Congruence modularity at 0

Benedek Skublics

University of Szeged
Bolyai Institute

International Conference on Algebras and Lattices
Prague, 2010

$\mathcal{V}$ a variety; Mal'tsev condition

$\mathcal{V}$ a variety; Mal'tsev condition
If $\mathcal{V}$ is

- congruence permutable (A.I. Mal'tsev)
- arithmetical (A.F. Pixley)
- congruence n-permutable (J. Hagemann and A. Mitschke)
- congruence distributive (B. Jónsson)
- congruence modular (A. Day)

then $\mathcal{V}$ can be characterized by a Mal'tsev condition.

$\mathcal{V}$ a variety with 0

the above congruence properties can be "localized" at 0

$\mathcal{V}$ a variety with 0
the above congruence properties can be "localized" at 0

Definition

$\lambda : p(x_1, \ldots, x_n) \leq q(x_1, \ldots, x_n)$ a lattice identity
λ holds for the congruences of $\mathcal{V}$ at 0 if for every $\mathbf{A} \in \mathcal{V}$ and for all $\alpha_1, \ldots, \alpha_n \in \mathrm{Con}\,\mathbf{A}$, we have

$$[0]p(\alpha_1, \ldots, \alpha_n) \subseteq [0]q(\alpha_1, \ldots, \alpha_n).$$

$\mathcal{V}$ a variety with 0

the above congruence properties can be "localized" at 0

Definition

$\lambda : p(x_1, \ldots, x_n) \leq q(x_1, \ldots, x_n)$ a lattice identity

λ holds for the congruences of $\mathcal{V}$ at 0 if for every $\mathbf{A} \in \mathcal{V}$ and for all $\alpha_1, \ldots, \alpha_n \in \operatorname{Con} \mathbf{A}$, we have

$$[0]p(\alpha_1, \ldots, \alpha_n) \subseteq [0]q(\alpha_1, \ldots, \alpha_n).$$

If $\mathcal{V}$ is

- congruence permutable at 0 (H.P. Gumm)
- arithmetical at 0 (J. Duda)
- congruence n-permutable at 0 (I. Chajda)
- congruence distributive at 0 (I. Chajda)
- congruence modular at 0 (B. S.)

then $\mathcal{V}$ can be characterized by a Mal'tsev condition.

Example (G. Czédli)

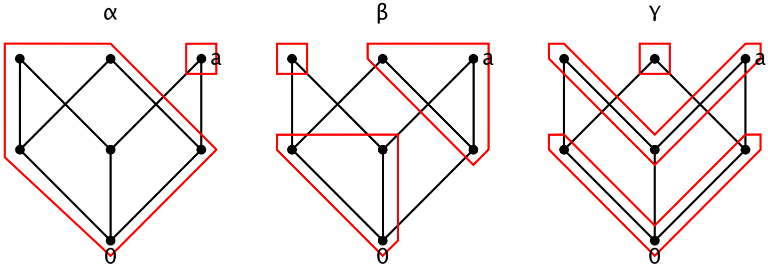
$\mathcal{S}$ the variety of meet semilattices with 0

- $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ holds for cong. of $\mathcal{S}$ at 0
- $x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$ does not hold for cong. of $\mathcal{S}$ at 0

Example (G. Czédli)

$\mathcal{S}$ the variety of meet semilattices with 0

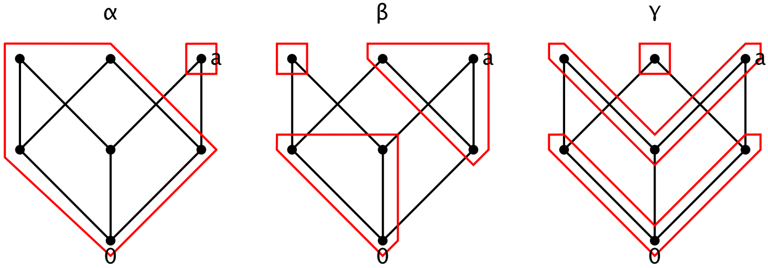
- $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ holds for cong. of $\mathcal{S}$ at 0
- $x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$ does not hold for cong. of $\mathcal{S}$ at 0



Example (G. Czédli)

$\mathcal{S}$ the variety of meet semilattices with 0

- $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ holds for cong. of $\mathcal{S}$ at 0
- $x \vee (y \wedge z) \geq (x \vee y) \wedge (x \vee z)$ does not hold for cong. of $\mathcal{S}$ at 0



$$a \in [0](\alpha \vee \beta) \wedge (\alpha \vee \gamma)$$
$$a \notin [0]\alpha \vee (\beta \wedge \gamma)$$

$\mathcal{V}$ variety with 0

$\mathcal{V}$ variety with 0

Theorem

The following conditions are equivalent:

1. *$\mathcal{V}$ is congruence modular at 0;*
2. *there are ternary terms $m_1, \ldots, m_n$ such that $\mathcal{V}$ satisfies:*

$$\begin{aligned} m_0(x,y,z) &= 0 \text{ and } m_n(x,y,z) = z; && && \text{(m1)} \\ m_i(x,x,0) &= 0 && \text{for all } i; && \text{(m2)} \\ m_i(x,x,z) &= m_{i+1}(x,x,z) && \text{for } i \text{ odd;} && \text{(m3)} \\ m_i(0,z,z) &= m_{i+1}(0,z,z) && \text{for } i \text{ even.} && \text{(m4)} \end{aligned}$$

$\mathcal{V}$ variety with 0

Theorem

The following conditions are equivalent:

1. $\mathcal{V}$ *is congruence modular at* 0;
2. *there are ternary terms* $m_1, \ldots, m_n$ *such that* $\mathcal{V}$ *satisfies:*

$$
\begin{aligned}
m_0(x,y,z) &= 0 \text{ and } m_n(x,y,z) = z; && && \text{(m1)}\\
m_i(x,x,0) &= 0 && \textit{for all } i; && \text{(m2)}\\
m_i(x,x,z) &= m_{i+1}(x,x,z) && \textit{for } i \textit{ odd;} && \text{(m3)}\\
m_i(0,z,z) &= m_{i+1}(0,z,z) && \textit{for } i \textit{ even.} && \text{(m4)}
\end{aligned}
$$

$1 \Rightarrow 2$ "as usual"

$\mathcal{V}$ variety with 0

Theorem

The following conditions are equivalent:

1. *$\mathcal{V}$ is congruence modular at 0;*
2. *there are ternary terms $m_1, \ldots, m_n$ such that $\mathcal{V}$ satisfies:*

$$\begin{aligned} m_0(x,y,z) &= 0 \ \textit{and}\ m_n(x,y,z) = z; && && \text{(m1)}\\ m_i(x,x,0) &= 0 && \textit{for all } i; && \text{(m2)}\\ m_i(x,x,z) &= m_{i+1}(x,x,z) && \textit{for } i \textit{ odd;} && \text{(m3)}\\ m_i(0,z,z) &= m_{i+1}(0,z,z) && \textit{for } i \textit{ even.} && \text{(m4)} \end{aligned}$$

$1 \Rightarrow 2$ "as usual"

$2 \Rightarrow 1$

Suppose 2 of Theorem.

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$,

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\Delta_k = \beta \circ \gamma \circ \cdots \circ \gamma \circ \beta$.

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\Delta_k = \beta \circ \gamma \circ \cdots \circ \gamma \circ \beta$.
Congruence modularity at 0: If $\alpha \geq \gamma$ then

- $[0]\alpha \wedge (\beta \vee \gamma) \subseteq [0](\alpha \wedge \beta) \vee \gamma$.

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\Delta_k = \beta \circ \gamma \circ \cdots \circ \gamma \circ \beta$.
Congruence modularity at 0: If $\alpha \geq \gamma$ then

- $[0]\alpha \wedge (\beta \vee \gamma) \subseteq [0](\alpha \wedge \beta) \vee \gamma$.
- $[0]\alpha \wedge \bigcup_{k=0}^{\infty} \Delta_k \subseteq [0](\alpha \wedge \beta) \vee \gamma$.

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\Delta_k = \beta \circ \gamma \circ \cdots \circ \gamma \circ \beta$.
Congruence modularity at 0: If $\alpha \geq \gamma$ then

- $[0]\alpha \wedge (\beta \vee \gamma) \subseteq [0](\alpha \wedge \beta) \vee \gamma$.
- $[0]\alpha \wedge \bigcup_{k=0}^{\infty} \Delta_k \subseteq [0](\alpha \wedge \beta) \vee \gamma$.

Lemma

Suppose $\alpha \geq \gamma$, $a, d \in A$, $k \in \mathbb{N}$

$$(0, a) \in (\alpha \wedge \beta) \vee \gamma, (a, d) \in \alpha \cap \Delta_k \Rightarrow (0, d) \in (\alpha \wedge \beta) \vee \gamma.$$

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\Delta_k = \beta \circ \gamma \circ \cdots \circ \gamma \circ \beta$.
Congruence modularity at 0: If $\alpha \geq \gamma$ then

- $[0]\alpha \wedge (\beta \vee \gamma) \subseteq [0](\alpha \wedge \beta) \vee \gamma$.
- $[0]\alpha \wedge \bigcup_{k=0}^{\infty} \Delta_k \subseteq [0](\alpha \wedge \beta) \vee \gamma$.

Lemma

Suppose $\alpha \geq \gamma$, $a, d \in A$, $k \in \mathbb{N}$

$$(0, a) \in (\alpha \wedge \beta) \vee \gamma, (a, d) \in \alpha \cap \Delta_k \Rightarrow (0, d) \in (\alpha \wedge \beta) \vee \gamma.$$

$k = 0$

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\Delta_k = \beta \circ \gamma \circ \cdots \circ \gamma \circ \beta$.
Congruence modularity at 0: If $\alpha \geq \gamma$ then

- $[0]\alpha \wedge (\beta \vee \gamma) \subseteq [0](\alpha \wedge \beta) \vee \gamma$.
- $[0]\alpha \wedge \bigcup_{k=0}^{\infty} \Delta_k \subseteq [0](\alpha \wedge \beta) \vee \gamma$.

Lemma

Suppose $\alpha \geq \gamma$, $a, d \in A$, $k \in \mathbb{N}$

$$(0, a) \in (\alpha \wedge \beta) \vee \gamma, (a, d) \in \alpha \cap \Delta_k \Rightarrow (0, d) \in (\alpha \wedge \beta) \vee \gamma.$$

$k = 0$
$k \to k + 1$:

Suppose 2 of Theorem. $\mathbf{A} \in \mathcal{V}$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$, $\Delta_k = \beta \circ \gamma \circ \cdots \circ \gamma \circ \beta$.
Congruence modularity at 0: If $\alpha \geq \gamma$ then

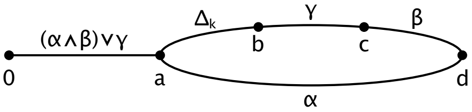
- $[0]\alpha \wedge (\beta \vee \gamma) \subseteq [0](\alpha \wedge \beta) \vee \gamma$.
- $[0]\alpha \wedge \bigcup_{k=0}^{\infty} \Delta_k \subseteq [0](\alpha \wedge \beta) \vee \gamma$.

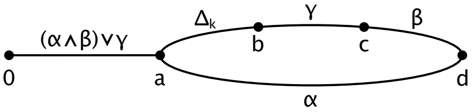
Lemma

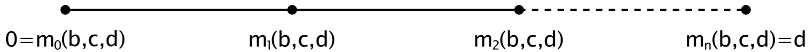
Suppose $\alpha \geq \gamma$, $a, d \in A$, $k \in \mathbb{N}$

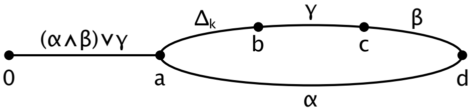
$$(0, a) \in (\alpha \wedge \beta) \vee \gamma, (a, d) \in \alpha \cap \Delta_k \Rightarrow (0, d) \in (\alpha \wedge \beta) \vee \gamma.$$

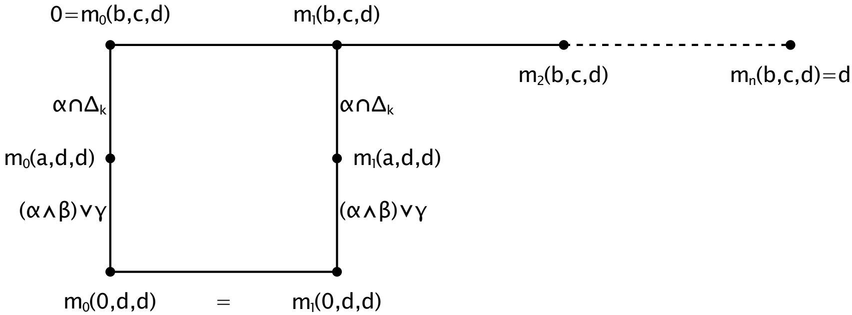
$k = 0$
$k \to k + 1$: $(a, d) \in \alpha \cap \Delta_{k+1} = \alpha \cap (\Delta_k \circ \gamma \circ \beta)$

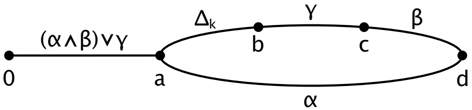

Δ_k
γ
β
$(\alpha\wedge\beta)\vee\gamma$
b
c
0
a
d
α

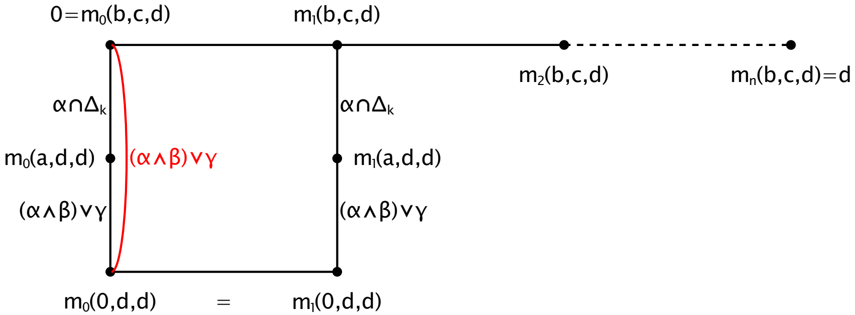

(α∧β)∨γ
Δ_k
γ
β
b
c
0
a
d
α

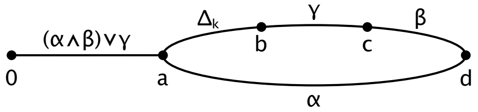

0=m_0(b,c,d)
m_1(b,c,d)
m_2(b,c,d)
m_n(b,c,d)=d


Δ_k
γ
β
$(\alpha\wedge\beta)\vee\gamma$
b
c
0
a
d
α

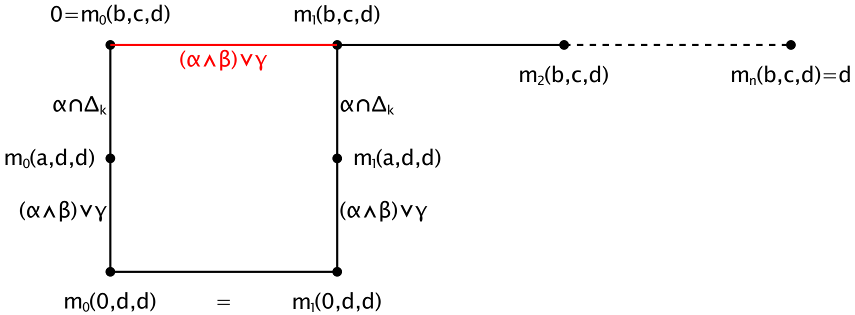

$0=m_0(b,c,d)$
$m_1(b,c,d)$
$m_2(b,c,d)$
$m_n(b,c,d)=d$
$\alpha\cap\Delta_k$
$\alpha\cap\Delta_k$
$m_0(a,d,d)$
$m_1(a,d,d)$
$(\alpha\wedge\beta)\vee\gamma$
$(\alpha\wedge\beta)\vee\gamma$
$m_0(0,d,d)$
$=$
$m_1(0,d,d)$


Δ_k
γ
β
$(\alpha \wedge \beta) \vee \gamma$
b
c
0
a
d
α

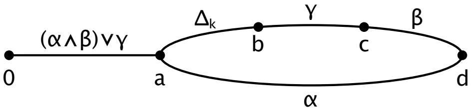

$0 = m_0(b,c,d)$
$m_1(b,c,d)$
$m_2(b,c,d)$
$m_n(b,c,d) = d$
$\alpha \cap \Delta_k$
$\alpha \cap \Delta_k$
$m_0(a,d,d)$
$(\alpha \wedge \beta) \vee \gamma$
$m_1(a,d,d)$
$(\alpha \wedge \beta) \vee \gamma$
$(\alpha \wedge \beta) \vee \gamma$
$m_0(0,d,d)$
=
$m_1(0,d,d)$

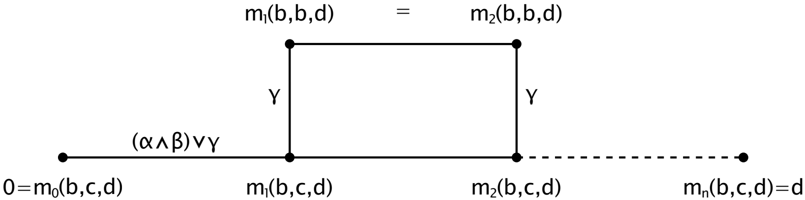


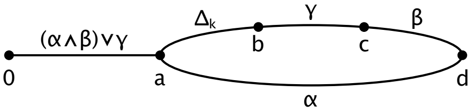
Δ_k
γ
β
$(\alpha\wedge\beta)\vee\gamma$
b
c
0
a
d
α

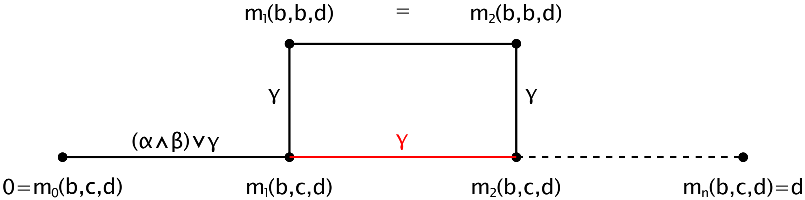


$0=m_0(b,c,d)$
$m_1(b,c,d)$
$(\alpha\wedge\beta)\vee\gamma$
$m_2(b,c,d)$
$m_n(b,c,d)=d$
$\alpha\cap\Delta_k$
$\alpha\cap\Delta_k$
$m_0(a,d,d)$
$m_1(a,d,d)$
$(\alpha\wedge\beta)\vee\gamma$
$(\alpha\wedge\beta)\vee\gamma$
$m_0(0,d,d)$
=
$m_1(0,d,d)$


Δ_k
γ
β
(α∧β)∨γ
b
c
0
a
d
α


m_1(b,b,d)
=
m_2(b,b,d)
γ
γ
(α∧β)∨γ
0=m_0(b,c,d)
m_1(b,c,d)
m_2(b,c,d)
m_n(b,c,d)=d


$(\alpha\wedge\beta)\vee\gamma$
Δ_k
γ
β
0
a
b
c
d
α


$m_1(b,b,d)$
$=$
$m_2(b,b,d)$
γ
γ
$(\alpha\wedge\beta)\vee\gamma$
γ
$0=m_0(b,c,d)$
$m_1(b,c,d)$
$m_2(b,c,d)$
$m_n(b,c,d)=d$

Example (G. Czédli)

$\mathcal{G}_0$ the variety of idempotent groupoids with 0

- $\mathcal{G}_0$ is congruence modular at 0, but
- $\mathcal{G}_0$ is not congruence modular in the usual sense

Example (G. Czédli)

$\mathcal{G}_0$ the variety of idempotent groupoids with 0

- $\mathcal{G}_0$ is congruence modular at 0, but
- $\mathcal{G}_0$ is not congruence modular in the usual sense

- applying Theorem with $n = 3$

$$m_0(x, y, z) = 0, \qquad m_1(x, y, z) = xz$$
$$m_2(x, y, z) = yz, \qquad m_3(x, y, z) = z.$$

Example (G. Czédli)

$\mathcal{G}_0$ the variety of idempotent groupoids with 0

- $\mathcal{G}_0$ is congruence modular at 0, but
- $\mathcal{G}_0$ is not congruence modular in the usual sense

- applying Theorem with $n = 3$

$$m_0(x,y,z) = 0, \qquad m_1(x,y,z) = xz$$
$$m_2(x,y,z) = yz, \qquad m_3(x,y,z) = z.$$

- the variety $\mathcal{S}$ of meet semilattices with 0 is a subvariety of $\mathcal{G}_0$, and R. Freese and J. B. Nation have proved that $\mathcal{S}$ satisfies no nontrivial congruence lattice identity

Thank you!