ON CATEGORICAL SKEW LATTICES

João Pita Costa

Department of Mathematics
University of Ljubljana

ICAL $\simeq$ Jardafest
Prague 2010

A definition for Skew Lattices

A nonempty set S

A definition for Skew Lattices

A nonempty set S and $\vee$, $\wedge$ associative, idempotent binary operations that satisfy the absorption laws

$$(y \wedge x) \vee x = x = x \vee (x \wedge y) \text{ and their duals.}$$

A definition for Skew Lattices

A nonempty set S and $\vee$, $\wedge$ associative, idempotent binary operations that satisfy the absorption laws

$$(y \wedge x) \vee x = x = x \vee (x \wedge y) \text{ and their duals.}$$

$$\Updownarrow$$

$$x \vee y = x \text{ iff } x \wedge y = y \text{ and } x \vee y = y \text{ iff } x \wedge y = x.$$

A definition for Skew Lattices

A nonempty set S and $\vee$, $\wedge$ associative, idempotent binary operations that satisfy the absorption laws

$$(y \wedge x) \vee x = x = x \vee (x \wedge y) \text{ and their duals.}$$

$$\Updownarrow$$

$$x \vee y = x \text{ iff } x \wedge y = y \text{ and } x \vee y = y \text{ iff } x \wedge y = x.$$

Skew lattices form a variety (Leech 1989).

Order, congruence and diagram

natural order: $x \geq y$ iff $x \wedge y = y = y \wedge x$ or $x \vee y = x = y \vee x$.

Order, congruence and diagram

natural order: $x \geq y$ iff $x \wedge y = y = y \wedge x$ or $x \vee y = x = y \vee x$.

natural pre-order: $x \succ y$ iff $x \vee y \vee x = x$ or $y \wedge x \wedge y = y$

Order, congruence and diagram

natural order: $x \geq y$ iff $x \wedge y = y = y \wedge x$ or $x \vee y = x = y \vee x$.
natural pre-order: $x \succ y$ iff $x \vee y \vee x = x$ or $y \wedge x \wedge y = y$
natural congruence: xDy iff $x \succ y$ and $y \succ x$

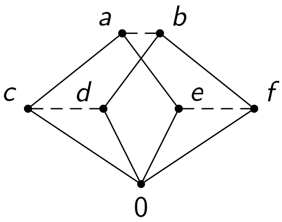
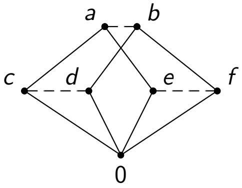


Figure: Diagram of a skew lattice

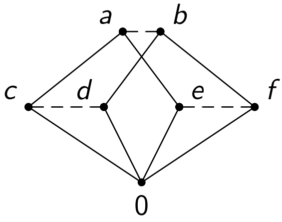
Cayley tables and diagram



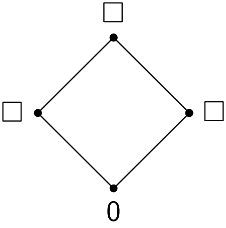
$\wedge$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	0	a	b	c	d	e	f
b	0	a	b	c	d	e	f
c	0	c	d	c	d	0	0
d	0	c	d	c	d	0	0
e	0	e	f	0	0	e	f
f	0	e	f	0	0	e	f

$\vee$	0	a	b	c	d	e	f
0	0	a	b	c	d	e	f
a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b
c	c	a	a	c	c	a	a
d	d	b	b	d	d	b	b
e	e	a	a	a	a	e	e
e	f	b	b	b	b	f	f

From a skew lattice S to a lattice $S/\mathcal{D}$



S



$S/\mathcal{D}$

Coset Structure

Consider two $\mathcal{D}$-classes $A > B$. For some $b \in B$, a subset $A \wedge b \wedge A = \{ a \wedge b \wedge a : a \in A \}$ is a coset of A in B. Similarly, a coset of B in A is any subset $B \vee a \vee B = \{ b \vee a \vee b : b \in B \}$ of A, for a fixed $a \in A$.

Coset Structure

Consider two $\mathcal{D}$-classes $A > B$. For some $b \in B$, a subset $A \wedge b \wedge A = \{ a \wedge b \wedge a : a \in A \}$ is a coset of A in B. Similarly, a coset of B in A is any subset $B \vee a \vee B = \{ b \vee a \vee b : b \in B \}$ of A, for a fixed $a \in A$.

(a) B is partitioned by cosets of A in B.

Coset Structure

Consider two $\mathcal{D}$-classes $A > B$. For some $b \in B$, a subset $A \wedge b \wedge A = \{\, a \wedge b \wedge a : a \in A \,\}$ is a coset of A in B. Similarly, a coset of B in A is any subset $B \vee a \vee B = \{\, b \vee a \vee b : b \in B \,\}$ of A, for a fixed $a \in A$.

(a) B is partitioned by cosets of A in B.

(c) Given cosets $B \vee a \vee B$ in A and $A \wedge b \wedge A$ in B, a natural bijection of cosets is given by the natural partial ordering:

$$x \mapsto y \text{ if, and only if, } x \geq y.$$

Coset bijections in $A > B$

Let $a \in A$ and $b \in B$ with $a > b$.

$$\begin{array}{ccc} B \vee a \vee B & \xrightarrow{\rho_{a,b}} & A \wedge b \wedge A \\ x & \longrightarrow & x \wedge b \wedge x \\ y \vee a \vee y & \longleftarrow & y \end{array}$$

A definition ...

Definition

A skew lattice is *categorical* if nonempty composites of coset bijections are coset bijections.

A definition ...

Let $A > B > C$ in S and $a > b > c$ with $a \in A$, $b \in B$ and $c \in C$.

A definition ...

Let $A > B > C$ in S and $a > b > c$ with $a \in A$, $b \in B$ and $c \in C$.Consider the coset bijections

$$C \vee a \vee C \xrightarrow{\chi_{a,c}} A \wedge c \wedge A$$

$$B \vee a \vee B \xrightarrow{\psi_{a,b}} (A \wedge b \wedge A), \quad (C \vee b \vee C) \xrightarrow{\varphi_{b,c}} B \wedge c \wedge B$$

A definition ...

Let $A > B > C$ in S and $a > b > c$ with $a \in A$, $b \in B$ and $c \in C$.Consider the coset bijections

$$C \vee a \vee C \xrightarrow{\chi_{a,c}} A \wedge c \wedge A$$

$$B \vee a \vee B \xrightarrow{\psi_{a,b}} (A \wedge b \wedge A), \quad (C \vee b \vee C) \xrightarrow{\varphi_{b,c}} B \wedge c \wedge B$$

$$((A \wedge b \wedge A) \cap (C \vee b \vee C)) \wedge c \wedge ((A \wedge b \wedge A) \cap (C \vee b \vee C)) \subseteq A \wedge b \wedge A \wedge c \wedge A \wedge b \wedge A = A \wedge c \wedge A$$

A definition ...

Let $A > B > C$ in S and $a > b > c$ with $a \in A$, $b \in B$ and $c \in C$.Consider the coset bijections

$$C \vee a \vee C \xrightarrow{\chi_{a,c}} A \wedge c \wedge A$$

$$B \vee a \vee B \xrightarrow{\psi_{a,b}} (A \wedge b \wedge A), \quad (C \vee b \vee C) \xrightarrow{\varphi_{b,c}} B \wedge c \wedge B$$

$$((A \wedge b \wedge A) \cap (C \vee b \vee C)) \wedge c \wedge ((A \wedge b \wedge A) \cap (C \vee b \vee C)) \subseteq A \wedge b \wedge A \wedge c \wedge A \wedge b \wedge A = A \wedge c \wedge A$$

S is categorical iff $\psi \circ \varphi \supseteq \chi$.

A category ...

If S is categorical, a category $Cat(S)$ can be defined by the following:

- The objects of S are the $\mathcal{D}$-classes of S ;
- $Hom(A, B)$ for $A > B$ are all coset bijections from all B-cosets in A to A-cosets in B (including ϕ). Otherwise, $Hom(A, B)$ is empty;
- $Hom(A, A)$ is the unique identity bijection in A ;
- morphism compositions are the *usual compositions*.

A variety ...

(Leech 1993) S is a categorical skew lattice iff the following identity hold

$$(x \wedge c \wedge x) \vee b \vee (x \wedge c \wedge x) =$$
$$((x \wedge c \wedge x) \vee a \vee (x \wedge c \wedge x)) \wedge b \wedge ((x \wedge c \wedge x) \vee a \vee (x \wedge c \wedge x))$$

given that $x\mathcal{D}a \geq b \geq c$ in S .

A variety ...

(Leech 1993) S is a categorical skew lattice iff the following identity hold

$$(x \wedge c \wedge x) \vee b \vee (x \wedge c \wedge x) =$$
$$((x \wedge c \wedge x) \vee a \vee (x \wedge c \wedge x)) \wedge b \wedge ((x \wedge c \wedge x) \vee a \vee (x \wedge c \wedge x))$$

given that $x \mathcal{D} a \geq b \geq c$ in S .

Categorical skew lattices form a subvariety of skew lattices.

A variety ...

(Cvetko-Vah 2005) Moreover,

$$\begin{aligned}(A \wedge b \wedge A) \cap (C \vee b \vee C) &= (C \vee a \vee C) \wedge b \wedge (C \vee a \vee C) \\ &= (A \wedge c \wedge A) \vee b \vee (A \wedge c \wedge A).\end{aligned}$$

is another characterization for Categorical Skew Lattices.

A criteria ...

(Leech 2010)

$$\begin{array}{ccc} a & \cdots & x = c' \vee a \vee c' \\ \vdots & & \vdots \\ b & \cdots & b' \\ \vdots & & \vdots \\ c & \cdots & c' = x \wedge c \wedge x \end{array}$$

S is categorical iff $b' = x \wedge b \wedge x = c' \vee b \vee c'$.

A criteria ...

$$
\begin{array}{ccc}
a & \cdots & x = \chi^{-1}(c') \\
\vdots & & \vdots \\
b & \cdots & b' \\
\vdots & & \vdots \\
c & \cdots & c' = \chi(c)
\end{array}
$$

S is categorical iff $b' = \psi_{a,b}^{-1}(c') = \varphi_{b,c}(x)$.

Coset bijections in $A > B > C$

Let $a \in A$, $b \in B$ and $c \in C$ with $a > b > c$.

$$B \vee a \vee B \xleftarrow{\psi_{a,b}^{-1}} (A \wedge b \wedge A), (C \vee b \vee C) \xrightarrow{\varphi_{b,c}} B \wedge c \wedge B$$

Coset bijections in $A > B > C$

Let $a \in A$, $b \in B$ and $c \in C$ with $a > b > c$.

$$B \vee a \vee B \xleftarrow{\psi_{a,b}^{-1}} (A \wedge b \wedge A) \cap (C \vee b \vee C) \xrightarrow{\varphi_{b,c}} B \wedge c \wedge B$$

A bijection ...

Let $a \in A$, $b \in B$ and $c \in C$ with $a > b > c$.

$$C \vee a \vee C \xleftarrow{\overline{\psi_{a,b}^{-1}}} (A \wedge b \wedge A) \cap (C \vee b \vee C) \xrightarrow{\varphi_{b,c}} B \wedge c \wedge B$$

as $C \vee a \vee C \subseteq B \vee a \vee B$.

A bijection ...

Let $a \in A$, $b \in B$ and $c \in C$ with $a > b > c$.

$$C \vee a \vee C \xleftarrow{\overline{\psi_{a,b}^{-1}}} (A \wedge b \wedge A) \cap (C \vee b \vee C) \xrightarrow{\overline{\varphi_{b,c}}} A \wedge c \wedge A$$

as $A \wedge c \wedge A \subseteq B \wedge c \wedge B$.

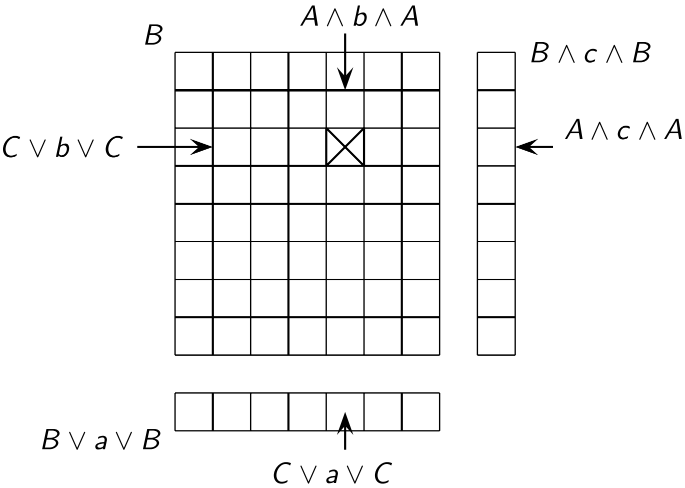
A bijection ...

Let $a \in A$, $b \in B$ and $c \in C$ with $a > b > c$.

$$C \vee a \vee C \xleftarrow{\overline{\psi_{a,b}^{-1}}} (A \wedge b \wedge A) \cap (C \vee b \vee C) \xrightarrow{\overline{\varphi_{b,c}}} A \wedge c \wedge A$$

S is categorical iff $\overline{\psi_{a,b}}$ and $\overline{\varphi_{b,c}}$ are bijections.

Egg box



Coset Laws

Theorem

Let $\mathbf{S}$ *be a skew lattice. The following statements are equivalent:*

(i) $\mathbf{S}$ *is categorical,*

Coset Laws

Theorem

Let **S** *be a skew lattice. The following statements are equivalent:*

(i) **S** *is categorical,*

(iii) *given any skew chain* $\{ A > B > C \}$ *in* **S**, $a \in A, b \in B, c \in C$ *and any* $x, x' \in A$,

$C \vee x \vee C = C \vee x' \vee C$ *if and only if* $B \vee x \vee B = B \vee x' \vee B$ *and, exist* $b \in B$ *such that*

$$C \vee (x \wedge b \wedge x) \vee A = A \vee (x' \wedge b \wedge x') \vee A.$$

Coset Laws

Theorem

Let **S** *be a skew lattice. The following statements are equivalent:*

(i) **S** *is categorical,*

(ii) *given any skew chain* $\{A > B > C\}$ *in* **S**, $a \in A, b \in B, c \in C$ *and any* $x, x' \in C$,

$A \wedge x \wedge A = A \wedge x' \wedge A$ *if and only if* $B \wedge x \wedge B = B \wedge x' \wedge B$ *and, exist* $b \in B$ *such that*

$$A \wedge (x \vee b \vee x) \wedge A = A \wedge (x' \vee b \vee x') \wedge A.$$

Coset Laws

Let **S** be a skew lattice. The following statements are equivalent:

(i) **S** is categorical,

(ii) given any skew chain $\{ A > B > C \}$ in **S**, $a \in A, b \in B, c \in C$ and any $x, x' \in C$,

$A \wedge x \wedge A = A \wedge x' \wedge A$ if and only if $B \wedge x \wedge B = B \wedge x' \wedge B$ and

$$A \wedge \overline{\varphi^{-1}}(x) \wedge A = A \wedge \overline{\varphi^{-1}}(x') \wedge A.$$

Index of a Coset

When we consider a skew lattice **S** with comparable $\mathcal{D}$ classes **A** $>$ **B** we may look at the size of the existing cosets.

$$b \vee A \vee b = \{\, a \in A : a \geq b \,\}, \text{ with } b \in B$$

and dually

$$a \wedge B \wedge a = \{\, b \in M : a \geq b \,\}, \text{ with } a \in A$$

Index of a Coset

When we consider a skew lattice **S** with comparable $\mathcal{D}$ classes **A** $>$ **B** we may look at the size of the existing cosets.

$$b \vee A \vee b = \{\, a \in A : a \geq b \,\}\text{, with } b \in B$$

and dually

$$a \wedge B \wedge a = \{\, b \in M : a \geq b \,\}\text{, with } a \in A$$

All image sets of elements from one class have equal size

Index of a Coset

When we consider a skew lattice **S** with comparable $\mathcal{D}$ classes **A** $>$ **B** we may look at the size of the existing cosets.

$$b \vee A \vee b = \{\, a \in A : a \geq b \,\}, \text{ with } b \in B$$

and dually

$$a \wedge B \wedge a = \{\, b \in M : a \geq b \,\}, \text{ with } a \in A$$

All image sets of elements from one class have equal size, ie,

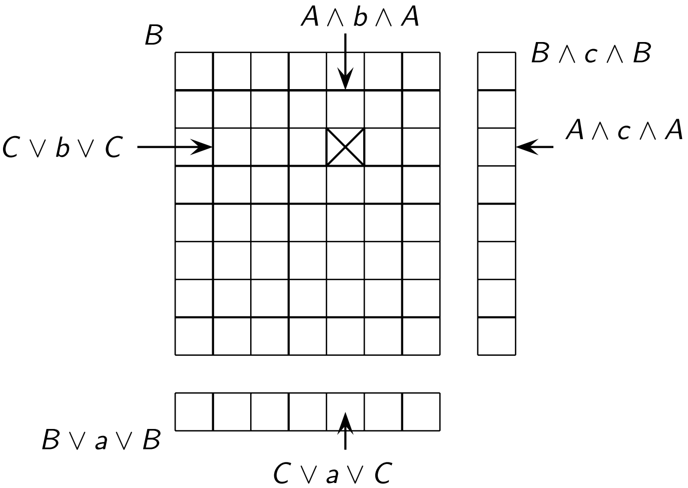
$$\text{for all } b, b' \in B,\ \left|\{\, a \in A : a \geq b \,\}\right| = \left|\{\, a \in A : a \geq b' \,\}\right|$$

Index of a Coset

Definition

The index of B in A, $[B : A] = |\{ a \in A : a \geq b \}|$, equals the number of B-cosets in A. Dually, we define the index of A in B, $[A : B] = |\{ b \in B : a \geq b \}|$.

Egg box


$A \wedge b \wedge A$
B
$B \wedge c \wedge B$
$C \vee b \vee C$
$A \wedge c \wedge A$
$B \vee a \vee B$
$C \vee a \vee C$

Index Theorem

$$[A : C]/[B : C] \in \mathbb{N}$$

Index Theorem

$$[A : C] = n.[B : C] \text{ with } n \in \mathbb{N}$$

Index Theorem

$$[A : C] = [A : B].[B : C]$$

Index Theorem

Theorem

Let **S** *be a cancellative skew lattice. Given* $\{ A > B > C \}$ *in* **S**,

$$[A : C] = [A : B].[B : C]$$

Index Theorem

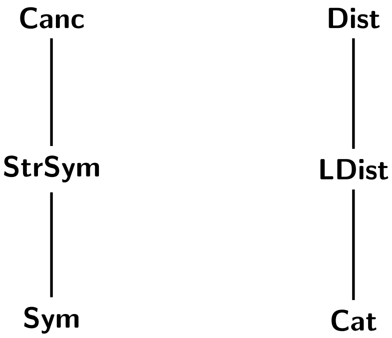
Theorem

Let **S** *be a cancellative skew lattice. Given* $\{ A > B > C \}$ *in* **S**,

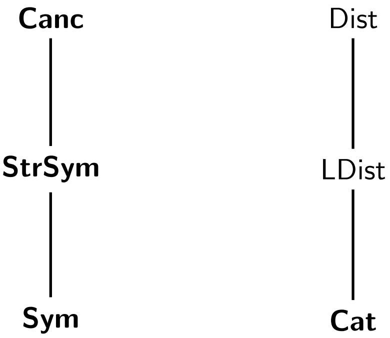
$$[A : C] = [A : B].[B : C]$$

and dually, $[C : A] = [C : B].[B : A]$.

Some subvarieties of Skew Lattices



Some subvarieties of Skew Lattices



De¡kujeme

An Example in Skew Lattices in Rings of Matrices

Let F be a field with characteristic different from 2, $n \in \mathbb{N}$ and **S** a right handed skew lattice in $M_n(F)$.

An Example in Skew Lattices in Rings of Matrices

Let F be a field with characteristic different from 2, $n \in \mathbb{N}$ and **S** a right handed skew lattice in $M_n(F)$.If **S** has two comparable $\mathcal{D}$-classes $A > B$ then given $a \in A$ and $b \in B$,

$$bA = \{ ba : a \in A \} \text{ is the coset of } A \text{ in } B \text{ and}$$

$$B \circ a = \{ b + a - ba : b \in B \} \text{ is the coset of } B \text{ in } A.$$

An Example in Skew Lattices in Rings of Matrices

The standard form for right handed skew lattices in $M_n(F)$ is described as follows:

An Example in Skew Lattices in Rings of Matrices

The standard form for right handed skew lattices in $M_n(F)$ is described as follows:Let $E_1 < ... < E_m$ be a maximal chain of $\mathcal{D}$-classes of the skew lattice **S**.

An Example in Skew Lattices in Rings of Matrices

The standard form for right handed skew lattices in $M_n(F)$ is described as follows:Let $E_1 < ... < E_m$ be a maximal chain of $\mathcal{D}$-classes of the skew lattice **S**. Then a basis for F^n exists such that in this basis:

(*i*) for any two matrices $a \in E_i$, $b \in E_j$, $i > j$, a block decomposition exists such that a and b have block forms

$$a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} I & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

An Example in Skew Lattices in Rings of Matrices

(*ii*) for non-comparable $\mathcal{D}$-classes A and B with the meet class M and the join class J a block decomposition exists such that we may assume that $m_0 \in M$, $a_0 \in A$, $b_0 \in B$ and $j_0 \in J$, where

$$m_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad j_0 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An Example in Skew Lattices in Rings of Matrices

Furthermore, given any matrices $m \in M$, $j \in J$, $a \in A$ and $b \in B$ they have block forms

$$m = \begin{bmatrix} I & m_{12} & m_{13} & m_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a = \begin{bmatrix} I & 0 & a_{13} & a_{14} \\ 0 & I & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} I & b_{12} & 0 & b_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad j = \begin{bmatrix} I & 0 & 0 & j_{14} \\ 0 & I & 0 & j_{24} \\ 0 & 0 & I & j_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An Example in Skew Lattices in Rings of Matrices

$$m = \begin{bmatrix} I & x & y & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} I & 0 & w & u \\ 0 & I & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An Example in Skew Lattices in Rings of Matrices

Thus

$$(m\nabla a)j_0 = \begin{bmatrix} I & 0 & y & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An Example in Skew Lattices in Rings of Matrices

Thus

$$(m \nabla a) j_0 = \begin{bmatrix} I & 0 & y & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

On the other hand,

$$m j_0 = \begin{bmatrix} I & x & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m a_0 = \begin{bmatrix} I & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An Example in Skew Lattices in Rings of Matrices

Thus

$$(m\nabla a)j_0 = \begin{bmatrix} I & 0 & y & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

On the other hand,

$$mj_0 = \begin{bmatrix} I & x & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \quad ma_0 = \begin{bmatrix} I & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Therefore,

$$mJ = m'J \text{ iff for all } a \in A,\ (m\nabla a)J = (m'\nabla a)J \text{ and } mA = m'A.$$

A short story

Pascual Jordan 1949 *"Uber Nichtkommutative Verbände"*

A short story

Pascual Jordan 1949 *"Uber Nichtkommutative Verbände"*
V. Slavík 1973 *"On skew lattices I & II"*

A short story

Pascual Jordan	1949 *"Uber Nichtkommutative Verbände"*
V. Slavík	1973 *"On skew lattices I & II"*
W. H. Cornish	1980 *"Boolean skew algebras"*

A short story

Pascual Jordan	1949 "*Uber Nichtkommutative Verbände*"
V. Slavík	1973 "*On skew lattices I & II*"
W. H. Cornish	1980 "*Boolean skew algebras*"
Johnathan Leech	1989 "*Skew lattices in rings*"

A short story

Pascual Jordan	1949 *"Uber Nichtkommutative Verbände"*
V. Slavík	1973 *"On skew lattices I & II"*
W. H. Cornish	1980 *"Boolean skew algebras"*
Johnathan Leech	1989 *"Skew lattices in rings"*
	1990 *"Skew Boolean algebras"*

A short story

Pascual Jordan	1949 *"Uber Nichtkommutative Verbände"*
V. Slavík	1973 *"On skew lattices I & II"*
W. H. Cornish	1980 *"Boolean skew algebras"*
Johnathan Leech	1989 *"Skew lattices in rings"*
	1990 *"Skew Boolean algebras"*
	1992 *"Normal skew lattices"*

A short story

Pascual Jordan	1949 "*Uber Nichtkommutative Verbände*"
V. Slavík	1973 "*On skew lattices I & II*"
W. H. Cornish	1980 "*Boolean skew algebras*"
Johnathan Leech	1989 "*Skew lattices in rings*"
	1990 "*Skew Boolean algebras*"
	1992 "*Normal skew lattices*"
	1993 "*The geometry of skew lattices*"

A short story

Pascual Jordan	1949 *"Uber Nichtkommutative Verbände"*
V. Slavík	1973 *"On skew lattices I & II"*
W. H. Cornish	1980 *"Boolean skew algebras"*
Johnathan Leech	1989 *"Skew lattices in rings"*
	1990 *"Skew Boolean algebras"*
	1992 *"Normal skew lattices"*
	1993 *"The geometry of skew lattices"*
Karin Cvetko-Vah	2005 *"Skew lattices in rings [PhD Thesis]"*

A short story

Pascual Jordan	1949 "*Uber Nichtkommutative Verbände*"
V. Slavík	1973 "*On skew lattices I & II*"
W. H. Cornish	1980 "*Boolean skew algebras*"
Johnathan Leech	1989 "*Skew lattices in rings*"
	1990 "*Skew Boolean algebras*"
	1992 "*Normal skew lattices*"
	1993 "*The geometry of skew lattices*"
Karin Cvetko-Vah	2005 "*Skew lattices in rings [PhD Thesis]*"
J. Leech & K. Cvetko-Vah & M. Kinyon & M. Spinks	2010 "*Cancellation in skew lattices*"

A short story

Pascual Jordan	1949 *"Uber Nichtkommutative Verbände"*
V. Slavík	1973 *"On skew lattices I & II"*
W. H. Cornish	1980 *"Boolean skew algebras"*
Johnathan Leech	1989 *"Skew lattices in rings"*
	1990 *"Skew Boolean algebras"*
	1992 *"Normal skew lattices"*
	1993 *"The geometry of skew lattices"*
Karin Cvetko-Vah	2005 *"Skew lattices in rings [PhD Thesis]"*
J. Leech & K. Cvetko-Vah & M. Kinyon & M. Spinks	2010 *"Cancellation in skew lattices"*
KCV & JPC	2010 *"On coset laws for skew lattices"*

A short story

Pascual Jordan	1949 "*Uber Nichtkommutative Verbände*"
V. Slavík	1973 "*On skew lattices I & II*"
W. H. Cornish	1980 "*Boolean skew algebras*"
Johnathan Leech	1989 "*Skew lattices in rings*"
	1990 "*Skew Boolean algebras*"
	1992 "*Normal skew lattices*"
	1993 "*The geometry of skew lattices*"
Karin Cvetko-Vah	2005 "*Skew lattices in rings [PhD Thesis]*"
J. Leech & K. Cvetko-Vah & M. Kinyon & M. Spinks	2010 "*Cancellation in skew lattices*"
KCV & JPC	2010 "*On coset laws for skew lattices*"
M. Spinks & M. Kinyon	2010 "*Distributivity in Skew Lattices*"