Monoids of languages, monoids of reflexive relations and ordered monoids

Ganna Kudryavtseva

June 22, 2010

$\mathcal{J}$-trivial monoids

A monoid S is called *$\mathcal{J}$-trivial* if the Green's relation $\mathcal{J}$ on it is the trivial relation, that is $a\mathcal{J}b$ implies $a = b$ for any $a, b \in S$, or, equivalently all $\mathcal{J}$-classes of S are one-element.
The class of finite $\mathcal{J}$-trivial monoids is closed with respect to taking finite direct products, finite submonoids and homomorphic images, that is it forms a *finite variety (quasivariety)* of monoids.
Examples:

- $\mathcal{C}_n$ — the monoid of all order-preserving and extensive maps from $\{1, \dots, n\}$ into itself.
- $\mathcal{R}_n$ — the monoid of reflexive relations of the set $\{1, \dots, n\}$, which we consider as a monoid of boolean matrices with diagonal entries equal to 1.
- $\mathcal{U}_n$ — the submonoid of $\mathcal{R}_n$ consisting of the upper triangular matrices.

Straubing's Theorem

Theorem (Straubing, 1980)

Let M be a finite monoid. The following conditions are equivalent:

- M *is* $\mathcal{J}$*-trivial.*
- M *divides* $\mathcal{C}_n$ *for some* n.
- M *divides* $\mathcal{R}_n$ *for some* n.
- M *divides* $\mathcal{U}_n$ *for some* n.

Positively ordered semigroups

A partial order $\leq$ on a semigroup S is called a *positive order* if

- it is compatible with the multiplication on S, that is, $a \geq b$ implies $ac \geq bc$ and $ca \geq cb$ for $a, b, c \in S$;
- $a \geq ab$, $a \geq ba$ for all $a, b \in S$.

If S is a monoid than the latter condition is equivalent to the condition that S satisfies the identity $x \leq 1$. If S is equipped with a positive order $\leq$ then S is called a *positively ordered semigroup*.

If S is a positively ordered semigroup then S is $\mathcal{J}$-trivial. The converse in not true.

Positively ordered semigroups

A partial order $\leq$ on a semigroup S is called a *positive order* if

- it is compatible with the multiplication on S, that is, $a \geq b$ implies $ac \geq bc$ and $ca \geq cb$ for $a, b, c \in S$;
- $a \geq ab$, $a \geq ba$ for all $a, b \in S$.

If S is a monoid than the latter condition is equivalent to the condition that S satisfies the identity $x \leq 1$. If S is equipped with a positive order $\leq$ then S is called a *positively ordered semigroup*.

If S is a positively ordered semigroup then S is $\mathcal{J}$-trivial. The converse in not true.

Theorem (Straubing and Thérien, 1988)

A finite monoid is J-trivial if and only if it is a divisor of a finite positively ordered monoid.

Positively ordered semigroups

A partial order $\leq$ on a semigroup S is called a *positive order* if

- it is compatible with the multiplication on S, that is, $a \geq b$ implies $ac \geq bc$ and $ca \geq cb$ for $a, b, c \in S$;
- $a \geq ab$, $a \geq ba$ for all $a, b \in S$.

If S is a monoid than the latter condition is equivalent to the condition that S satisfies the identity $x \leq 1$. If S is equipped with a positive order $\leq$ then S is called a *positively ordered semigroup*.

If S is a positively ordered semigroup then S is $\mathcal{J}$-trivial. The converse in not true.

Theorem (Straubing and Thérien, 1988)

A finite monoid is J-trivial if and only if it is a divisor of a finite positively ordered monoid.

Some definitions

Let X be a poset.
A transformation φ of X is called *order-preserving* if $a \geq b$ implies $a\varphi \geq b\varphi$, and *extensive* if $a\varphi \geq a$ for all $a, b \in X$.

- $\mathcal{OE}(X)$ — the monoid of all order-preserving and extensive transformations of X.
- $\mathcal{OE}_{chains}$ and $\mathcal{OE}_{posets}$ — the classes of monoids embeddable into $\mathcal{OE}(X)$ for some linearly ordered set X or some poset X, respectively.

Let X be a set.

- $\mathcal{R}(X)$ — the monoid of all reflexive binary relations over X.
- $\mathcal{R}$ — the class of monoids which can be embedded into $\mathcal{R}(X)$ for some X.

Some definitions

Let X be a poset.
A transformation φ of X is called *order-preserving* if $a \geq b$ implies $a\varphi \geq b\varphi$, and *extensive* if $a\varphi \geq a$ for all $a, b \in X$.

- $\mathcal{OE}(X)$ — the monoid of all order-preserving and extensive transformations of X.
- $\mathcal{OE}_{chains}$ and $\mathcal{OE}_{posets}$ — the classes of monoids embeddable into $\mathcal{OE}(X)$ for some linearly ordered set X or some poset X, respectively.

Let X be a set.

- $\mathcal{R}(X)$ — the monoid of all reflexive binary relations over X.
- $\mathcal{R}$ — the class of monoids which can be embedded into $\mathcal{R}(X)$ for some X.

More definitions

A reflexive binary relation $\sim$ over a poset X is called *upper-triangular* if $a \sim b$ implies $b \geq a$.

- $\mathcal{U}(X)$ — the monoid of all reflexive binary relations over X.
- $\mathcal{U}_{posets}$ and $\mathcal{U}_{chains}$ — the classes of monoids which can be embedded into $\mathcal{U}(X)$ for some poset X or some chain X, respectively.

All the monoids above are positively ordered with the anti-inclusion relation.

Volkov (2004) proved that $\mathcal{R}_n$, $\mathcal{U}_n$ and $\mathcal{OE}_n$ satisfy the same set of identities.

More definitions

A reflexive binary relation $\sim$ over a poset X is called *upper-triangular* if $a \sim b$ implies $b \geq a$.

- $\mathcal{U}(X)$ — the monoid of all reflexive binary relations over X.
- $\mathcal{U}_{posets}$ and $\mathcal{U}_{chains}$ — the classes of monoids which can be embedded into $\mathcal{U}(X)$ for some poset X or some chain X, respectively.

All the monoids above are positively ordered with the anti-inclusion relation.
Volkov (2004) proved that $\mathcal{R}_n$, $\mathcal{U}_n$ and $\mathcal{OE}_n$ satisfy the same set of identities.

Semigroups of languages

Let A be an alphabet. A *semigroup of languages* over A is a subsemigroup of the power monoid $P(A^*)$.
Let λ denote the empty word.
We denote by $P_1(A^*)$ the submonoid of $P(A^*)$ consisting of all languages which contain λ. We call such languages *positive languages.*
The monoid $P_1(A^*)$ is positively ordered with the reversed inclusion relation, that is $C \leq B$ if and only if $C \supseteq B$.
Notation: $\mathcal{PL}$ — the class of all semigroups of positive languages over A. $\mathcal{FL}$ — the class of all finite submonoids of $P(A^*)$.
Observation. $\mathcal{FL} \subset \mathcal{PL}$.

Semigroups of languages

Let A be an alphabet. A *semigroup of languages* over A is a subsemigroup of the power monoid $P(A^*)$.
Let λ denote the empty word.
We denote by $P_1(A^*)$ the submonoid of $P(A^*)$ consisting of all languages which contain λ. We call such languages *positive languages.*
The monoid $P_1(A^*)$ is positively ordered with the reversed inclusion relation, that is $C \leq B$ if and only if $C \supseteq B$.
Notation: $\mathcal{PL}$ — the class of all semigroups of positive languages over A. $\mathcal{FL}$ — the class of all finite submonoids of $P(A^*)$.
Observation. $\mathcal{FL} \subset \mathcal{PL}$.

Semigroups of languages as power semigroups of the complement of an ideal

Let S be a subsemigroup of $P_1(A^*)$ and I an ideal of A^*. Then the map $L \mapsto L \cup I$ is homomorphism from S to $P_1(A^*)$. It is one-to-one if and only if for every $L, M \in S$, $L \neq M$, the set $\big((L \setminus M) \cup (M \setminus L)\big) \setminus I$ is not empty.

Let $I \neq A^*$ be an ideal of A^*. Then $\lambda \notin I$. Let $P_1(A^* \setminus I)$ be the subset of $P(A^* \setminus I)$ consisting of all subsets of $A^* \setminus I$ containing λ. To turn it into a semigroup we define the multiplication on it as follows: for $A, B \in P_1(X^* \setminus I)$ we set

$$A \cdot B = \{ab : a \in A, b \in B \text{ and } ab \in A^* \setminus I\}.$$

Denote the semigroup $(P_1(A^* \setminus I), \cdot)$ by $P_1'(A^* \setminus I)$.

Semigroups of languages as power semigroups of the complement of an ideal

Let S be a subsemigroup of $P_1(A^*)$ and I an ideal of A^*. Then the map $L \mapsto L \cup I$ is homomorphism from S to $P_1(A^*)$. It is one-to-one if and only if for every $L, M \in S$, $L \neq M$, the set $\big((L \setminus M) \cup (M \setminus L)\big) \setminus I$ is not empty.

Let $I \neq A^*$ be an ideal of A^*. Then $\lambda \notin I$. Let $P_1(A^* \setminus I)$ be the subset of $P(A^* \setminus I)$ consisting of all subsets of $A^* \setminus I$ containing λ. To turn it into a semigroup we define the multiplication on it as follows: for $A, B \in P_1(X^* \setminus I)$ we set

$$A \cdot B = \{ab : a \in A, b \in B \text{ and } ab \in A^* \setminus I\}.$$

Denote the semigroup $(P_1(A^* \setminus I), \cdot)$ by $P_1'(A^* \setminus I)$.

Semigroups of factor-words

If w is a word, then $S(w)$ denotes the set of factor-words of w. Let $w_j, j \in J$, be non-empty words. Let $I(w_j, j \in J) = A^* \setminus \left(\cup_{j \in J} S(w_j)\right)$.
$I = I(w_j, j \in J)$ is an ideal of A^*. Denote the semigroup $P_1'(X^* \setminus I)$ by $P_1'(w_j, j \in J)$.
Example. The elements of $P_1'(xy)$ are $\{\lambda\}$, $\{\lambda, x\}$, $\{\lambda, y\}$, $\{\lambda, xy\}$, $\{\lambda, x, y\}$, $\{\lambda, x, xy\}$, $\{\lambda, y, xy\}$, $\{\lambda, x, y, xy\}$.

Theorem

- *Any semigroup of positive languages is isomorphic to some subsemigroup of $P_1'(w_i, i \in I)$.*
- *Any finite semigroup of languages is isomorphic to some $P_1'(w)$, such that all letters of w are pairwise different.*

The connection of $\mathcal{OE}_{posets}$ with $\mathcal{PL}$ and $\mathcal{U}_{posets}$

Theorem

1. $\mathcal{U}_{posets} \subseteq \mathcal{OE}_{posets}$.
2. $\mathcal{PL} \subseteq \mathcal{U}_{posets}$.
3. *Every semigroup from $\mathcal{OE}_{posets}$ is a factor-semigroup of some semigroup from $\mathcal{PL}$.*

Remark. Item 3 was first published by Vernitski (2008).
Illustration of 2. $S = S(xzy, ztxz)$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof of 3

Let $\leq$ be a positive order on S. Let S' be the set disjoint with S, which has the same cardinality as S, and let $' : S \to S'$ be a fixed bijection. For $s \in S$ by L_s denote the set of all languages L over S' satisfying the following conditions:

- $\lambda \in L$,
- if $s_1' s_2' \dots s_k' \in L$ for some $k \geq 1$ then $s_1 s_2 \cdots s_k \leq s$ in S,
- L contains some word $s_1' s_2' \dots s_k'$ such that $s_1' s_2' \dots s_k' = s$ in S.

Let $T = \cup_{s \in S} L_s$. T is a semigroup and the map sending all elements of L_s to s is an onto homomorphism from T to S.

Two more economic constructions

Construction A. Let $S \in \mathcal{OE}(X)$, X a poset, and $\varphi \in S$. K_φ — the set of all such relations $\sim$ over X that for all $x \in X$:

$$x \sim x,\ x \sim x\varphi \tag{1}$$

$$\text{if } x \sim y \text{ then } x \leq y \leq x\varphi. \tag{2}$$

Set $T = \cup_{\varphi \in S} K_\varphi$. Obviously, $T \subset \mathcal{U}(X)$. Define $\alpha : T \to S$: for $\sim \in T$ set $\sim \alpha = \varphi$, where $\varphi \in S$ is such that $\sim \in K_\varphi$. Then T is a semigroup and α is an onto homomorphism.

Construction B. Let $\varphi \in S$. By K'_φ denote the set of all such relations $\sim$ that for all $x \in X$ (1), (2) hold, and, in addition:

$$\text{for at least one chain } C \text{ between } x \text{ and } x\varphi : x \sim y \text{ for all } y \in C.$$

Set $R = \cup_{\varphi \in S} K'_\varphi$. Define $\beta : R \to S$ in the same manner as α above. Then R is a semigroup and β is an onto homomorphism.

Two more economic constructions

Construction A. Let $S \in \mathcal{OE}(X)$, X a poset, and $\varphi \in S$. K_φ — the set of all such relations $\sim$ over X that for all $x \in X$:

$$x \sim x,\ x \sim x\varphi \tag{1}$$

$$\text{if } x \sim y \text{ then } x \leq y \leq x\varphi. \tag{2}$$

Set $T = \cup_{\varphi \in S} K_\varphi$. Obviously, $T \subset \mathcal{U}(X)$. Define $\alpha : T \to S$: for $\sim \in T$ set $\sim \alpha = \varphi$, where $\varphi \in S$ is such that $\sim \in K_\varphi$. Then T is a semigroup and α is an onto homomorphism.
Construction B. Let $\varphi \in S$. By K'_φ denote the set of all such relations $\sim$ that for all $x \in X$ (1), (2) hold, and, in addition:

$$\text{for at least one chain } C \text{ between } x \text{ and } x\varphi : x \sim y \text{ for all } y \in C.$$

Set $R = \cup_{\varphi \in S} K'_\varphi$. Define $\beta : R \to S$ in the same manner as α above. Then R is a semigroup and β is an onto homomorphism.

Corollary

Let X be a linearly ordered set. Then the semigroup $\mathcal{OE}(X)$ can be embedded into $\mathcal{U}(X)$.

Remark. Let X be a linearly ordered set. If we take $S = \mathcal{OE}(X)$ then Construction B outputs an isomorphism between $\mathcal{OE}(X)$ and the subsemigroup R of $\mathcal{U}(X)$ of all *consistent* upper-triangular reflexive binary relations on X, that is such upper-triangular reflexive relations $\sim$ that $x \sim y$ implies $x \sim z$ for all $x \leq z \leq y$.

Finite semigroups of languages

Theorem

$\mathcal{FL} = \mathcal{U}^f_{chains}$.

It is enough to show that the semigroup $\mathcal{P}'_1(w)$, where $w = a_1 a_2 \dots a_n$ and all its letters are different, is isomorphic to $\mathcal{U}_{n+1}$. Let $w = a_1 a_2 \dots a_n$ and let $A \in \mathcal{P}'_1(w)$. We define $A' \in \mathcal{U}_{n+1}$:

$$A'_{ij} = \begin{cases} 1, & \text{if } i = j \text{ or } a_i \cdots a_{j-1} \in A, \\ 0, & \text{otherwise.} \end{cases}$$

This map is an isomorphism.

Finite semigroups of languages

Corollary

Any semigroup in $\mathcal{OE}^f_{posets}$ can be covered by a semigroup in $\mathcal{FL}$.

Remark. The claim of Corollary 7 was first proved by Vernitski (2008). In our construction the covering semigroup is over the alphabet of the cardinality $|X|$. In the case when X is a chain our homomorphism is one-to-one.

Consistent semigroups

Call $A \in \mathcal{P}_1'(w)$ *consistent* if from $u \in A$ it follows that all factor-words of u are also in A. Call a subsemigroup of $\mathcal{P}_1'(w)$ *consistent* if all its elements are consistent. Call a boolean upper-triangular matrix $A \in \mathcal{U}_n$ *consistent*, if from $A_{ij} = 1$ with $j > i$ it follows that $A_{kl} = 1$ for any $i \le k \le l \le j$. Call a subsemigroup of $\mathcal{U}_n$ *consistent* if all its elements are consistent.

Theorem

Let S be a finite semigroup. The following statements are equivalent:

1. $S \in \mathcal{OE}^f_{chains}$.
2. *S is isomorphic to a consistent subsemigroup of some $\mathcal{P}_1'(w)$.*
3. *S is isomorphic to a consistent subsemigroup of $\mathcal{U}^f_{chains}$.*

A problem

Suppose S is a non-consistent subsemigroup of some $\mathcal{P}'_1(w)$. It is natural to ask if it is possible to find another word u such that S is isomorphic to a consistent subsemigroup of $\mathcal{P}'_1(u)$. This question is equivalent to the asking if the inclusion $\mathcal{OE}^f_{chains} \subseteq \mathcal{FL}$ is strict. The answer is negative.

Therefore, in the following chain of inclusions

$$\mathcal{OE}^f_{chains} \subseteq \mathcal{U}^f_{chains} = \mathcal{FL} \subseteq \mathcal{R}^f \subseteq \mathcal{OE}^f_{posets}. \tag{3}$$

the first inclusion is strict.

Problem. Are the remaining inclusions strict?

A problem

Suppose S is a non-consistent subsemigroup of some $\mathcal{P}_1'(w)$. It is natural to ask if it is possible to find another word u such that S is isomorphic to a consistent subsemigroup of $\mathcal{P}_1'(u)$. This question is equivalent to the asking if the inclusion $\mathcal{OE}^f_{chains} \subseteq \mathcal{FL}$ is strict. The answer is negative. Therefore, in the following chain of inclusions

$$\mathcal{OE}^f_{chains} \subseteq \mathcal{U}^f_{chains} = \mathcal{FL} \subseteq \mathcal{R}^f \subseteq \mathcal{OE}^f_{posets}. \qquad (3)$$

the first inclusion is strict.
Problem. Are the remaining inclusions strict?

The last frame

Thank you
for your attention!!!