Minimal and minimal compact left distributive groupoids

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Minimal vs minimal compact

Definition. A universal algebra is:
(i) *minimal* if it includes no proper subalgebras,
(ii) *minimal compact* if it is compact and includes no proper compact subalgebras.

Fact. *Any compact Hausdorff algebra includes a minimal compact subalgebra.*

Proof. Apply Zorn's Lemma.

In the sequel, topological spaces are Hausdorff.

Both types of minimality can display a similarity. We discuss some examples.

Example 1: Semigroups.

Fact. *If a semigroup is minimal, then it consists of a unique element.*

A groupoid is *left topological* if all its left translations are continuous.

Theorem. *If a left topological semigroup is minimal compact, then it consists of a unique element.*

Corollary (Ellis). *Any compact left topological semigroup has an idempotent.*

This leads to *idempotent ultrafilters*, which are important for applications.

Ultrafilters: topology and algebra.

The set βX of ultrafilters over a set X carries a natural topology generated by sets

$$\{u \in \beta X : A \in u\}$$

for all $A \subseteq X$.

Fact. *The space βX is the Stone–Čech compactification of the discrete space X.*

Letting $X \subseteq \beta X$, every unary operation F on X extends to a continuous operation on βX. One can compute $F(u)$ explicitly:

$$F(u) = \Big\{S : \{x : F(x) \in S\} \in u\Big\}.$$

If $\cdot$ is a binary operation, the extension can be fulfilled in two steps: first one extends right translations, then left ones. Explicitly:

$$uv = \Big\{S : \{x : \{y : xy \in S\} \in u\} \in v\Big\}.$$

Fact. *The groupoid $(\beta X, \cdot)$ is left topological. Moreover, its right translations by principal ultrafilters are continuous, and such an extension is unique.*

[Similarly for all universal algebras.]

Many algebraic properties are not stable under β. However, associativity *is* stable:

Lemma. *If X is a semigroup, so is βX.*

Thus any semigroup X extends to the compact left topological semigroup βX of ultrafilters. Applying Ellis' result, one gets

Theorem. *Any semigroup carries an idempotent ultrafilter.*

Idempotent ultrafilters are crucial for applications in number theory, algebra, topological dynamics, and ergodic theory.

Popular examples:

van der Waerden's and Szemerédi's theorems on arithmetic progressions,

Hindman's theorem on finite sums,

Hales–Jewett's theorem on free semigroups,

Furstenberg's theorem on common recurrence,

etc. Many results have no (known) elementary proofs.

Example 2: Semirings.

$(X,+,\cdot)$ is a *left semiring* if each of its groupoids is a semigroup, and $\cdot$ is left distributive w.r.t. $+$:

$$x(y+z) = xy + xz.$$

Right semirings: defined dually.
Semirings: left and right simultaneously.

$(X,+,\cdot)$ is *left topological* if so is each of its groupoids. The following generalizes Ellis' result:

Theorem. *If a left topological left semiring is minimal compact, then it consists of a unique element.*

Corollary. *Any compact left topological left semiring has a common (i.e. additive and multiplicative simultaneously) idempotent.*

Algebraic counterpart.

Question. *Can a minimal left semiring have more than one element?*

I was able to produce the expected answer *No* only in partial cases:

(i) *if the left semiring is finite,*
(ii) *if it is a semiring.*

(i): by Theorem (consider the discrete topology),
(ii): by different arguments.

How large can other “minimal” algebras be?

Minimal groupoids. Any size $\leq \aleph_0$ is possible. E.g. the following is a countable minimal quasi-group:

$*$	0	1	2	3	4	5	6	...
0	1	0	2	3	4	5	6	
1	0	2	1	5	3	4	8	
2	2	1	3	0	6	7	4	
3	3	5	0	4	1	2	9	
4	4	3	6	1	5	0	2	
5	5	4	7	2	0	6	1	
6	6	8	4	9	2	1	7	
...								

Minimal compact groupoids. An expected value:

$$2^{2^{\aleph_0}}$$

(= the largest cardinality of a separable space). We shall see that this size is possible.

Left distributivity

Definition. A groupoid is *left distributive* if its operation is left distributive w.r.t. itself:

$$x(yz) = (xy)(xz).$$

Right distributive groupoids: defined dually.
Distributive groupoids: left and right distributive.

Investigated from 80s by:

Matveev, Joyce (knot theory),

Ježek, Kepka, Jeřábek, Jedlička, Stanovský
(distributivity, left distributive left quasigroups),

Laver, Dehornoy, Dougherty, Jech
(set theory, free left distributive groupoids).

The most intriguing problem: *Can large cardinals be eliminated from the proof of the freeness of the inverse limit of Laver groupoids?* It remains widely open.

Minimal

left distributive groupoids

A simple construction.
Given any groupoid X and $a \in X$, put

$$x * y = ay.$$

The groupoid $(X, *)$ is left distributive.

Taking the additive groups $\mathbb{Z}_n$ and their units 1 as X and a, we get a series of left distributive groupoids

$$\begin{array}{c|c} * & 0 \\ \hline 0 & 0 \end{array} \qquad \begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{array} \qquad \begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 3 & 0 \end{array} \qquad \ldots$$

Obviously, all they are minimal.
The converse is less obvious:

Theorem. *Any minimal left distributive groupoid is (isomorphic to) one of these instances.*

In particular, there exist no infinite minimal left distributive groupoids.

Proof (scetch). Based on the following facts:

(i) *Any left distributive groupoid satisfies*

$$(x^m)^n = x^{m+n-1},$$

where x^n denotes the nth *right power* of x, defined inductively: $x^1 = x$, $x^{n+1} = xx^n$.

(ii) *Any minimal left distributive groupoid is left divisible*, i.e. satisfies $\exists y\, xy = z$.

(iii) *Any left divisible left distributive groupoid is left idempotent*, i.e. satisfies $x^2y = xy$.

(iv) *Any left idempotent groupoid satisfies* $x^m y = xy$ *and so* $x^m x^n = x^{n+1}$.

[*Remark.* All left distributive groupoids satisfy this for $m \leq n$.]

It follows

(v) *If a left distributive* X *is minimal and* $a \in X$*, then* $X = \{a^n : n \geq 1\}$*.*

The rest of the proof:

Pick any $a, b \in X$.

By (v), $a = b^n$ and $b = a^m$.

By (i), $a = (a^m)^n = a^{m+n-1}$.

Therefore

$$|X| \leq m + n - 1$$

and the mapping

$$a^i \mapsto i$$

is an isomorphism of $(X, \cdot)$ onto $(|X|, *)$ where $i * j = 1 + j \bmod |X|$.

This completes the proof.

Minimal compact left distributive groupoids

Here a similarity between minimal and minimal compact groupoids loses.

Theorem. *There exists a topological minimal compact left distributive groupoid of size* $2^{2^{\aleph_0}}$*. Besides, it includes no minimal subgroupoids.*

Proof (scetch). Consider $\beta\mathbb{N}$ with the operation

$$u * v = 1 + v$$

where $+$ extends the usual addition on $\mathbb{N}$.

The groupoid is left distributive and topological (the operation is continuous since 1 is a principal ultrafilter).

Easy facts:

(i) *For any term* t *one has* $t(v, \ldots) = n + u$ *where* u *is in the right-most position in* t*, and* n *equals the depth of the occurrence of* u *in* t*.*

(ii) *For any* $u \in \beta\mathbb{N}$ *the subgroupoid generated by* u *is* $\{n + u : n \in \mathbb{N}\}$*.*

A fact of general topology: Any unary operation on X has a fixed point iff its continuous extension to βX has a fixed point.

(iii) *For any* $u \in \beta\mathbb{N}$ *all the ultrafilters* $n + u$ *are distinct.*

(iv) *Any one-generated subgroupoid of* $(\beta\mathbb{N}, *)$ *is isomorphic to* $(\mathbb{N}, *)$*.*

Consequently, $(\beta\mathbb{N}, *)$ has no minimal subgroupoids.

The rest of the proof:

Pick a minimal compact subgroupoid $(S, *)$.
By (iv), S is infinite.

A standard fact of general topology: Any infinite closed subset of $\beta\mathbb{N}$ includes a topological copy of $\beta\mathbb{N}$.

A fortiori, $|S| = 2^{2^{\aleph_0}}$.

This completes the proof.

Remark. $(\beta\mathbb{N}, *)$ is not minimal compact.

Let $D \subseteq \beta\mathbb{N}$ consist of ultrafilters whose elements are "algebraically big" in a sense (e.g. contain arbitrarily long arithmetic progressions).

It can be shown: The set D is closed nowhere dense in $\beta\mathbb{N}$, and it forms a subgroupoid of $(\beta\mathbb{N}, *)$.

Question. Can a topological minimal compact quasigroup be of size $2^{2^{\aleph_0}}$?

Question. Exists there a groupoid X such that βX or $\beta X \setminus X$ is a minimal compact groupoid?