QCSPs for Temporal Relations

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ICAL 2010, Prague

Outline

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Constraint Satisfaction Problems

Primitive Positive Formula

Let τ be a signature of relational symbols.
Primitive positive formula over τ is of the form:

$$\exists x_1 \ldots \exists x_m \bigwedge_{i=1}^{n} R_i(v_1^i, \ldots, v_{k_i}^i),$$

where $R_i \in \tau$ for all $1 \leq i \leq n$.

CSP(Γ)

Γ is a τ-structure.
Instance: A primitive positive sentence ϕ over τ.
Question: Is ϕ true in Γ?

CSPs for temporal languages.

temporal language $\langle Q; R_1, \ldots, R_k \rangle$ — fo-definable over $\langle Q, < \rangle$:

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Scheduling with AND/OR precedence constraints.

$\Gamma_{ANDOR} = \langle \mathbb{Q}, \{\langle a, b, c \rangle \in \mathbb{Q}^3 | a > b \vee a > c\} \rangle$

Example: CSP-instance of $\mathbf{CSP}(\Gamma_{ANDOR})$
Instance: $(x > y \vee x > z) \wedge (z > y \vee z > x)$
Solution: $s(x) = 1; s(y) = 0; s(z) = 2$

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Ord-Horn relations

$x_1 = y_1 \wedge \ldots \wedge x_k = y_k \rightarrow z_1 R z_2$ where $R \in \{<, \leq, =\}$

Point Algebra

Polymorphisms of Temporal Relations.

Definition of a polymorphism

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If $\langle A, B, C \rangle, \langle a, b, c \rangle \in R_{ANDOR}$ and $A \geq a$, then $\langle \mathbf{min}(A, a), \mathbf{min}(B, b), \mathbf{min}(C, c) \rangle \in R_{ANDOR}$.

Understanding by classification. Bodirsky, Kára 2008

Theorem

Γ — *a relation FO-definable over* $\langle \mathbb{Q}, < \rangle$, *then*

- *either* Γ *is closed under* **min**, **dual-min** *(***max***)*, **mi**, **dual-mi**, **mx**, **dual-mx**, **ll**, **dual-ll**, **constant** *and* **CSP**(Γ) *is in P, or*
- **CSP**(Γ) *is NP-complete.*

$-f(-x_1, \ldots, -x_k)$ *is the dual of* $f(x_1, \ldots, x_k)$

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Example

- AND/OR constraints closed under **min**
- Ord-Horn relations closed under **ll**

Polymorphisms imply algorithms.

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Algorithm for min-closed temporal languages.

Input: An instance Φ of $\mathbf{CSP}(\Gamma)$.

Output: A solution s to Φ or false.

i:= 0

while $V(\Phi) \neq \emptyset$ do begin

S:= MaxMinSet(Φ)

If $S = \emptyset$, then return false

for each $x \in MaxMinSet(\Phi)$ do $s(x) := i$

$i = i + 1$

$\Phi = \exists S\ \Phi \wedge \bigwedge_{x_i \in S, x_j \notin S} x_i < x_j$

MaxMinSet is a maximal set of variables that may be set to the least value in some solution s to Φ.

Quantified Constraint Satisfaction Problems

Quantified Positive Formula

Let τ be a signature of relational symbols.
Quantified positive formula over τ is of the form:

$$Q_1 x_1 \dots Q_m x_m \bigwedge_{i=1}^{n} R_i(v_1^i, \dots, v_{k_i}^i),$$

where $Q_i \in \{\forall, \exists\}$, and $R_i \in \tau$ for all $1 \leq i \leq n$.

QCSP(Γ)

Γ is a τ-structure.
Instance: A quantified positive sentence ϕ over τ.
Question: Is ϕ true in Γ?

Quantified Characterization of Equality Languages

Bodirsky and Chen.

Γ — a structure FO-definable over $\langle \mathbb{Q}, = \rangle$. Then holds exactly one of the following.

- Negative languages. Relations of Γ may be defined as:
  $$\bigwedge_{i=1}^{n} (x_i = y_i) \wedge \bigwedge_{i=1}^{k} (z_1^i \neq v_1^i \vee \ldots \vee z_{k_i}^i \neq v_{k_i}^i),$$
  and then **QCSP**(Γ) is in LOGSPACE.
- Positive languages. Relations of Γ' may be defined as:
  $$\bigwedge_{i=1}^{n} (x_1^i = y_1^i \vee \ldots \vee x_{k_i}^i = y_{k_i}^i),$$
  and then **QCSP**(Γ) is NP-complete.
- In any other case, **QCSP**(Γ) is PSPACE-complete.

Positive Temporal Languages

Charatonik and W.

Γ — a structure positive definable over $\langle \mathbb{Q}, \leq \rangle$.
Then holds exactly one of the following.

1. Relations of Γ are definable as: $\bigwedge_{i=1}^{n} x_i = y_i$ and then **QCSP**(Γ) is in LOGSPACE.
2. Relations of Γ are definable as: $\bigwedge_{i=1}^{n} x_i \leq y_i$ and then **QCSP**(Γ) is NLOGSPACE-complete.
3. Definable by: $\bigwedge_{i=1}^{n}(x_{i_1} \leq x_{i_2} \vee \ldots \vee x_{i_1} \leq x_{i_k})$ and then **QCSP**(Γ) is P-complete.
4. $\bigwedge_{i=1}^{n}(x_{i_2} \leq x_{i_1} \vee \ldots \vee x_{i_k} \leq x_{i_1})$ — P-complete.
5. Positive equality languages — NP-complete.
6. The problem **QCSP**(Γ) is PSPACE-complete.

Our goal: 'quantified' analog of Bodirsky-Kára theorem

We are expecting a theorem of the form:

Theorem

Let Γ be a temporal language, then one of the following holds:

- *if Γ is closed under $\mathbf{Pol}_1$, then $\mathbf{QCSP}(\Gamma)$ is in P;*
- *else if Γ is closed under $\mathbf{Pol}_2$, then $\mathbf{QCSP}(\Gamma)$ is NP-complete;*
- *else if Γ is closed under $\mathbf{Pol}_3$, then $\mathbf{QCSP}(\Gamma)$ is PSPACE-complete;*

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Will this theorem be substantially different from csps classification?
will compare with classifications for **CSP** *and* **QCSP** *for relations over two element domain*

Collapsibility for **min** : $\{0,1\} \to \{0,1\}$.

Polymorphisms may simplify quantifier prefix. (Chen.)

Example

$\Psi := \exists x \forall v_1 \exists y \forall v_2 \exists z \Phi(x, v_1, y, v_2, z)$ is true iff

1. $\exists x \forall v_1 \exists y \exists z \Phi(x, v_1, y, 1, z)$
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1 and 2 and **min** give strategy for Ψ

$\exists x$	$\forall v_1$	$\exists y$	$\forall v_2$	$\exists z$
$\mathbf{min}(t'_x, s'_x)$	0	$\mathbf{min}(t''_y, s'_y)$	0	$\mathbf{min}(t''_z, s''_z)$
			1	$\mathbf{min}(t''_z, s'_z)$
	1	$\mathbf{min}(t'_y, s'_y)$	0	$\mathbf{min}(t'_z, s''_z)$
			1	$\mathbf{min}(t'_z, s'_z)$

Collapsibility tells the full story for QCSPs over $\{0, 1\}$

- **CSP**(Γ) is in P if Γ closed under **min**, **max**, **majority**, **minority**, **constant**.

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- classification for temporal **QCSP**s may need other polymorphisms than the one for temporal **CSP**s

Polynomially Generated Powers

collapsibility implies PGP. (Chen)

Definition

$\langle \Gamma, D \rangle$ has PGP if there is polynomial $p(n)$ such that for each n there is $X_n \subseteq D^n$ of size $|X_n| < p(n)$ generating $\langle D, \mathbf{IPol}(\Gamma) \rangle^n$.

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$\mathbf{min} : \{0,1\} \to \{0,1\}$ gives PGP

$$
\begin{array}{ccccc}
1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 0 \\
 & & \mathbf{min} & & \\
0/1 & 0/1 & 0/1 & \ldots & 0/1
\end{array}
$$

One more slide on PGP

We can use PGP to solve Π_2 *HORN-SAT* (closed under **min**) in P.

Example

$\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \; \Phi(x_1, \dots, x_n, y_1, \dots, y_m)$ is true iff

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- $\Phi(0, 0, 1, \dots, 1, \mathbf{min}(t_1, t'_1), \dots, \mathbf{min}(t_m, t'_m))$.

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- under **constant** (tractable CSP)
- **QCSP**$(\mathbb{Q}, x = y \vee z = v)$ is NP-complete
- $x = y \to z > v$ also has PGP

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- aiming at: complete classification for temporal quantified CSPs,
- it cannot be simply obtained from the classification for temporal CSPs,
- known methods for finite domain QCSPs are not directly applicable