Maltsev Conditions for Omitting Types

Jardafest
Charles University

Matt Valeriote

(joint work with M. Kozik, A. Krokhin, M. Maroti, R. Willard)

McMaster University

24 June 2010
Six Interesting Families

Quote from “The Structure of Finite Algebra”, by Hobby & McKenzie:

“Our theory reveals a sharp division of locally finite varieties of algebras into six interesting new families, each of which is characterized by the behaviour of congruences in the algebras.”

Goals of this talk:

- Describe these six families,
- Present various old and new characterizations of them,
- Show that some characterizations are simpler than expected and that some of them can not be significantly simplified.
Hobby and McKenzie have developed a notion of neighbourhood, or minimal set of a finite algebra. They show that the behaviour of minimal sets is limited to one of the following five types:

1. Unary
2. Affine
3. 2-element Boolean algebra
4. 2-element Lattice
5. 2-element Semi-lattice

**Definition**

- We say that a finite algebra $A$ omits a particular type if no neighbourhoods of that type occur in $A$.
- A variety $V$ omits a particular type if each finite member of it does.
Neighbourhoods

Definition

The type of $\alpha$ is equal to the type of any one of the $\alpha$-neighbourhoods.
Remark

There is a natural order on the five types, determined by the "richness" of the associated algebraic structure:

\[ 1 < 2 < 3 > 4 > 5 > 1 \]
Remark

With respect to the type ordering, there are six proper order ideals, and for each, Hobby and McKenzie define an associated family of locally finite varieties:

\[
\begin{align*}
\mathcal{M}_1 &= \text{omit } \{1\} \\
\mathcal{M}_2 &= \text{omit } \{1, 5\} \\
\mathcal{M}_3 &= \text{omit } \{1, 4, 5\} \\
\mathcal{M}_4 &= \text{omit } \{1, 2\} \\
\mathcal{M}_5 &= \text{omit } \{1, 2, 5\} \\
\mathcal{M}_6 &= \text{omit } \{1, 2, 4, 5\}
\end{align*}
\]
### Some Properties of the Six Families

<table>
<thead>
<tr>
<th>Name</th>
<th>Type Omitting Condition</th>
<th>Other Defining Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{M}_1$</td>
<td>${1}$</td>
<td>largest non-trivial idempotent Maltsev class</td>
</tr>
<tr>
<td>$\mathcal{M}_2$</td>
<td>${1, 5}$</td>
<td>equivalent to satisfying a non-trivial congruence identity</td>
</tr>
<tr>
<td>$\mathcal{M}_3$</td>
<td>${1, 4, 5}$</td>
<td>$n$-permutable varieties</td>
</tr>
<tr>
<td>$\mathcal{M}_4$</td>
<td>${1, 2}$</td>
<td>congruence meet semi-distributive varieties</td>
</tr>
<tr>
<td>$\mathcal{M}_5$</td>
<td>${1, 2, 5}$</td>
<td>congruence join semi-distributive varieties</td>
</tr>
<tr>
<td>$\mathcal{M}_6$</td>
<td>${1, 2, 4, 5}$</td>
<td>$n$-permutable and congruence join semi-distributive varieties</td>
</tr>
</tbody>
</table>
Remark

*Each of the six families can be defined in terms of idempotent Maltsev Conditions.*

Example

A locally finite variety $\mathcal{V}$ belongs to the class $\mathcal{M}_3$ if and only if for some $n > 0$ there are $\mathcal{V}$-terms $p_i(x, y, z)$, for $1 \leq i \leq n$ such that

\[
x \approx p_1(x, y, y),
\]

\[
p_i(x, x, y) \approx p_{i+1}(x, y, y) \text{ for each } i,
\]

\[
p_n(x, x, y) \approx y
\]

A locally finite variety $\mathcal{V}$ belongs to the class $\mathcal{M}_4$ if and only if for some $n > 0$ there are $\mathcal{V}$-terms $d_i(x, y, z)$ and $e_i(x, y, z)$, for $1 \leq i \leq n$ such that

\[
x \approx d_1(x, x, y),
\]
Some Special Terms

Definition

A term \( t(x_1, \ldots, x_n) \) of a variety \( \mathcal{V} \) is:

- **idempotent** if the equation \( t(x, x, \ldots, x) \approx x \) holds in \( \mathcal{V} \),
- a **Taylor term** if it is idempotent and for each \( 1 \leq i \leq n \), an equation in the variables \( \{x, y\} \) of the form \( t(a_1, \ldots, a_n) \approx t(b_1, \ldots, b_n) \) holds in \( \mathcal{V} \), where \( a_i = x \) and \( b_i = y \).
- a **weak near unanimity term** if it is idempotent and the equations
  \[
  t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, x, \ldots, x, y)
  \]
  hold in \( \mathcal{V} \),
- a **cyclic term** if it is idempotent and the equation
  \[
  t(x_1, x_2, \ldots, x_{n-1}, x_n) \approx t(x_2, x_3, \ldots, x_n, x_1)
  \]
  holds in \( \mathcal{V} \).
Theorem (Hobby, Maroti, McKenzie)

Let $\mathcal{V}$ be a locally finite variety. The following are equivalent:

- $\mathcal{V} \in \mathcal{M}_1$
- $\mathcal{V}$ omits the unary type
- $\mathcal{V}$ has a Taylor term
- $\mathcal{V}$ has a weak near unanimity term

Theorem (Barto, Kozik)

Let $\mathbb{A}$ be a finite algebra and let $\mathcal{V} = \text{HSP}(\mathbb{A})$. Then $\mathcal{V}$ omits the unary type if and only if for all prime numbers $p > |\mathbb{A}|$, $\mathbb{A}$ has a cyclic term of arity $p$. 
Siggers’ Result

Remarks

- For all $n > 0$ one can find a finite algebra $A_n$ that has a weak near unanimity term of arity $n$ but of no smaller arity.
- From this, it appears that the Maltsev condition for locally finite varieties that omit the unary type is not strong.
- but ...

Theorem (Siggers)

Let $\mathcal{V}$ be a locally finite variety. Then $\mathcal{V}$ omits the unary type if and only if it has a 6-ary idempotent term $t$ such that $\mathcal{V}$ satisfies the equations

\[
\begin{align*}
t(x, x, x, x, y, y) & \approx t(x, y, x, y, x, x) \\
t(y, y, x, x, x, x) & \approx t(x, x, y, x, y, x).
\end{align*}
\]
Siggers’ Result

Remark

Shortly after Siggers announced his result, Markovic and McKenzie observed that 4-ary versions of Siggers’ term exist. Here is one version:

Theorem

A locally finite variety $\mathcal{V}$ omits the unary type if and only if it has a 4-ary idempotent term operation $t$ that satisfies the identities:

\[ t(y, y, x, x) \approx t(x, y, y, x) \approx t(x, x, x, y). \]

Corollary

The class $\mathcal{M}_1$ is defined by a strong Maltsev condition.
A short proof

**Theorem**

Let $\mathbb{A}$ be a finite algebra such that $\mathcal{V} = \text{HSP}(\mathbb{A})$ omits the unary type. Then $\mathbb{A}$ has an idempotent term $t$ such that

$t(y, y, x, x) \approx t(x, y, y, x) \approx t(x, x, x, y)$.

**Proof.**

- Let $p$ be some prime number $> |A|$ of the form $5k + 3$ for some $k$,
- let $c(x_1, \ldots, x_p)$ be a cyclic term of $\mathbb{A}$ of arity $p$,
- Let $t(x, y, z, w) = c(x, x, \ldots, x, y, y, \ldots, y, z, z, \ldots z, w, w, \ldots w)$, where the variables
  - $x$ and $z$ occur $k + 1$ times,
  - $y$ occurs $k$ times and
  - $w$ occurs $2k + 1$ times.
- $c$ cyclic implies that $t$ satisfies the stated equations.
Remarks

- Recall that the class $\mathcal{M}_4$ consists of all locally finite varieties that omit the unary and affine types.
- It was noted earlier that this class is definable by a complicated Maltsev condition.

Theorem

A locally finite variety omits the unary and affine types if and only if it has 3-ary and 4-ary weak near unanimity terms $v(x, y, z)$ and $w(x, y, z, w)$ that satisfy the equation $v(y, x, x) \approx w(y, x, x, x)$.

Proof.

Uses results of McKenzie and Barto and Kozik on weak near unanimity terms, a result of Barto and Kozik on the constraint satisfaction problem, and a construction of weak near unanimity terms due to Kiss.
What about the other four families?

**Theorem**

*Of the classes of locally finite varieties \( \mathcal{M}_i \), \( 1 \leq i \leq 6 \), only \( \mathcal{M}_1 \) and \( \mathcal{M}_4 \) can be defined by strong Maltsev conditions.*

**Sketch of Proof**

- For each \( n \), we construct a finite idempotent algebra \( A_n \) such that \( \mathcal{V}_n = \text{HSP}(A_n) \) omits all types except the Boolean type (type 3).
- Thus \( \mathcal{V}_n \) belongs to all six families.
- Establish that if \( \Sigma \) is any strong Maltsev condition that is satisfied by \( \mathcal{V}_n \) for all \( n \), then the variety of semilattices also satisfies \( \Sigma \).
- Therefore none of the families that omit the semilattice type (type 5) can be defined by a strong Maltsev condition.
The algebras $\mathbb{A}_n$

**Definition**

Let $n > 0$ and $1 \leq i \leq n$.

- Let $\mathbb{A}[i, n]$ be the algebra with universe $\{0, 1\}$ and whose only basic operation is the $2n + 1$-ary operation $t_{(i,n)}$ defined by:

\[
t_{(i,n)}(x_0, x_1, \ldots, x_{2n-1}, x_{2n}) = x_0 \land (x_1 \land x_2) \land \cdots \land (x_{2i-3} \land x_{2i-2}) \land (\overline{x_{2i-1}} \lor x_{2i}).
\]

- Let $\mathbb{A}_n$ be the cartesian product $\prod_{i=1}^{n} \mathbb{A}[i, n]$ and let $\mathcal{V}_n$ be the variety generated by $\mathbb{A}_n$.$^a$

---

$a$This construction is based on one found in a recent paper by Carvalho, Dalmau, and Krokhin.
Can we do better?

- We’ve seen that the class $\mathcal{M}_1$ can be characterized by the existence of a 4-ary term. Is it possible that it could also be characterized by the existence of some kind of 3-ary term?
  - No, but
  - it can be characterized by the existence of two 3-ary idempotent terms $p(x, y, z)$ and $q(x, y, z)$ such that

$$p(x, x, y) \approx p(y, x, x) \approx q(x, y, y) \text{ and } p(x, y, x) \approx q(x, y, x).$$

- Something similar happens with $\mathcal{M}_4$, namely, it can be characterized by the existence of three 3-ary idempotent terms that satisfy certain equations. This was observed by M. Maroti and A. Janko.
Conclusions

- Finding “nice” Maltsev conditions for $\mathcal{M}_1$ and $\mathcal{M}_4$ has led to computationally more efficient algorithms to determine if a given finite algebra generates a variety that belongs to one of these classes.
- The study of these Maltsev classes has advanced work on the constraint satisfaction problem (and vice versa).
- In their new book, “The Shape of Congruence Lattices”, Kearnes and Kiss study in detail the extensions of the six families to the general case, i.e., the non-locally finite case.
- Question: Can the other four families be defined by better Maltsev conditions than the standard ones?
- Question: Are any other familiar Maltsev conditions equivalent to strong Maltsev conditions for locally finite varieties?