform of F. That's the Fourier Transform of F reversed.

So just look at this one and this one. This is my favorite statement, actually, of the duality statements 'cause it's the easiest one for me to remember.

It says that the Fourier Transform of the reverse signal is the reversal of the Fourier Transform. In words, it's so easy, and it's so nice. The Fourier Transform of the reverse signal is the reverse of the Fourier Transform, all right? That's another statement of duality, all right?

There are no inverses in there. I short circuited by putting the inverse in the middle. The Fourier Transform of F reversed is the inverse Fourier Transform of F. The Fourier inverse Transform of F is the Fourier Transform of the reverse signal. See, I can say all these words fast enough, of course I say everything fast, but I can say all these things because I've been through it a couple times, all right? And because I am not afraid not to write my variables.

So you have this statement. That's another statement of duality. So, really, there's this statement. There is the first statement that I have, the Fourier Transform of F reversed is the inverse Fourier Transform of F, all right? We use this statement to get this statement.

There is one other formula – they're all basically the same. There's one other formula that people often lump into, sort of, duality statements, and that is if you take the Fourier Transform of the Fourier Transform. Now, watch this; this is cute. One more, and that is the Fourier Transform of the Fourier Transform of F, okay? Now, you can derive this a number of ways. I think I will only give you the result of this thing and not give a derivation, but the result is you can take the Fourier Transform of the Fourier Transform and get back the reverse – you don't get back the function; you get back the reverse signal. You get back F reversed, okay?

So you can derive this; you derive this. You can derive it as a constant 'cause of what we've already shown over there. You can try to do it from scratch, just with the definition of the integral; don't worry about convergence or anything like that, just see what happens if you plug it into the definition.

And, again, people have a hard time with this because they're wedded to this interpretation of the Fourier Transform as taking you from the time domain to the frequency domain. That's not that that interpretation is wrong. I mean, in many cases you want to think in terms of two domains. You want to think in terms of times and frequency, but not always, all right, not always.

Sometimes you just want to think of the Fourier Transform as an operator taking one function to another function. I have to plug in a variable in order to write down the formula, but it doesn't matter what I call the variable. It doesn't matter how I interpret the variable, all right? It's just a statement. It's just a mathematical statement of how the thing is defined, all right?

So, say, if you think that way, then you have no trouble – it's one thing to derive this formula, but there's certainly no inconsistency here. There's nothing here that shakes your faith in the world, or shakes your faith in the distinction between the time domain and the frequency domain, you know, screw that. Excuse me. All right, it's just a mathematical statement.

Now, these are very useful, all right? These come up a fair amount, and I'll give you one example; I'll give you a quick application, all right? Quick application, this application is actually finding Fourier Transforms, all right? So, for example, let's find the Fourier Transform of the sync function, all right? We found the Fourier Transform of a rectangle function is equal to the sync function. We haven't found the Fourier Transform of the sync function.

So we know – do it over here. Now, I do this, actually, one way in the notes; let me do it a slightly different way here, all right? We know the Fourier Transform of a rectangle function is the sync function, all right? So the Fourier Transform of the sync function is the Fourier Transform of the Fourier Transform of a rectangle function, all right?

Fourier Transform of the rec – that's the first calculation we did. Fourier Transform of the rec function is the sync function. Fourier Transform of the sync function is the Fourier Transform of the Fourier Transform of the rec function, but the Fourier Transform of the Fourier Transform of the rec function is, according to this formula, the reversal of the rec function, but the rec function is even, all right? The rectangle function is an even function. So this is the rec function again. Lighting fast, we have derived that formula.

Now, you would be hard pressed, all right, to show – let me just indicate here what the issue is. If I were to say to you, "Find the Fourier Transform of the sync function." And you were to say to me, "I have only one recourse, and that is to the definition." All right? You would have to write down the Fourier Transform of the sync function at S is the integral from -8 to 8, E to the -2p I, ST, sign of pS ÷ pS – Oh, it's not a pT ÷ pT, DT.

Good luck. All right? Good luck. There are real serious issues of convergence here for this integral. I mean, and trying to evaluate this, and imagine, this evaluates to, if we believe this, the function which is 1 from -½ to ½ and 0 elsewhere, and – really? No. Oh, kidding? All right? You would be hard pressed, but the duality makes it easy, all right?

Now, like I said, this works because the rigor police are off duty, all right? This works because I'm assuming that there's actually no problem with the definitions, with the formulas, and so on. That all the integrals work and so on. Takes a lot of – it actually takes some work to do this, all right? I mean, it is true. This statement is true, that the Fourier Transform of the sync function is the rectangle function, but actually to do that completely rigorously requires a certain amount of effort which we're not gonna do actually. I mean, we're gonna do it but, sort of, by indirection.

But, nevertheless, with the rigor police off duty, and it's okay, you find very simply how it works, and you find a very similar – the same sort of argument will give you what the Fourier Transform of the sync² is the triangle function, same duality argument. Same duality argument gives that the Fourier Transform of sync² is the triangle function, okay? Same thing. All right.

I tell you what. I usually keep you overtime. I think today, maybe you'll even get out on time because, god forbid, we get out early. Next time what I want to do is I want to continue the discussion of general properties. I want to talk about the so-called stretch theorem, the shift theorem, and talk about convolution next time. So I want to move ahead a little bit, and make a little bit of a push to get some of the more general properties of this, okay? See you on Wednesday.

[End of Audio]

Duration: 48 minutes