Amalgamation for Quasi-Stone Algebras

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A generalisation of Stone algebras

Recall:

Definition

A *Stone algebra* is a pseudocomplemented distributive lattice $(L; \wedge, \vee, ^*, 0, 1)$ where for all $a \in L$

$$a^* \vee a^{**} = 1 \quad \text{(Stone identity)}.$$

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De Morgan Laws:

$$\begin{aligned} (a \vee b)' &= a' \wedge b' \\ (a \wedge b)' &= a' \vee b' \end{aligned}$$

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Definition (Sankappanavar and Sankappanavar, 1993)

An algebra $(L; \wedge, \vee, ', 0, 1)$ is a *quasi-Stone algebra* (QSA) if $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the unary operation $'$ satisfies the following conditions:

(QS1) $0' = 1$ and $1' = 0$,
(QS2) $(a \vee b)' = a' \wedge b'$ (the $\vee$-DeMorgan law),
(QS3) $(a \wedge b')' = a' \vee b''$ (the weak $\wedge$-DeMorgan law),
(QS4) $a \wedge a'' = a$,
(QS5) $a' \vee a'' = 1$ (Stone identity),

for all $a, b \in L$.

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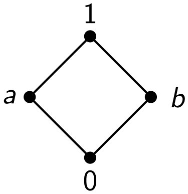
- Stone $\Longrightarrow$ QSA
- QSA + $(a \wedge b)' = a' \vee b' \Longrightarrow$ Stone

Examples



$0' = 1,\ a' = b,\ b' = a,\ 1' = 0$

Boolean algebra

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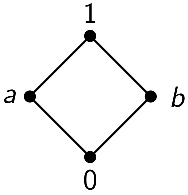

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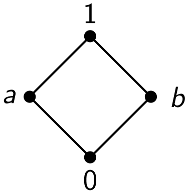

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"Special QSAs": $(L,')$

L any bounded distributive lattice, $a' = \begin{cases} 0 & \text{if } a \neq 0\,, \\ 1 & \text{if } a = 0\,. \end{cases}$

Duality for QSAs

Priestley Duality:

$(L; \wedge, \vee, 0, 1)$ bounded distributive lattices	$\longleftrightarrow$	$(X; \tau, \leq)$ compact, totally order-disconnected spaces
L	$\longrightarrow$	$D(L)$ prime filters
$E(X)$ clopen increasing sets	$\longleftarrow$	X
$f : L_1 \to L_2$ $\{0, 1\}$-lattice homomorphisms	$\longleftrightarrow$	$\varphi : X_2 \to X_1$ continuous, order preserving functions

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Definition

A *quasi-Stone space* (*QS-space*) is a pair $(X;\mathcal{E})$ such that X is a Priestley space and $\mathcal{E}$ is an equivalence relation on X satisfying the following three conditions:

(1) The equivalence classes of $\mathcal{E}$ are closed in X,

(2) $\mathcal{E}(U) \in E(X)$ for each $U \in E(X)$,

(3) $X \setminus \mathcal{E}(U) \in E(X)$ for each $U \in E(X)$.

$\mathcal{E}(Y) := \bigcup_{x\in Y}[x]_{\mathcal{E}}$, for each $Y \subseteq X$

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- Let $(X; \mathcal{E})$ be a QS-space. Then $(E(X);')$ is a QSA with $U' = X \setminus \mathcal{E}(U)$ for each $U \in E(X)$.

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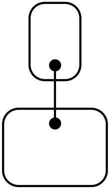
- Let $(L;')$ be a QSA. Then $(D(L); \mathcal{E})$ is a QS-space with $\mathcal{E} = \{(P, Q) \in D(L) \times D(L) \mid P \cap B(L) = Q \cap B(L)\}$. $B(L) = \{x' \mid x \in L\}$ is the skeleton of L.

Lemma

If $(X, \mathcal{E})$ is a QS-space and $x, y \in X$ are non-equivalent, then they are incomparable.



can *not* occur

Subvarieties of **QSA**

Finite subdirectly irreducible QSA's:

Q_0 denotes the one-element QSA and
$Q_{m,n} := (\widehat{B}_m \times B_n; \wedge, \vee, ', (0,0), (u_m, 1))$, where B_n denotes the Boolean lattice with n atoms, $\widehat{B}_m := B_m \oplus \{u_m\}$, and the operation $'$ is *special*, i.e.

$$(x,y)' = \begin{cases} (0,0) & \text{if } (x,y) \neq (0,0)\,, \\ (u_m, 1) & \text{if } (x,y) = (0,0)\,. \end{cases}$$

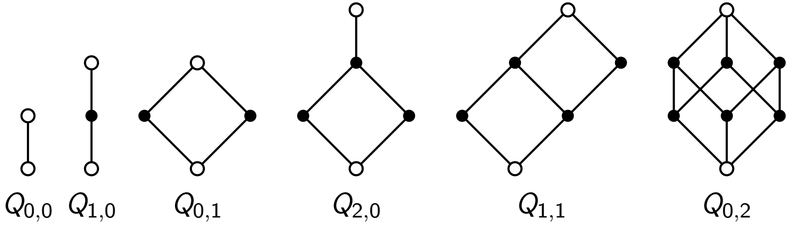
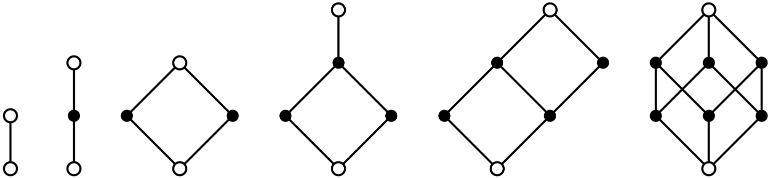


Figure: Some subdirectly irreducible QSAs.



$Q_{0,0}$ $Q_{1,0}$ $Q_{0,1}$ $Q_{2,0}$ $Q_{1,1}$ $Q_{0,2}$

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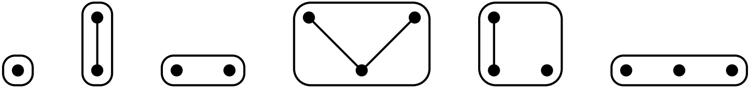

$D(Q_{0,0})$ $D(Q_{1,0})$ $D(Q_{0,1})$ $D(Q_{2,0})$ $D(Q_{1,1})$ $D(Q_{0,2})$

Figure: The corresponding QS-spaces.

$(\omega + 1)$-chain of subvarieties:

$$
\begin{array}{lllllllll}
\mathbb{V}(Q_0) & \subset & & & & & & & \\
\mathbb{V}(Q_{0,0}) & \subset & & & & & & & \\
\mathbb{V}(Q_{1,0}) & \subset & \mathbb{V}(Q_{0,1}) & \subset & & & & & \\
\mathbb{V}(Q_{2,0}) & \subset & \mathbb{V}(Q_{1,1}) & \subset & \mathbb{V}(Q_{0,2}) & \subset & & & \\
\mathbb{V}(Q_{3,0}) & \subset & \mathbb{V}(Q_{2,1}) & \subset & \mathbb{V}(Q_{1,2}) & \subset & \mathbb{V}(Q_{0,3}) & \subset & \\
\ldots & & & & & & & & \\
\ldots & & & & & & & & \subset \mathbf{QSA}
\end{array}
$$

Lemma

Let $L \in \mathbb{V}(Q_{0,n})$ with dual QS-space X. Then for all $x \in X$

- *either $[x]$ has at most n maximal elements*
- *or $[x]$ is an $(n+1)$-antichain.*

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- *either* $[x]$ *has at most* $m + n - 1$ *maximal elements*
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Special case:

If $L \in \mathbb{V}(Q_{1,n})$, then, for all $x \in X$, $[x]$ has at most $n+1$ maximal elements.

The Amalgamation Property

Definition

A class $\mathbb{K}$ of algebras has the *amalgamation property* (AP) if, for all A, B, $C \in \mathbb{K}$ and all embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$, there is $D \in \mathbb{K}$ and embeddings $k : B \hookrightarrow D$ and $l : C \hookrightarrow D$ such that $k \circ f = l \circ g$.

$$\begin{tikzcd}
 & D & \\
B \arrow[ur, hook, "k"] & & C \arrow[ul, hook', "l"'] \\
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An algebra A is an *amalgamation base* if, for all B, $C \in \mathbb{K}$ and $f : A \hookrightarrow B$, $g : A \hookrightarrow C$, there is $D \in \mathbb{K}$ as above.

- Variety of bounded distributive lattices: has AP

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$$\underbrace{\overset{\text{BAs}}{\mathbb{B}_0} \subset \overset{\text{Stone algebras}}{\mathbb{B}_1} \subset \mathbb{B}_2}_{\text{AP}} \subset \underbrace{\mathbb{B}_3 \subset \dots}_{\text{no AP}} \subset \underbrace{\mathbb{B}_\omega}_{\text{AP}}$$

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- Quasi-Stone algebras:

$$\underbrace{\overset{\text{BAs}}{\mathbb{V}(Q_{0,0})} \subset \overset{\text{Stone algebras}}{\mathbb{V}(Q_{1,0})}}_{\text{AP}} \subset \underbrace{\mathbb{V}(Q_{0,1}) \subset \dots \subset \textbf{QSA}}_{\text{no AP}}$$

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- We have $\hat{B}_0 \leq \hat{B}_1 \leq \hat{B}_2 \leq \hat{B}_3 \leq \ \ldots$, but $Q_{0,0} \leq Q_{1,0} \leq Q_{0,1} \not\leq Q_{2,0} \ \ldots$.

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- Congruence extension property holds for p-algebras but fails already in $\mathbb{V}(Q_{0,1})$.

Finite amalgamation bases

An algebra $A \in \mathbb{V}(Q_{m,n})$ is an *amalgamation base* of $\mathbb{V}(Q_{m,n})$ if, for all B, $C \in \mathbb{V}(Q_{m,n})$ and $f : A \hookrightarrow B$, $g : A \hookrightarrow C$, there is some $D \in \mathbb{V}(Q_{m,n})$ and QSA-embeddings $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that

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Equivalently,

$$\begin{tikzcd} & X_D \arrow[dl, "\lambda"'] \arrow[dr, "\rho"] & \\ X_B \arrow[dr, "\psi"'] & & X_C \arrow[dl, "\varphi"] \\ & X_A & \end{tikzcd}$$

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I.e. iff A is a direct product with factors $Q_{0,0}$ and $Q_{0,n}$.

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A finite $A \in \mathbb{V}(Q_{m,n})$ is an amalgamation base iff each equivalence class of X_A

- contains $m+n$ maximal elements and each non-maximal element is covered by exactly m maximal elements.

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