On lattices with a compact top congruence

Gillibert

Here

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- Equivalently $\operatorname{Con}_{\mathrm{c}} f$ is an isomorphism, where $f\colon K \hookrightarrow L$ is the inclusion map.

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- Let $\mathcal{V}$ be a non-distributive variety of lattices. The lattice $F_{\mathcal{V}}(\aleph_1)$ has no congruence-permutable congruence-preserving extension (G., Wehrung, 2009).

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- Morphisms in $\mathcal{V}^{\mathrm{b}}$ are morphisms of lattices $f \colon K \to L$ such that $(\operatorname{Con}_{\mathrm{c}} f)(\mathbf{1}_K) = \mathbf{1}_L$.

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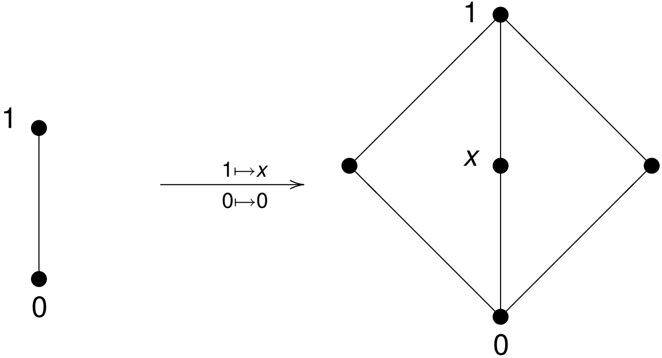

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- Here L is *condensate* of the arrow f.

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- Moreover if any of those assertion is true, can the construction be made functorial ?

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- (3) $\implies$ (1) is obvious.

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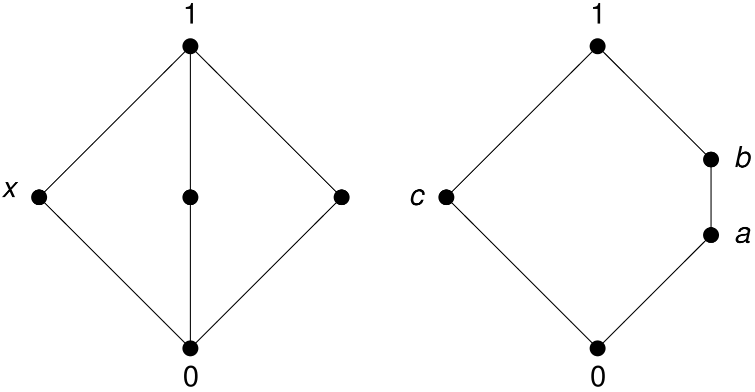
FIG.: The lattices M_3 and N_5.



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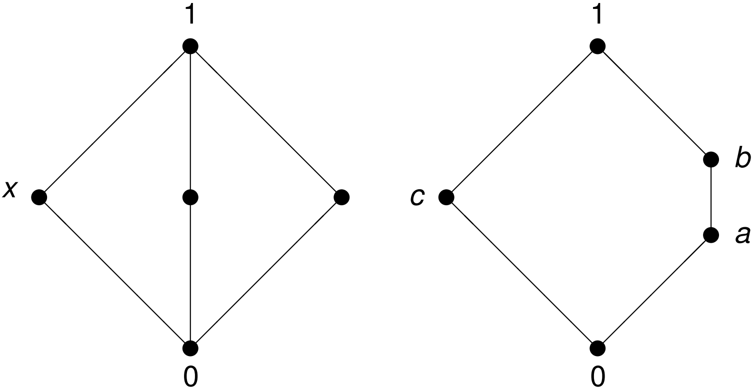

- Notice that N_5 satisfies (2), but M_3 fails (2).

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- Similarly y is the largest element of L.
- Hence L is bounded.

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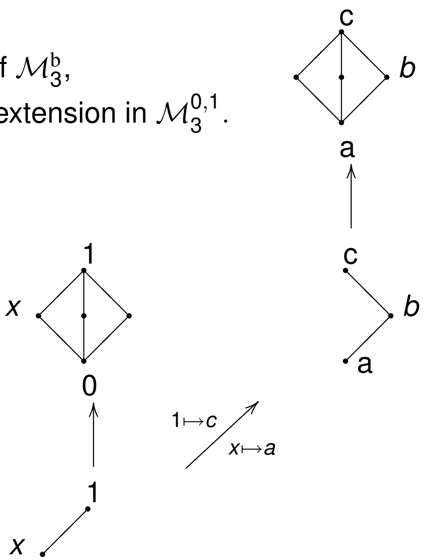

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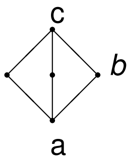

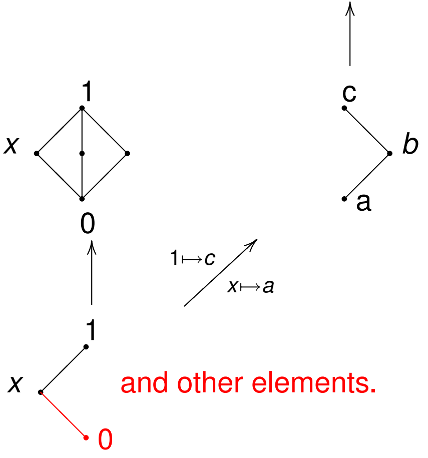

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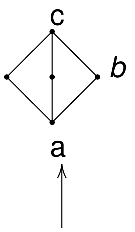
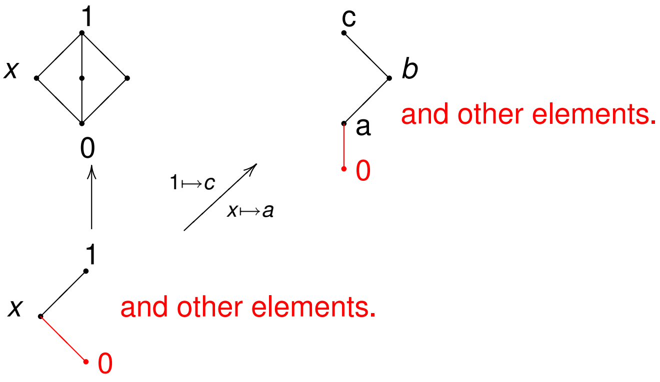
A diagram of $\mathcal{M}_3^{\mathrm{b}}$,
with no CP-extension in $\mathcal{M}_3^{0,1}$.





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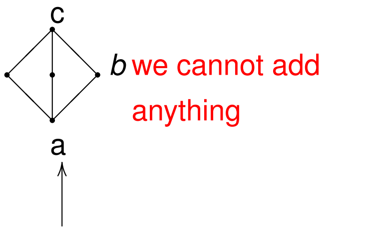

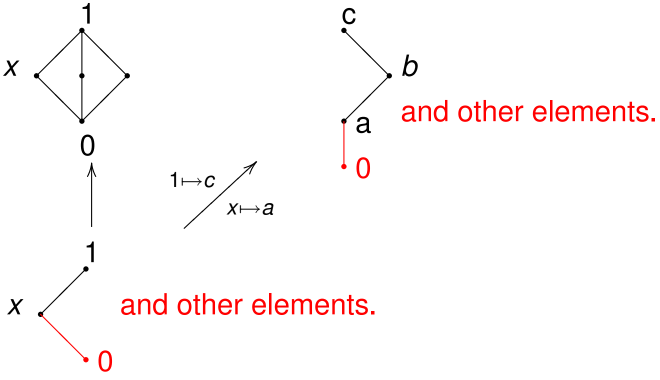

Proof of (1) $\Longrightarrow$ (2)

- There is a diagram in $\mathcal{M}_3^{\mathrm{b}}$ that has no congruence-preserving extension into $\mathcal{M}_3^{0,1}$.

Proof of (1) $\implies$ (2)

- There is a diagram in $\mathcal{M}_3^{\mathrm{b}}$ that has no congruence-preserving extension into $\mathcal{M}_3^{0,1}$.
- Using a *condensate* construction we obtain a countable lattice $L \in \mathcal{M}_3^{\mathrm{b}}$ that has no congruence-preserving extension into $\mathcal{M}_3^{0,1}$.

A functor

Theorem

There is a functor $\Psi\colon \mathcal{M}_3^{\mathrm{b}} \to \mathcal{M}_3^{0,1}$,

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There is a functor $\Psi\colon \mathcal{M}_3^{\mathrm{b}} \to \mathcal{M}_3^{0,1}$, such that $\operatorname{Con_c} \circ \Psi \cong \operatorname{Con_c}$. In particular for all $K \in \mathcal{M}_3^{\mathrm{b}}$ there is $L \in \mathcal{M}_3^{0,1}$ such that $\operatorname{Con_c} K \cong \operatorname{Con_c} L$.

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- Basically only the morphisms $f\colon \mathbf{2} \to M_3$ can cause problems, change them to the only possible morphism that preserves bounds.

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- We have a functor $\Psi\colon \mathcal{M}_3^{\mathrm{b}\dagger} \to \mathcal{M}_3^{0,1}$, such that $\mathrm{Con_c} \circ \Psi \cong \mathrm{Con_c}$.
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- Hence we can extend Ψ to a functor $\mathcal{M}_3^{\mathrm{b}} \to \mathcal{M}_3^{0,1}$ that preserves directed colimits.
- Moreover, as $\mathrm{Con_c}$ preserves directed colimits, $\mathrm{Con_c} \circ \Psi \cong \mathrm{Con_c}$.

That is all !

Thank you for your attention

Have you any questions ?