Clones on Ramsey structures and Schaefer's theorem for graphs

An opera in a prologue, three acts, and an epilogue

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Synopsis

- **Prologue:** The graph satisfiability problem
  (Bodirsky, Pinsker)
- **Act I:** Reducts of homogeneous structures
  (Cameron, Thomas)
- **Act II:** The Ramsey property
  (Nešetřil, Rödl)
- **Act III:** Topological dynamics
  (Kechris, Pestov, Todorcevic, Tsankov)
- **Epilogue:** Schaefer’s theorem for graphs

Prologue

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Theorem (Schaefer '78)

Boolean-SAT(Ψ) is either in P or NP-complete, for all Ψ.

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Γ_Ψ is a *reduct of* the random graph, i.e.,
a structure with a first-order definition in G.

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Let's study the reducts of the random graph!

Act I

Reducts of homogeneous structures


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Problem

Classify the reducts of Γ.

Possible classifications

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Denote by $(\mathbb{Q}; <)$ be the dense linear order, and set

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Theorem (Junker, Ziegler '08)

$(\mathbb{Q}; <, 0)$ has 116 reducts, up to f.o.-interdefinability.

Thomas' conjecture

Conjecture (Thomas '91)

Let Γ be homogeneous in a finite language.

Then Γ has finitely many reducts up to f.o.-interdefinability.

Act II

The Ramsey property

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Groups, Monoids, Clones

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Clone... set of finitary operations which contains all projections and which is closed under composition

The reducts of the random graph, revisited

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5. The full symmetric group S_V.

How to find all reducts up to . . .-interdefinability?

Climb up the lattice!

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$f : V \to V$ is *canonical on* $F \subseteq V$ iff its restriction to F is canonical.

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Conclusion: Every finite graph has a copy in G on which f is canonical.

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The structure $(V; E, c, d)$ has similar Ramsey properties as $(V; E)$.

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Theorem (Thomas '96)

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We thus know the *minimal closed monoids* containing $\mathrm{Aut}(G)$.

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Theorem (Bodirsky, P. '09)

Let $f : V^n \to V$, $f \notin \mathrm{Aut}(G)$.
Then f generates one of the following:

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A class $\mathcal{C}$ of structures of the same signature is called a *Ramsey class* iff
for all $H, P \in \mathcal{C}$ there is N in $\mathcal{C}$ such that $N \to (H)^P$.

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We would like to fix $c_1, \ldots, c_n$ witnessing $f \notin \mathrm{Aut}(\Gamma)$,
and have canonical behavior on $(\Gamma, c_1, \ldots, c_n)$.

Act III

Topological dynamics

Adding constants to Ramsey classes

Problem

If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

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Theorem (Kechris, Pestov, Todorcevic '05)

An ordered homogeneous structure Δ is Ramsey iff
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Corollary

If Γ is ordered Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$,
then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$
which behaves like f on $\{c_1, \ldots, c_n\}$.

Minimal monoids above Ramsey structures

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- Every closed supermonoid of $\mathrm{Aut}(\Gamma)$ contains a minimal closed supermonoid of $\mathrm{Aut}(\Gamma)$.

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- Every closed clone above $\mathrm{Aut}(\Gamma)$ contains a minimal one.

Epilogue

Schaefer's theorem for graphs

The Graph Satisfiability Problem

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Let Ψ be a finite set of graph formulas.

Computational problem: Graph-SAT(Ψ)

INPUT:

- A set W of variables (vertices), and
- statements $\phi_1, \ldots, \phi_n$ about the elements of W, where each ϕ_i is taken from Ψ.

QUESTION: Is there a graph on the vertex set W that satisfies all ϕ_i?

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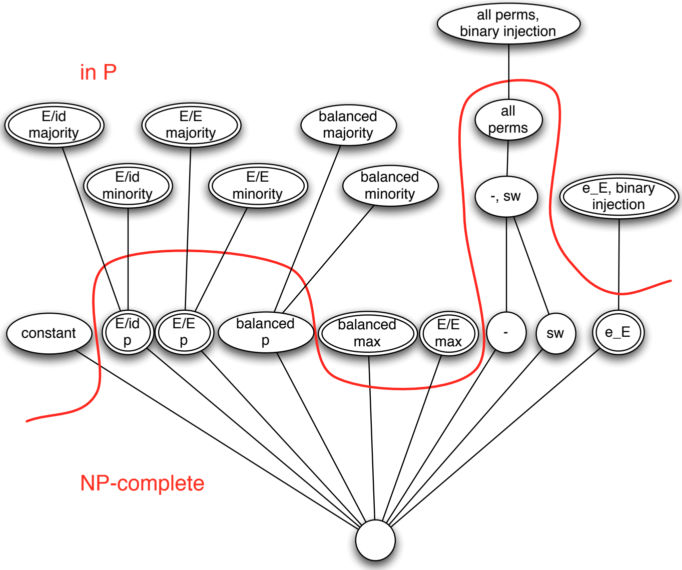
Theorem (Bodirsky, P. '10)

Graph-SAT(Ψ) is either in P or NP-complete, for all Ψ.

Classification



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Can a clone containing the automorphism group of an ordered Ramsey structure Γ have infinitely many superclones?

Open problems II

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Open problems II

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