On finite distributive congruence lattices

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Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all lattices isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K}$ = Con $\mathcal{L}$
(Con $\mathcal{K} \subseteq$ Con $\mathcal{L}$)

Especially, for finitely generated varieties $\mathcal{K}$, $\mathcal{L}$ we have an algorithmic problem.

Necessary condition

In the sequel: $\mathcal{V}\ldots$ a finitely generated CD variety;
SI$(\mathcal{V})\ldots$ the family of subdirectly irreducible members;
$M(L)\ldots$ completely $\wedge$-irreducible elements of a lattice L.

Lemma

Let $L\in$Con$\mathcal{V}$. Then for every $x \in M(L)$, the lattice $\uparrow x$ is isomorphic to $\mathrm{Con}\, T$ for some $T \in SI(\mathcal{V})$.

Congruence-maximal varieties

On the finite level (for finite L), the necessary condition is sometimes also sufficient. In such a case we say that $\mathcal{V}$ is *congruence-maximal*. Formally, $\mathcal{V}$ is congruence-maximal, if for every finite distributive lattice L the following two conditions are equivalent:

(i) $L \in \operatorname{Con} \mathcal{V}$;

(ii) for every $x \in M(L)$, the lattice $\uparrow x$ is isomorphic to $\operatorname{Con} T$ for some $T \in SI(\mathcal{V})$.

Easy case

Theorem

Let $\mathcal{V}$ be a congruence-distributive variety with the property that $\operatorname{Con} C$ is a finite chain for every $C \in \mathrm{SI}(\mathcal{V})$ and $n = \max\{\mathrm{length}(\operatorname{Con} C) \mid C \in \mathrm{SI}(\mathcal{V})\}$. Let L be a finite distributive lattice. The following conditions are equivalent.

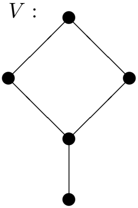
(i) $L \in \operatorname{Con} \mathcal{V}$;

(ii) *For every $x \in M(L)$, the set $\uparrow x$ is a chain of the length at most n.*

Examples: distributive lattices, Stone algebras ...

The simplest of the difficult cases

In the sequel, suppose that every algebra in $\mathrm{SI}(\mathcal{V})$ is simple or has the congruence lattice isomorphic to



Necessary condition specified

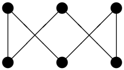
For every $A \in \mathcal{V}$, $L = \operatorname{Con} A$,

(NC) $M(L)$ is a disjoint union of two antichains $D \cup N$ and for every $n \in N$ there are exactly two $d, e \in D$ with $n < d, e$.

So, $\mathcal{V}$ is congruence-maximal iff every finite distributive lattice L satisfying (NC) belongs to Con$\mathcal{V}$.

Example: the variety $\mathcal{N}_5$ generated by the 5-element nonmodular lattice N_5 is congruence-maximal.

Let $\mathcal{V}$ contain only one algebra A with Con$A = V$, such that the two nontrivial subdirectly irreducible quotients of A are not isomorphic. Then L with $M(L)$ equal to



Compatible families

Let A be a subset of $B \times B$ for some set B. Let X be a set and let $\mathcal{F}$ be a set of functions $X \to B$. We say that $\mathcal{F}$ is *A-compatible* if $\{f(x), g(x)) \mid x \in X\} = A$ or $\{(g(x), f(x)) \mid x \in X\} = A$ for every $f, g \in \mathcal{F}$, $f \neq g$.

Lemma

*(*P. Gillibert*) Suppose that $A \subseteq B \times B$ contains a pair (a, b) with $a \neq b$. Then the following condition are equivalent.*

(i) *There exist arbitrarily large finite A-compatible sets of functions.*

(ii) *For every $(a, b) \in A$ there are $x, y, z \in B$ such that $(x, x), (y, y), (z, z), (x, y),$ $(x, z), (y, z), (x, a), (x, b), (a, y), (y, b), (a, z), (b, z) \in A$.*

Let $\mathcal{V}$ satisfy the above conditions.

Theorem

$\mathcal{V}$ is congruence-maximal iff there exist $B, C \in \mathcal{V}$ and surjective homomorphisms $h_0, h_1 : C \to B$ such that

(i) *B is simple, $\operatorname{Con} C = V$;*

(ii) $\operatorname{Ker}(h_0) \neq \operatorname{Ker}(h_1)$*;*

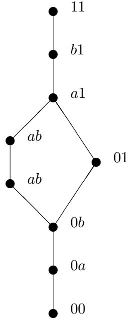
(iii) *there are arbitrarily large A-compatible sets of functions for $A = \{(h_0(x), h_1(x)) \mid x \in C\} \subseteq B \times B$.*

Positive example

For $\mathcal{V} = \mathcal{N}_5$ we have $B = \{0,1\}$, $A = \{(0,0),(0,1),(1,0),(1,1)\}$ so almost every family of functions is compatible and $\mathcal{V}$ is congruence-maximal.

Negative example

Consider the following lattice C with two additional unary operations.



$f(00) = 00, f(0a) = 0b, f(0b) = f(01) = 01$

$f(ab) = f(a1) = b1, f(b1) = f(11) = 11$

$g(11) = 11, g(b1) = a1, g(a1) = g(01) = 01$

$g(ab) = g(0b) = 0a, g(0a) = g(00) = 00$

Negative example 2

For the variety $\mathcal{C}$ generated by C we have $B = \{0, 1, a, b\}$, $A = \{(0, 0), (0, a), (0, b), (a, b), (0, 1), (a, 1), (b, 1), (1, 1)\}$ (the labels on the elements of C), and the pair (a, b) violates Gillibert's condition. Thus, $\mathcal{C}$ is not congruence-maximal.

Problem

Find a finitely generated CD-variety $\mathcal{V}$ such that one SI-member has the congruence lattice isomorphic to V and all other SI-members are simple, which is not congruence-maximal, but $\operatorname{Con} \mathcal{V}$ contains L with $M(L)$ equal to