Quasialgebras, Lattices and Matrices

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References

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A G-graded algebra $A = \oplus_{g \in G} A_g$ is a quasialgebra if,
$(ab)c = \phi(\bar{a}, \bar{b}, \bar{c})a(bc), \forall_{a \in A_{\bar{a}}, b \in A_{\bar{b}}, c \in A_{\bar{c}}}$,
for any invertible group cocycle $\phi : G \times G \times G \to K^*$,

that is,

$\forall x, y, z, t \in G$,
$\phi(x, y, z)\phi(y, z, t) = \frac{\phi(xy,z,t)\phi(x,y,zt)}{\phi(x,yz,t)}$
$\phi(x, e, y) = 1$.

($\Rightarrow A_0$ is an associative algebra and A_g is an A_0- bimodule)

(A G-graded quasialgebra is an algebra in the monoidal category of G-graded vector spaces)

Examples of quasiassociative algebras are $K_F G$ algebras: $K_F G$ is the same vector space as the group algebra KG but with a different product $a.b = F(a,b)ab, \forall_{a,b \in G}$, where F is any 2-cochain on G.
Some examples of $K_F G$ algebras are Cayley algebras as octonions and Clifford algebras as quaternions or complex numbers, ...

A algebra with identity over K

σ involutive anti-automorphism]

α non zero element of K

$\bar{A} = A + vA$ is an algebra with identity for the multiplication defined by

$$(a + vb)(c + vd) = (ac + \alpha d\sigma(b)) + v(\sigma(a)d + cb)$$

endowed with an involutive automorphism

$$\bar{\sigma}(a + vb) = \sigma(a) - vb$$

We say that $\bar{A}$ is obtained from A by Cayley-Dickson process.

$$\mathbb{R} \xrightarrow{-1} \mathbb{C} \xrightarrow{-1} \text{Quaternions} \xrightarrow{-1} \text{Octonions}$$

1844 - John Graves (Arthur Cayley) Algebra of Octaves (Octonions)

	1	a	b	c	x	y	z	t
1	1	a	b	c	x	y	z	t
a	a	-1	$-c$	b	$-y$	x	t	$-z$
b	b	c	-1	$-a$	$-z$	$-t$	x	y
c	c	$-b$	a	-1	$-t$	z	$-y$	x
x	x	y	z	t	-1	$-a$	$-b$	$-c$
y	y	$-x$	t	$-z$	a	-1	c	$-b$
z	z	$-t$	$-x$	y	b	$-c$	-1	a
t	t	z	$-y$	$-x$	c	b	$-a$	-1

$$K \xrightarrow{\alpha} K[a] \xrightarrow{\beta} \text{Generalized quaternion algebra} \xrightarrow{\gamma} \text{Cayley algebra}$$

	1	a	b	c	x	y	z	t
1	1	a	b	c	x	y	z	t
a	a	$\alpha 1$	$-c$	$-\alpha b$	$-y$	$-\alpha x$	t	αz
b	b	c	$\beta 1$	βa	$-z$	$-t$	$-\beta x$	$-\beta y$
c	c	αb	$-\beta a$	$-\alpha\beta 1$	$-t$	$-\alpha z$	βy	$\alpha\beta x$
x	x	y	z	t	$\gamma 1$	γa	γb	γc
y	y	αx	t	αz	$-\gamma a$	$-\alpha\gamma 1$	$-\gamma c$	$-\alpha\gamma b$
z	z	$-t$	βx	$-\beta y$	$-\gamma b$	γc	$-\beta\gamma 1$	$\beta\gamma a$
t	t	$-\alpha z$	βy	$-\alpha\beta x$	$-\gamma c$	$\alpha\gamma b$	$-\beta\gamma a$	$\alpha\beta\gamma 1$

Let G be a finite abelian group, F a cochain on it (so $k_F G$ is a G-graded quasialgebra). For any $s : G \to k^*$ with $s(e) = 1$ we define $\bar{G} = G \times Z_2$ and on it the cochain $\bar{F}$ and function $\bar{s}$,
$\bar{F}(x, y) = F(x, y), \bar{F}(x, vy) = s(x)F(x, y),$
$\bar{F}(vx, y) = F(y, x), \bar{F}(vx, vy) = \alpha s(x)F(y, x),$
$\bar{s}(x) = s(x), \bar{s}(vx) = -1$ for all $x, y \in G$.
Here $x \equiv (x, e)$ and $vx \equiv (x, v)$ denote elements of $\bar{G}$, where $Z_2 = \{e, v\}$ with product $v^2 = e$.
If $\sigma(x) = s(x)x$ is a strong involution, then $k_{\bar{F}}\bar{G}$ is the algebra obtained from Cayley-Dickson process applied to $k_F G$.

If $G = (Z_2)^n$ and $F = (-1)^f$ then the standard Cayley-Dickson process has $\bar{G} = (Z_2)^{n+1}$ and $\bar{F} = (-1)^{\bar{f}}$.
We use a vector notation $\vec{x} = (x_1, \cdots, x_n) \in (Z_2)^n$ where $x_i \in \{0, 1\}$ (and the group Z_2 is now written additively).
Then, $\bar{f}((\vec{x}, x_{n+1}), (\vec{y}, y_{n+1})) =$
$f(\vec{x}, \vec{y})(1 - x_{n+1}) + f(\vec{y}, \vec{x})x_{n+1} + y_{n+1}f(\vec{x}, \vec{x}) + x_{n+1}y_{n+1}.$

Complex number algebra

$G = Z_2, f(x, y) = xy, x, y \in Z_2$ where we identify G as the additive group Z_2 but also make use of its product.

Quaternion algebra

$\bar{G} = Z_2 \times Z_2, \bar{f}(\vec{x}, \vec{y}) = x_1 y_1 + (x_1 + x_2) y_2$ where $\vec{x} = (x_1, x_2) \in \bar{G}$ is a vector notation.

Octonion algebra

$\bar{\bar{G}} = Z_2 \times Z_2 \times Z_2, \bar{\bar{f}}(\vec{x}, \vec{y}) = \sum_{i \leq j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3.$

16-onion algebra

$\bar{\bar{\bar{G}}} = Z_2 \times Z_2 \times Z_2 \times Z_2$ and $\bar{\bar{\bar{f}}}(\vec{x}, \vec{y}) = \sum_{i \leq j} x_i y_j + \sum_{i \neq j \neq k \neq i} x_i x_j y_k + \sum_{\text{distinct } i,j,k,l} x_i x_j y_k y_l + \sum_{i \neq j \neq k \neq i} x_i y_j y_k x_4.$

Let $\mathbf{q}$ be a nondegenerate quadratic form on a vector space V over k of dimension n. It is known that there is an orthogonal basis $\{e_1, \cdots, e_n\}$, say, of V such that $\mathbf{q}(e_i) = q_i$ for some $q_i \neq 0$. The Clifford algebra $C(V, \mathbf{q})$, is the associative algebra generated by 1 and $\{e_i\}$ with the relations $e_i^2 = q_i.1, e_ie_j + e_je_i = 0, \forall i \neq j$. The dimension of $C(V, \mathbf{q})$ is 2^n and it has a canonical basis $\{e_{i_1} \cdots e_{i_p} | \, 1 \leq i_1 < i_2 \cdots < i_p \leq n\}$

The algebra $k_F Z_2^n$ can be identified with $C(V, \mathbf{q})$, where $F \in Z^2(G, k)$ is defined by $F(x, y) = (-1)^{\sum_{j<i} x_i y_j} \prod_{i=1}^n q_i^{x_i y_i}$ where $x = (x_1, \cdots x_n) \in Z_2^n$.

A algebra with identity over K

σ involutive automorphism

α non zero element of K

$\bar{A} = A + Av$ is an algebra with identity for the multiplication defined by

$$(a + bv)(c + dv) = (ac + \alpha b\sigma(d)) + (ad + b\sigma(c))v$$

endowed with an involutive automorphism

$$\bar{\sigma}(a + bv) = \sigma(a) - \sigma(b)v$$

We say that $\bar{A}$ is obtained from A by Clifford process.

Let G be a finite Abelian group and F a cochain as above. So $k_F G$ is a G-graded quasialgebra and for any $s : G \to k^*$ with $s(e) = 1$ and any $q \in k^*$, define $\bar{G} = G \times Z_2$ and
$\bar{F}(x, yv) = F(x, y) = \bar{F}(x, y),$
$\bar{F}(xv, y) = s(y)F(x, y), \bar{F}(xv, yv) = qs(y)F(x, y),$
$\bar{s}(x) = s(x), \bar{s}(xv) = -s(x)$ for all $x, y \in G$.
Here $x \equiv (x, e)$ and $xv \equiv (x, \eta)$ where η with $\eta^2 = e$ is the generator of the Z_2. If $\sigma(e_x) = s(x)e_x$ is an involutive automorphism then $k_{\bar{F}}\bar{G}$ is the Clifford process applied to $k_F G$.

We now look at the case where F and s are of the form $F(x,y)=(-1)^{f(x,y)}, s(x)=(-1)^{\xi(x)}, q=(-1)^{\zeta}$ for some Z_2-valued functions f, ξ and $\zeta \in Z_2$. We also suppose that $G=Z_2^n$ and use a vector notation.
For G, F, s based on Z_2, the generalised Clifford process yields the same form with $G=Z_2^{n+1}$ and $\bar{f}((x,x_{n+1}),(y,y_{n+1}))$-
$=f(x,y)+(y_{n+1}\zeta+\xi(y))x_{n+1}, \bar{\xi}(x,x_{n+1})=\xi(x)+x_{n+1}$.

Starting with $C(r,s)$, the Clifford process with $q = 1$ yields $C(r+1,s)$ and with $q = -1$ gives $C(r,s+1)$. Hence any $C(m,n)$ with $m \geq r, n \geq s$ can be obtained from $C(r,s)$ by successive applications of the Clifford process.

Starting with $C(0,0) = k$ and iterating the Clifford process with a choice of $q_i = (-1)^{\zeta_i}$ at each step, we arrive at the standard $C(V,\mathbf{q})$ and the standard automorphism $\sigma(e_x) = (-1)^{\rho(x)} e_x$ $(\rho(x) = \sum x_i)$.

Lattices in $\mathbb{R}^n$

An n-dimensional lattice in $\mathbb{R}^n$ is a set of points of the form $\Omega = Z\omega_1 + \cdots + Z\omega_n$ where $\omega_1, \ldots, \omega_n$ are some $\mathbb{R}$-linear independent vectors from $\mathbb{R}^n$. (A priori, such a lattice is only endowed with the algebraic structure of a Z-module, that means, if $\omega, \eta \in \Omega$, then $\omega \pm \eta \in \Omega$ and $\alpha\Omega \subseteq \Omega$ for any $\alpha \in Z$.)

If one defines a further multiplication operation on the underlying vector space $\mathbb{R}^n$, then special lattices,called *lattices with multiplication*, have the additional property that there are also non integers elements a such that $a \cdot \Omega \subset \Omega$. Furthermore, we say that the lattice Ω is *closed under multiplication* if for all $\omega, \eta \in \Omega$ holds $\omega \cdot \eta \in \Omega$.

If we consider a group G with n elements, say $G = \{g_1, \ldots, g_n\}$, then we can identify each element $a_1 g_1 + \ldots + a_n g_n, a_i \in \mathbb{R}$ of the algebra $\mathbb{R}_F G$ with the element $(a_1, \ldots, a_n) \in \mathbb{R}^n$. With this identification the multiplication defined in $\mathbb{R}_F G$ will introduce a special multiplication on $\mathbb{R}^n$. This is called *the multiplication of $\mathbb{R}^n$ induced by the group G using the cochain F*. In this case we say that $\mathbb{R}^n$ is embedded in $\mathbb{R}_F G$.

The simplest non-trivial explicit examples are lattices in the plane with complex multiplication that are extremely well studied (Fueter,Lang,Schoeneberg).

A two-dimensional lattice (in $\mathbb{R}^2$) in the normalized form $Z + Z\tau$ ($\Im(\tau) > 0$) has $\mathbb{R}_F Z_2$ (induced by the complex algebra or by Z_2 group algebra) , if and only if $\tau \in Q[e_1\sqrt{D}]$, where D is a positive square-free integer. (Square-free means that no prime number appears more than once in the prime factorization.) In the case where $\tau \in Z[e_1\sqrt{D}]$, one even has $\omega \cdot \eta \in \Omega$ for all $\omega, \eta \in \Omega$.

Let's identify $\mathbb{R}^{2^n}$ with the Clifford algebra $Cl_{0,n}$. Let $\Omega = Z\omega_1 + \cdots + Z\omega_{2^n}$ be a lattice where the generators w_i ($i = 1, \ldots, 2^n$) have the form

$$\omega_i = \omega_0^{(i)} + \sum_{j=1}^{n} \omega_j^{(i)} \sqrt{D_j} e_j + \sum_{j,k \in 1,\ldots,n, i<j} \omega_{jk}^{(i)} \sqrt{D_j D_k} e_j e_k + \cdots +$$

$$\omega_{12\ldots n}^{(i)} \sqrt{D_1 D_2 \cdots D_n} e_{12\ldots n}.$$

If each $\omega_A^{(i)}$ ($A \subset P(1, 2, \ldots, n)$) is an integer and $D_1, \ldots D_n$ are mutually distinct positive square-free integers, then Ω has a non-trivial $Cl_{0,n}$ multiplication. Here again one can show that the class of 2^n-dimensional lattices that have Clifford multiplication are exactly those whose real components of the generators stem up to conjugation from the multiquadratic number fields $Q[\sqrt{D_1}, \cdots \sqrt{D_n}]$.

Let $k \in N$ and $n_1, \ldots, n_k, m_1, \ldots m_k$ be some positive integers. Let $D_1^{(n_j)}, \ldots, D_{m_j}^{(n_j)}$ be n_j-power free positive integers for all $j = 1, \ldots, k$. Suppose that $F_1, \ldots, F_k$ are integer valued cochains. Lattices with generators whose components stem from the algebraic field

$$Q\left[\sqrt[n_1]{D_1^{(n_1)}}, \ldots, \sqrt[n_1]{D_{m_1}^{(n_1)}}, \ldots, \sqrt[n_k]{D_1^{(n_k)}}, \ldots, \sqrt[n_k]{D_{m_k}^{(n_k)}}\right]$$

have an $\mathbb{R}_{F_1} Z_{n_1}^{m_1} \times \cdots \times \mathbb{R}_{F_k} Z_{n_k}^{m_k}$ multiplication.

Let G be a group with n elements and F be a cochain in G. Further, let $\omega_1, \ldots, \omega_n$ be some $\mathbb{R}$-linearly independent vectors from $\mathbb{R}^n$. Let

$$\Omega = Z\omega_1 + \cdots + Z\omega_n$$

be the associated lattice embedded in $\mathbb{R}_F G$. Let $\mathbb{R}_{\tilde{F}}\tilde{G}$ be the algebra obtained from the algebra $\mathbb{R}_F G$ by the Clifford (Cayley- Dickson) process.

Consider the lattice

$\tilde{\Omega} = Z(\omega_1, 0) + \cdots + Z(\omega_n, 0) + Z(0, \omega_1) + \cdots + Z(0, \omega_n)$

embedded in $\mathbb{R}_{\tilde{F}}\tilde{G}$.

Then we can establish that the lattice Ω is closed under multiplication induced by $\mathbb{R}_F G$ and stable under the involutive automorphism σ if and only if the lattice $\tilde{\Omega}$ is closed under multiplication induced by $_{\tilde{F}}\tilde{G}$ and stable under the involutive automorphism $\tilde{\sigma}$.

We can define lattices in $\mathbb{R}^{2^n}$ that are closed under the multiplication and the involutive automorphism of the Clifford algebras $Cl_{p,q}$ (for any $p, q \in N_0$ with $p + q = n$) and for Cayley algebras.

More general we can define a larger class of closed lattices in $\mathbb{R}^n$ closed for $\mathbb{R}_F^G$ multiplication:
Let $G = \{g_1, \cdots, g_n\}$ and F be a cochain with integer values,
Let $\Omega = Z\omega_1 + \cdots + Z\omega_n$ be a lattice in $\mathbb{R}^n$ embedded in the twisted group algebra $\mathbb{R}_F G$, where the generators are each of the form

$$\omega_i = a_1^{(i)} g_1 + a_2^{(i)} g_2 + \cdots a_n^{(i)} g_n$$

with $a_1^{(i)}, \ldots, a_n^{(i)}$ integers.

Let

$$A := \begin{pmatrix} a_1^{(1)} & a_2^{(1)} & \dots & a_n^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \dots & a_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{(n)} & a_2^{(n)} & \dots & a_n^{(n)} \end{pmatrix}$$

The lattice Ω is closed for the multiplication if det(A) is 1 or -1.

For each unimodular matrix A we have a closed lattice in $\mathbb{R}^n$.

Let's consider the isomorphism between the group algebra of Z_n and the set of circulant matrices of order n with real elements, $C^n(\mathbb{R})$.

Let $\Omega = ZX_1 + \cdots + ZX_n$ where $X_1, \cdots, X_n$ are in $C^n(\mathbb{R})$ each of the form, form

$$\omega_i = a_1^{(i)} I + a_2^{(i)} \Pi + \cdots a_n^{(i)} \Pi^{n-1}$$

with $a_1^{(i)}, \ldots, a_n^{(i)}$ integers.

The product of any elements of Ω is in Ω if the matrix

$$A := \begin{pmatrix} a_1^{(1)} & a_2^{(1)} & \ldots & a_n^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \ldots & a_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{(n)} & a_2^{(n)} & \ldots & a_n^{(n)} \end{pmatrix}$$

is unimodular.