On direct decompositions of sd-Artinian modules

In treating the problem of understanding the behaviour of direct decompositions of a given module one usually tries to find a set of direct decompositions of the module that on the one hand is easy to understand, and on the other hand determines the rest of the direct decompositions of the module. It is a rare thing to find a module in which one can achieve this completely. In the present work we do so, to some extent, on a certain class of modules, the so called sd-Artinian modules. We will start with some conventions.

- $\mathcal{L}(M)$ will donote the lattice of all submodules of a module M.
- $S(M)$ will denote the set of all direct summands of a module M as a subposet of $\mathcal{L}(M)$.

Definition. Let M be a module. We say that M is sd-Artinian if $S(M)$ satisfies the ascending chain condition (eq. the descending chain condition).

Our main interest is in studying the behaviour of direct decompositions of sd-Artinian modules. It is very easy to see that sd-Artinian modules admit finite indecomposable direct decompositions, our first result says that on an sd-Artinian module we can obtain any such direct decomposition just by refining any of its direct decompositions. First the following definition.

Definition. Let $M = \bigoplus_A M_\alpha$ and $M = \bigoplus_B N_\beta$ be two drect decompositions of the module M. We say that $M = \bigoplus_A M_\alpha$ refines into $M = \bigoplus_B N_\beta$ if for each $\alpha \in A$ there exists a direct decomposition $M_\alpha = \bigoplus_{C_\alpha} M_{\alpha_\gamma}$ of M_α such that the two direct decompositions $M = \bigoplus_B N_\beta$ and $M = \bigoplus_A \left(\bigoplus_{C_\alpha} M_{\alpha_\gamma}\right)$ of M are equal.

Proposition. Let M be a module. If M is sd-Artinian then all its direct decompositions refine into finite indecomposable direct decompositions.

At first glance the condition described in the proposition above says more than the fact that sd-Artinian modules admit finite indecomposable direct decompositions, in what follows we prove that it actually does, we do this by exhibiting a ring R, and an R-module M, with finite indecomposable direct decompositions, but such that not all its direct decompositions refine into finite indecomposable direct decompositions. We start with the following lemma, due to Facchini and Herbera.

Lemma (Facchini, Herbera). Let D be an additive submonoid of $\mathbb{N}$. There exist a ring R and an R-module M such that for each $n \in \mathbb{N}$ the R-module M^n admits finite indecomposable direct decompositions if and only if $n \in D$.

If we put $D = 2\mathbb{N}$, then the lemma above gives us a ring R and an R-module M such that M^2 admits finite indecomposable direct decompositions, while M does not. Clearly the module M^2 is an example of a module that admits finite indecomposable direct decompositions, but such that not all its direct decompositions refine into finite indecomposable direct decompositions.

Note that the existence of a module with finite indecomposable direct decompositions but such that not all its direct decompositions refine into finite indecomposable direct decompositions implies that niether the class of all modules such that all their direct decompositions refine into finite indecomposable direct decompositions, nor the class of all sd-Artinian modules is closed under taking finite direct sums.

When studying idecomposable direct decompositions of sd-Artinian modules, one of the dificulties that we might encounter is that these may be very different from one to another. For example, we could have an sd-Artinian module which has indecomposable direct decompositions of cardinality n for almost all n. Thus we wish to characterize those sd-Artinian modules such that their indecomposable direct decompositions have only a finite number of cardinalities. We have the following definition.

Definition. Let M be a module and let $\aleph$ be a cardinal. We say that M has sd-length $\aleph$ ($sd.l\,(M) = \aleph$), if $\aleph$ is the maximum cardinality of a subset of $S\,(M)$ which is independent in $\mathcal{L}(M)$.

The following proposition says that modules with finite sd-length are precisely those modules such that their indecomposable direct decompositions have only a finite number of cardinalities.

Proposition. Let M be a module. The following conditions are equivalent.

1. M has finite sd-length.
2. There exists $n \in \mathbb{N}$ such that M has no indecomposable direct decompositions of cardinality larger than n.
3. There exists $n \in \mathbb{N}$ such that M has no direct decompositions of cardinality larger than n.
4. There exists $n \in \mathbb{N}$ such that $S(M)$ has no irreductible chains of length larger than n.
5. There exist order preserving functions from $S(M)$ to $\mathbb{N} \cup \{0\}$.

Moreover, if M satisfies the conditions above, then $sd.l(M)$ is the largest cardinality of direct decompositions of M and the largest length of irreductible chains in $S(M)$.

Furthermore, if M has finite sd-length and $\mathcal{F}$ is the set of all order preserving functions from $S(M)$ to $\mathbb{N} \cup \{0\}$ ordered in the obvious way (i.e. $\alpha \leq \beta$ if and only if $\alpha(N) \leq \beta(N)$ for each $N \in S(M)$), then the function

$$sd.l\_ : S(M) \longrightarrow \mathbb{N} \cup \{0\}$$

that associates its sd-length to each element of $S(M)$ is the minimum of $\mathcal{F}$.

Definition. Let M be a module. We say that a function $\psi : S(M) \to \mathbb{N} \cup \{0\}$ is additive if for each finite direct decomposition $M = \bigoplus_{i=1}^{n} M_i$ of M the following holds:

$$\psi(M) = \sum_{i=1}^{n} \psi(M_i).$$

We already have many examples of additive functions on any module M. For example the functions $G.\text{dim}\_$, $\text{codim}\_$, $K.\text{dim}\_$, $K.\text{codim}$, and $\ell\_$, that associate to each $N \in S(M)$ its Goldie dimention, dual Goldie dimention, Krull dimention, dual Krull dimention, and its length when they exist, are all additive.

Notice that since additive functions are order preserving, then modules with finite Goldie dimention, and modules with finite dual Goldie dimention all have finite sd-length, and in that case the following inequality holds.

$$sd.l(M) \leq m\acute{\imath}n \{G \dim M, \text{codim} M, \ell(M)\}.$$

The following proposition characterizes those sd-Artinian modules such that all their indecomposable direct decompositions have the same cardinality.

Proposition. Let M be an sd-Artinian module. All indecomposable direct decompositions of M have the same cardinality if and only M has finite sd-length and the function $sd.l\_$ is additive.

Apart from the problem of determining the behaviour of direct decompositions of sd-Artinian modules, we are also interested in the problem of decomposing a module in terms of a given class of modules. The following is our first result in this direction.

Proposition. Let M be a module, and let Ω be a class of modules closed under finite direct sums. If $S(M) \cap \Omega \neq \varnothing$, and has maximal elements, then there exists a direct decomposition $M = A \oplus B$ of M, such that $A \in \Omega$, and such that B has no nontrivial direct summands in Ω.

Corollary. Let M be an sd-Artinian module, and let Ω be a class of modules closed under finite direct sums. If $S(M) \cap \Omega \neq \varnothing$ then there exists a direct decomposition $M = A \oplus B$ of M, such that $A \in \Omega$, and such that B has no nontrivial direct summands in Ω.

The proposition above says that given an sd-Artinian module M and a class of modules Ω, if Ω is closed under taking finite direct sums, then we can decompose M in terms of Ω. We are also concerned with the problem of uniqueness of direct decompositions as the one in the proposition above. We first have to precise on the notion of uniqueness of direct decompositions of a module. The following definition does this.

Definition. We say that two direct decompositions $M = \bigoplus_A M_\alpha$ and $M = \bigoplus_B N_\beta$ of a module M are decomposition isomorphic if there exists a biyection $\psi : A \to B$ such that for each $\alpha \in A$, M_α and $N_{\psi(\alpha)}$ are isomorphic modules.

The following proposition says that under certain extra conditions on the class of modules Ω, we can guarentee that direct decompositions as the ones in the proposition above are unique up to decomposition isomorphisms.

Proposition. Let M be an sd-Artinian module, and let Ω be a class of modules with the following properties

1. Ω is closed under taking finite direct sums.
2. Ω is closed under taking direct summands.
3. $End\,(N)$ is a local ring for each indecomposable $N \in \Omega$.

If $S\,(M) \cap \Omega \neq \varnothing$, then there exists a direct decomposition $M = A \oplus B$ of M, unique up to decomposition isomorphisms, such that $A \in \Omega$, and such that B has no nontrivial direct summands in Ω.

An example of a class of modules that satisfies conditions 1,2, and 3 above is the class of all modules with finite length, thus we have the following corollary.

Corollary. Let M be a module. If M is sd-Artinian, then there exists a direct decomposition $M = A \oplus B$ of M, unique up to decomposition isomorfisms, such that A has finite length, and such that B has no nontrivial direct summands with finite length.

Finally, since Artinian and Noetherian modules are sd-Artinian, and since modules with finite length are precisely those modules that are Artinian and Noetherian, we have the following corollary, which says that in a unique fashion Artinian modules can be put in terms of Noeatherian modules, and the other way around.

Corollary. Let M be a module.

1. If M is Noetherian, there exists a direct decomposition $M = A \oplus B$ of M, unique up to decomposition isomorphisms, such that A is Artinian and such that B has no nontrivial Artinian direct summands.

2. If M is Artinian, there exists a direct decomposition $M = A \oplus B$ of M, unique up to decomposition isomorphisms, such that A is Noetherian and such that B has no nontrivial Noetherian direct summands.

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