# Week 3: Adaptive smoothing (cont.)

# Adaptive approaches: moving averages (linear filters)

$$\widehat{Tr}_t = \sum_{i=-m}^{m} w_i y_{t-i},$$

where $\sum_{i=-m}^{m} w_i = 1$ are weights

What did you read about

- derivation of $w_i$ under local polynomial trend of order $r$

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This week:

- criterion for selection of $r$
- how to compute the edges
- other linear filters:
  - simple $w_i = \frac{1}{2m+1}$
  - filter for seasonal data $\rightsquigarrow$ centered moving average
  - robust approach: take the median instead of the mean

See some R examples.

# Exponential smoothing

Assumption

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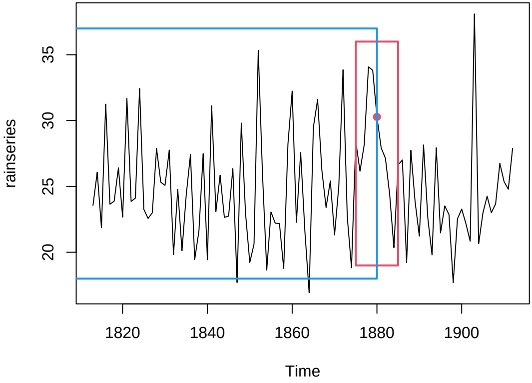
### Linear filters

- Aim: series decomposition (trend elimination)
- $\widehat{Tr}_t$ computed from $Y_{t-m}, \ldots, Y_{t+m}$

### Exponential smoothing

- Aim: **forecast** of future values
- $\widehat{L}_t$ computed only from the past $Y_{t-i}$

# Exponential smoothing



# Exponential smoothing: motivation

Series $Y_1, \ldots, Y_n$, locally constant

- Naive forecasts of future $Y_{n+h}$

  $$\widehat{Y}_{n+h}(n) = \frac{1}{n} \sum_{t=1}^{n} Y_t$$

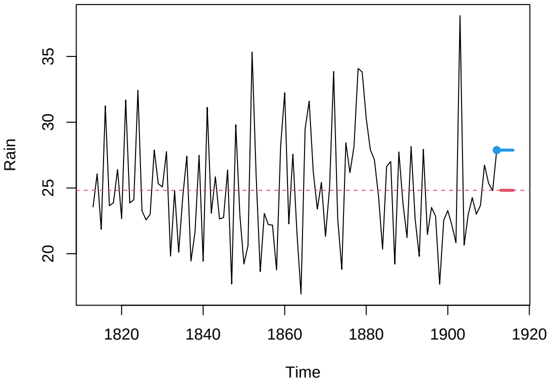
  gives the same weight to all observations. Or

  $$\widehat{Y}_{n+h}(n) = Y_n$$

  gives weight 1 to the last observation
- we typically want something in between

  $$\widehat{Y}_{n+h}(n) = \widehat{L}_n,$$

  where $\widehat{L}$ gives more weight to recent observations, but at the same time takes into account all observations



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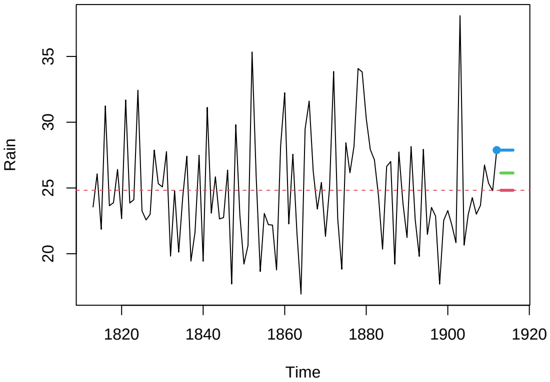
Time



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# Idea of exponential smoothing

- Compute

$$\widehat{L}_t = \widehat{Y}_{t+1}(t) = \sum_{i=0}^{\infty} w_i Y_{t-i}$$

as a weighted average of $Y_t, Y_{t-1}, \dots$ with geometrically decaying weights $w_i = const \cdot \beta^i \quad 0 < \beta < 1$.

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- Then

$$\begin{aligned}
\widehat{L}_t &= \alpha Y_t + \alpha(1-\alpha) Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \ldots \\
&= \alpha Y_t + (1-\alpha) \underbrace{[\alpha Y_{t-1} + \alpha(1-\alpha) Y_{t-2} + \ldots]}_{\widehat{L}_{t-1}} \\
&= \alpha Y_t + (1-\alpha)\widehat{L}_{t-1}
\end{aligned}$$

a recursive formula

# Another point of view

Let say we want to find $\theta$ such that

$$\min_{\theta} \sum_{j=0}^{\infty} (Y_{t-j} - \theta)^2 w_j.$$

The solution is

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Also see

$$\widehat{Y}_{t+1}(t) = \widehat{L}_t = \widehat{L}_{t-1} + \alpha(Y_t - \widehat{L}_{t-1}) = \widehat{L}_{t-1} + \alpha \underbrace{(Y_t - \widehat{Y}_t(t-1))}_{e_t}$$

$e_t$ one-step-ahead error

# Practical issues

Recall $\widehat{L}_t = \widehat{Y}_{t+1}(t)$ and $\widehat{L}_t = \alpha Y_t + (1-\alpha)\widehat{L}_{t-1} \rightsquigarrow$

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\widehat{L}_1 &= \alpha Y_1 + (1-\alpha)\widehat{L}_0, \\
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&\vdots \\
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and $\widehat{L}_n$ will be used for predictions

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See R examples

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- e.g. $\widehat{L}_0 = \frac{1}{K}\sum_{t=1}^{K} Y_t$ for small $K$, e.g. $K = 6$
- $\alpha$ chosen to minimize sum of squared errors

$$SSE = \sum_{t=2}^{n} e_t^2 = \sum_{t=2}^{n}(Y_t - \widehat{Y}_t(t-1))^2 = \sum_{t=2}^{n}(Y_t - \widehat{L}_{t-1})^2$$

See R examples

# Prediction intervals

Recall that $e_t = Y_t - \widehat{L}_{t-1}$ and $\widehat{L}_t = \widehat{L}_{t-1} + \alpha e_t$. Assume that

$$Y_t = L_{t-1} + \varepsilon_t,$$
$$L_t = L_{t-1} + \alpha \varepsilon_t.$$

So called state space model.

Prediction intervals

- can be constructed under **normality** and **independence** assumptions for $\varepsilon_t$
- modern approach: use bootstrap

# Double exponential smoothing

Locally linear trend

$$Tr_{t+j} = L_t + B_t \cdot j$$

$L_t$ level, $B_t$ slope

Then

$$\widehat{Y}_{t+h}(t) = \widehat{L}_t + \widehat{B}_t h$$

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Basic idea:

$$\min_{\beta_{0t},\beta_{1t}} \sum_{j=0}^{\infty} [Y_{t-j} - \beta_{0t} - \beta_{1t}(-j)]^2 \cdot (1-\alpha)^j$$

$\rightsquigarrow$ recursive formulas (see the book)

More general approach: Holt's linear trend method (next week)