Finite axiomatization of quasivarieties of relational structures

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Proof

Elements of M_k		Elements of $\mathbb{Z}_2^{n+6}$		
a_0		1100	$000\cdots000\cdots000$	0
a_1		0011	$000\cdots000\cdots000$	0
a'_0	$\mapsto$	1010	$000\cdots000\cdots000$	0
a'_1		0101	$000\cdots000\cdots000$	0
b	$\mapsto$	1111	$000\cdots000\cdots000$	0
c_0		0000	$100\cdots000\cdots000$	0
c_1		0000	$010\cdots000\cdots000$	0
$\cdots$			$\cdots\cdots\cdots\cdots\cdots$	
c_{k-1}	$\mapsto$	0000	$000\cdots100\cdots000$	0
c_{k+1}		0000	$000\cdots001\cdots000$	0
$\cdots$			$\cdots\cdots\cdots\cdots\cdots$	
c_n		0000	$000\cdots000\cdots001$	0
d_0		0011	$100\cdots000\cdots000$	0
d_1		0011	$110\cdots000\cdots000$	0
$\cdots$			$\cdots\cdots\cdots\cdots\cdots$	
d_{k-1}		0011	$111\cdots100\cdots000$	0
d_k	$\mapsto$	0011	$111\cdots110\cdots000$	0
d_{k+1}		0011	$111\cdots111\cdots000$	0
$\cdots$			$\cdots\cdots\cdots\cdots\cdots$	
d_n		0011	$111\cdots111\cdots111$	0
d'_0		0101	$100\cdots000\cdots000$	0
d'_1		0101	$110\cdots000\cdots000$	0
$\cdots$			$\cdots\cdots\cdots\cdots\cdots$	
d'_{k-1}		0101	$111\cdots100\cdots000$	0
d_k	$\mapsto$	0101	$111\cdots110\cdots000$	1
d'_{k+1}		0101	$111\cdots111\cdots000$	1
$\cdots$			$\cdots\cdots\cdots\cdots\cdots$	
d'_n		0101	$111\cdots111\cdots111$	1
e	$\mapsto$	1111	$111\cdots111\cdots111$	0

TABLE: The mapping $\jmath_k$. Elements of $\mathbb{Z}_2^{n+6}$ are represented as words over $\mathbb{Z}_2$. For the sake of clarity we divided these words into 3 segments of length 4, $n+1$ and 1 respectively. In the second segment $(k-1)$th, kth and $(k+1)$th digits are placed between dots.

Quasivarieties

Quasi-identities look like

$$(\forall \bar{x})\,[\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \to \varphi(\bar{x})],$$

where $\varphi_i(\bar{x})$, $\varphi(\bar{x})$ are atomic formulas.

Quasivarieties look like

$$\mathrm{Mod}(\text{quasi-identities}).$$

The smallest quasivariety containing a class $\mathcal{K}$ (generated by) equals

$$\mathrm{Q}(\mathcal{K}) = \mathrm{SPP_U}(\mathcal{K})$$

.

Quasivariety $\mathcal{Q}$ is finitely axiomatizable (finitely based) if $\mathcal{Q} = \mathrm{Mod}(\Sigma)$ for some finite set Σ of quasi-identities.

Forbidden substructures

Observation $\Downarrow$

Assume that $\mathcal{K}$ is a class of relational structures axiomatized by a finite set Φ of universal sentences. Let n be the maximal number of variables occurring in sentences from Φ. Then for each relational structure $\mathbf{M}$ we have

$$\mathbf{M} \in \mathcal{K} \quad \text{iff} \quad (\forall \mathbf{N} \leq \mathbf{M})\,[|N| \leq n \rightarrow \mathbf{N} \in \mathcal{K}]. \qquad (A_n)$$

Observation $\Uparrow$

Conversely, if the language of $\mathcal{K}$ is finite and there exists a finite n such that (A_n) holds for all $\mathbf{M}$, then $\mathcal{K}$ is finitely axiomatizable.

Meet of observations $\Updownarrow$

An universal class (quasivariety) $\mathcal{K}$ of relational structures in a finite language is finitely axiomatizable if and only if it admits a finite set of finite forbidden substructures.

Graphs

A graph is a relational structure with one binary symmetric and irreflexive relation.

Theorem (Nešetřil, Pultr '78)

Let $\mathcal{K}$ be a quasivariety generated by a finite number of finite graphs. Then $\mathcal{K}$ is finitely axiomatizable only in the following cases:

- $\mathcal{K} = \{$  $\}$;
- $\mathcal{K} = \{$  $, \bullet \}$;
- $\mathcal{K} =$ discrete graphs $\cup \{$  $\}$;
- $\mathcal{K} = \{$disjoint unions of $\bullet$ —— $\bullet$ and $\bullet\} \cup \{$  $\}$;
- $\mathcal{K} = \{$disjoint unions of complete bipartite graphs$\} \cup \{$  $\}$.

Antivarieties

Anitivariety is a $\mathsf{H}^{-1}\mathsf{S}$-closed elementary class or, equivalently, a class defined by anti-idetities.

$\mathsf{A}(\mathcal{K})$ = the smallest antivariety containing $\mathcal{K}$.

Fact

If $\mathcal{A}$ is an antivariety, then $\mathcal{A} \cup \{\text{loop}\}$ is a quasivariety. Moreover, $\mathcal{A}$ is finitely axiomatizable iff $\mathcal{A} \cup \{\text{loop}\}$ does.

Antivariety $\mathcal{A}$ admits a finite duality if there is a finite family of finite structures $\mathbf{O_1}, \ldots, \mathbf{O_n}$ such that

$$(\forall \mathbf{M})\ [\mathbf{M} \in \mathcal{A} \quad \text{iff} \quad \mathbf{O_1}, \ldots, \mathbf{O_n} \notin \mathsf{A}(\mathbf{M})].$$

Let $\mathsf{CSP}(\mathcal{K}) = \mathsf{A}(\mathcal{K})_{fin}$.

CSPs

Therem (Atserias, Larose, Loten, Rossman, Tardif '08)

Let $\mathbf{M}$ be a finite relational structure. TFAE

- $A(\mathbf{M}) \cup \{\text{loop}\}$ is finitely axiomatizable;
- $A(\mathbf{M})$ is finitely axiomatizable;
- $A(\mathbf{M})$ admits a finite duality;
- $\mathrm{CSP}(\mathbf{M})$ is finitely axiomatizable (relative to finite structures);
- $\mathrm{CSP}(\mathbf{M})$ admits a finite duality (relative to finite structures);
- $\mathrm{Core}(\mathbf{M})^{\mathbf{2}}$ dismantles to the diagonal.

Semigroups

The graph of an algebra $\mathbf{A} = (A, \Omega)$ is NOT a graph. It is the relational structure

$$\mathrm{G}(\mathbf{A}) = (\mathrm{A}, \{\mathrm{R}_\omega\}_{\omega \in \Omega}),$$

where

$$(a_0, \ldots, a_k) \in R_\omega \quad \text{iff} \quad \omega(a_0, \ldots, a_{k-1}) = a_n.$$

For a class $\mathcal{C}$ of algebras let $\mathrm{G}(\mathcal{C}) = \{\mathrm{G}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{C}\}$.

Theorem (Gornostaev, Stronkowski '09)

Let $\mathcal{C}$ be a class of semigroups possessing a nontrivial member with a neutral element. Then $\mathrm{QG}(\mathcal{C})$ is not finitely axiomatizable.

Corollary

Let $\mathcal{C}$ be a class of monoids or groups possessing a nontrivial member. Then $\mathrm{QG}(\mathcal{C})$ is not finitely axiomatizable.

Proof

Recall

Observation $\Downarrow$

Let $\mathcal{K}$ be a finitely axiomatizable quasivariety of relational structures. Then there is a finite n such that for each relational structure $\mathbf{M}$ we have

$$\mathbf{M} \in \mathcal{K} \quad \text{iff} \quad (\forall \mathbf{N} \leq \mathbf{M})\,[|N| \leq n \rightarrow \mathbf{N} \in \mathcal{K}].$$

Thus it is enough to construct for each n a model $\mathbf{M}$ such that

- $\mathbf{M} \notin \mathrm{QG}(\text{Semigroups})$,
- if $\mathbf{N} \leq \mathbf{M}$ and $|N| \leq n$, then $\mathbf{N} \in \mathrm{QG}(\mathcal{C})$.

Proof

We can do it easily with the aid of the quasi-identity

$$(\forall \bar{x}, \bar{x}', y, \bar{z}, \bar{u}, \bar{u}', v)\ \big[R(x_0, x_1, y) \wedge R(x_0', x_1', y)$$
$$\wedge\, R(x_1, z_0, u_0) \wedge R(u_0, z_1, u_1) \wedge \cdots \wedge R(u_{n-1}, z_n, u_n) \wedge R(x_0, u_n, v)$$
$$\wedge\, R(x_1', z_0, u_0') \wedge R(u_0', z_1, u_1') \wedge \cdots \wedge R(u_{n-1}', z_n, u_n') \ \rightarrow\ R(x_0', u_n', v)\big]$$

i.e. **M** is given by

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d_n		0011	$111\cdots111\cdots111$	0
d'_0		0101	$100\cdots000\cdots000$	0
d'_1		0101	$110\cdots000\cdots000$	0
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d_k	$\mapsto$	0101	$111\cdots110\cdots000$	1
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