Congruence lattices of finite intransitive group acts

Steve Seif

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Finite group acts

A finite **group act** is a **unary** algebra $\mathbf{X} = \langle X, G \rangle$, where

- G is closed under composition, and
- G consists of permutations of X.

If G acts transitively on X, $\mathbf{X}$ is said to be a **transitive group act**. Otherwise, $\mathbf{X} = \langle X_1 \sqcup \ldots \sqcup X_n, G \rangle$ is an **intransitive group act**, having $n > 1$ components $X_1, \ldots, X_n$, with each component being a minimal subalgebra of $\mathbf{X}$.

A (transitive) monoid act $\langle X; M \rangle$ can be defined in like manner, but with M a monoid rather than a group.

Background

The following are well known.

(1) L is the congruence lattice of some finite algebra implies L is the congruence lattice of a finite monoid act.

(2) If $\mathbf{X} = \langle X, G\rangle$ *is a transitive group act, theren there exists a subgroup* H *of* G *such that* $Con(\mathbf{X})$ *is isomorphic to* $I[H, G]$*, an interval in* $Sub(G)$*, the lattice of subgroups of* G*.*

A theorem of P.P. Palfy and P. Pudlak states that

(3) Every finite lattice is isomorphic to the congruence lattice of some finite algebra if and only if every finite lattice is isomorphic to the congruence lattice of some finite **transitive** *group act.*

Palfy-Pudlak

> *Every finite lattice is isomorphic to the congruence lattice of some finite algebra if and only if every finite lattice is isomorphic to the congruence lattice of some finite* **transitive** *group act.*

The proof of their theorem indicates that

> *(4) If there exist finite lattices that are not isomorphic to the congruence lattice of some transitive group act, then there are finite lattices not isomoprhic to the congruence lattice of* **any** *finite group act.*

Apparently congruence lattices of finite **transitive** group acts are of special interest.

What (if anything) of interest can be said about congruence lattices of finite intransitive group acts?

Finite lattices that force transitivity

Apparently congruence lattices of finite **transitive** group acts are of special interest. But what (if anything) of interest can be said about congruence lattices of finite intransitive group acts?

Not every finite lattice can be represented as the congruence lattice of a finite **intransitive** group act. This follows from a more general result of the speaker's concerning monoid acts and their congruence lattices.

> *(5) There exists a finite lattice L such that if $L \cong Con(\langle X; M\rangle)$, then M acts transitively on X.*

Transitivity forcing

In fact, the speaker has proven that

> *(5) If a finite lattice L is* **not semimodular** *but every proper subinterval of L is semimodular, then $L \cong Con(\langle X; M \rangle)$ implies that M acts transitively on X.*

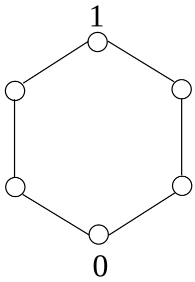

Figure: Not representable as the congruence lattice of a transitive monoid act

(5) If a finite lattice L is **not semimodular** *but every proper subinterval of L is semimodular, then $L \cong Con(\langle X; M \rangle)$ implies that M acts transitively on X.*

(5) above limits lattices that can be congruence-represented by a finite intransitive group act.

Within which classes of lattices (e.g. distributive, modular, ..) are the lattices that are congruence–representable by finite intransitive group acts decidable? Perhaps some classes are decidable with some help– e.g. an oracle that can determine if a finite lattice is congruence- representable by a finite **transitive** group act.

Distributive lattices

Under what assumptions, within which classes of lattices, are the lattices that are congruence–representable by finite intransitive group acts decidable?
If we restrict to finite distirbutive lattices, there's very smooth sailing.

> *A finite distributive lattice is congruence-representable by a finite intransitive group act if and only if it has a unique co-atom.*

The above follows as a special case of a more general result, one that will be described.

Preparation for main definitions, results

Let $\mathbf{X} = \langle Y \sqcup Z; G \rangle$ be a group act having **two components**. For $c, d \in X$, let's examine the **principal congruence** $Cg(c, d)$.

Suppose c and d are in the **same component** say in Y. Then $Cg(c, d) \in Con(\mathbf{Z})$ corresponds to the obvious congruence of $\mathbf{Y}$ (namely $Cg(c, d)|_Y$).

This leads to the trivial observation that $Con(\mathbf{Y}) \times Con(\mathbf{Z})$ is isomorphic to an ideal $I[\Delta, \kappa]$ of $Con(\mathbf{X})$, where κ is a maximal congruence of $\mathbf{X}$ that collapses each component to a point.

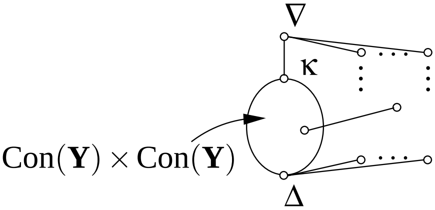

Figure: Con($\mathbf{X}$)

Two components, continued

$\mathbf{X} = \langle Y \sqcup Z; G \rangle$ still. But now c and d are in **different components**.

Lemma

Suppose $c, d \in X$ *and* $X_c \neq X_d$. *Then* $\mathbf{X}/Cg(c,d) \cong \mathbf{Y}/Cg(c,d)|_Y \cong \mathbf{Z}/Cg(c,d)|_Z$.

Proof: We show that there's an isomorphism from $\mathbf{X}/Cg(s,t)|_X$ to $\mathbf{Z}/Cg(s,t)$. Look quickly. Here it is: for all $x \in X$, let $x/Cg(s,t)|_X \to x/Cg(s,t)$. $\square$

The transitivity of the two actions is all that's needed in the proof: The lemma is valid when $\langle Y; G \rangle$ and $\langle Z; G \rangle$ are transitive monoid acts. I'll come back to this theme— most of the results here are valid for certain intransitive monoid acts.

Two classes of examples

Recall the Lemma: Lemma: Suppose $c, d \in X$ and $X_c \neq X_d$. Then $\mathbf{X}/Cg(c,d) \cong \mathbf{Y}/Cg(c,d)|_Y \cong \mathbf{Z}/Cg(c,d)|_Z$.

First class of examples: If If $|Y|$, $|Z|$ are relatively prime and $s, t \in X$ with $X_s \neq X_t$, then $Cg(s,t) = \nabla$.

Proof. Since transitive group acts are congruence regular, that $|Y|, |Z|$ are rel prime implies that $\mathbf{Y}$ and $\mathbf{Z}$ have only one common homomorphic image, namely the trivial group act. By the Lemma, if c, d are in distinct components, then $Cg(c,d)$ contains $Y \times Y \cup Z \times Z$; it contains (c,d), so it must be ∇.

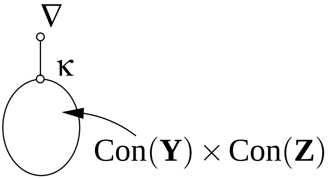

Figure: $|\mathbf{Y}|$, $|\mathbf{Z}|$ rel prime

Second class of examples: If $Y = Z$ and the action of G on both copies is the same, then $\mathbf{X}$ has two kinds of minimal congruences: Those arising from minl congruences of $\langle Y; G\rangle$, and those coming from automorphisms of $\langle Y; G\rangle$.

Proof sketch. If $\alpha \in Con(\mathbf{X})$ is minimal and not below κ, it follows that $\alpha = Cg(c, d)$, where $c \in Y$ and $d \in Z$ and that $Cg(c, d)|_Y = \Delta_Y$ and $Cg(c, d)|_Z = \Delta_Z$. By the Lemma, $\mathbf{Y} \cong \mathbf{Z}$, and α is associated with the automorphism that sends c to d. The other inclusion is just as easy.

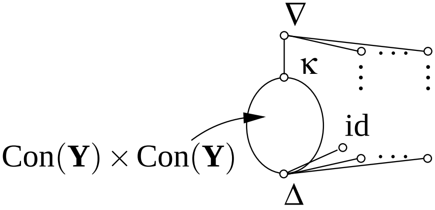

Figure: Automorphisms encoded

Definitions

Definition: **Property K**

$\mathbf{X} = \langle X_1 \sqcup \ldots \sqcup X_n; G \rangle$ is said to satisfy **Property K** if for all $c, d \in X$ with $X_c \neq X_d$, the only common homomorphic image (up to isomorphism) of $\mathbf{X_c}$ and $\mathbf{X_d}$ is the trivial group act.

Lemma 2: $\mathbf{X}$ satisfies Property K iff for all $c, d \in X$ such that $X_c \neq X_d$, the congruence $Cg(c, d)$ contains $X_c \times X_c$.

Definition: Π–**product lattices**

Let $L_1, \ldots, L_n$ be a sequence of finite lattices; let the bottom and top of L_i be denoted $0_i, 1_i$ respectively. Let $\Pi(n)$ be the lattice of partitions of $\{1, \ldots, n\}$. The Π–**product** sublattice of $L_1 \ldots L_n \times \Pi(n)$, denoted $\Pi(L_1, \ldots, L_n)$, is defined:

> *Let* $\Pi(L_1, \ldots, L_n)$ *consist of all tuples of the form* $(a_1, \ldots, a_n, \alpha)$ *where* i *and* j *are identified by* α *implies* $a_i = 1_i$ *and* $a_j = 1_j$.

Congruence lattices that are Π-product lattices

A. $\mathbf{X}$ satisfies Property K iff for all $c, d \in X$ such that $X_c \neq X_d$, the congruence $Cg(c, d)$ contains $X_c \times X_c$.

B. $\Pi(L_1, \ldots, L_n)$ consist of all tuples of the form $(a_1, \ldots, a_n, \alpha)$ where i and j are identified by α implies $a_i = 1_i$ and $a_j = 1_j$.

The next observation is easy to prove.
Observation If $\mathbf{X} = \langle X_1 \sqcup \ldots X_n, G \rangle$ satisfies Property K, then $Con(\mathbf{X})$ is isomorphic to $\Pi(Con(\mathbf{X_1}), \ldots, Con(\mathbf{X_n}))$.
That the converse is true is a bit surprising.

Theorem
A finite intransitive group act has congruence lattice isomorphic to a Π-product lattice if and only if it satisfies Property K.

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The above theorem correlates a property of a congruence lattices with a property of the algebras under discussion. The second theorem above has a long statement but is easier to prove, and indicates that algebras with congruence lattices that are Π–product lattices can be easily constructed.

The above results, while nice enough, still do not say anything really interesting about finite lattices. We need a lattice property that "forces Property K", without mention of Π–product lattices or of Property K.

The 2–Chain condition on lattices

A class of lattices that generalizes the so-called graded finite lattices is defined.

> *A finite lattice L satisfies the* **2-Chain condition** *if $a \succ b \succ c$ in L implies that the interval $I[a, c]$ is isomorphic to M_n, some $n \geq 1$.*

Most of the classical lattices are graded lattices, so satisfy the 2-Chain condition.

Let $Y = \langle \{0, 1\}, C_2 \rangle$, the 2-element cyclic grouip's transitive act, and $\mathbf{X} = \langle Y \sqcup Y, C_2 \rangle$.
DIAGRAM 5: Congruence lattice of $\mathbf{X}$

Theorem

Any finite intransitive group action **X** *whose congruence lattice satisfies the 2–Chain condition satisfies Property K and (therefore) has a congruence lattice isomorphic to a* Π*–product lattice.*

Corollary

A finite latice L satisfying the 2–Chain condition is congruence-representable by a finite intransitive group act if and only if L is isomorphic to a Π*-product lattice* $\Pi(L_1, \ldots, L_n)$*, and for* $i = 1, \ldots, n$*,* L_i *satisfies the 2–Chain condition and is congruence-representable by a finite transitive group act.*

Corollary

With the help of the oracle O that determines if a finite lattice is congruence-representable by a transitive group act, the problem with instances finite 2-Chain condition satisfying lattices and question "Is the lattice congruence-representable by a finite intransitive group act?" is decidable.

The general case

Given a finite lattice L, it turns out there is a computable function that returns

1. nothing, or
2. a Π-product lattice $\Pi(L_1, \ldots, L_n)$, one that is isomorphic to a densely embedded $0, 1$ sublattice of L

such that if L actually is the congruence lattice of some finite intransitive group act, then $\mathbf{X}$ has n components, and if $n > 2$, then $L_i \cong Con(\mathbf{X_i})$, for $i = 1, \ldots, n$ (after possibly some reordering).

The above still does not lead the speaker to make any positive conjectures.

Given a finite lattice L, is there a transitive group action $\langle X; G\rangle$ such that $Con(\langle Y \sqcup Y; G\rangle) \cong L$, where the action of G is the same on the two copies of X?. This problem is undecidable, I conjecture, even given with the oracle O.

Lemma If the problem above is undecidable, the problem of determining whether a finite lattice is congruence-representable by a of a finite intransitive group act is undecidable, even with oracle O.

Conclusion

It has been shown that questions regarding lattices that are congruence-representable by finite intransitive group acts revolve around Π-product lattices.

- The 2-Chain condition, Property K, and Π-product lattices are intimately related in finite intransitive group acts.
- Π-product lattices are the "skeleton" for the congruence lattices of y finite intransitive group acts .
- Automorphisms of components and their homomorphic images play a role in fleshing out their congruence lattices.
- Someone who knows more than the speaker about finite groups will show that that the problem of deciding whether a finite lattice is congruence-representable by a finite intransitive group act is undecidable, even with oracle O.

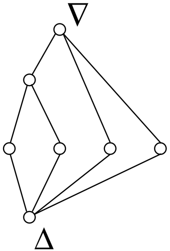

Figure: Con(**X**): Fails 2–chain condition