Categorically-algebraic topology

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Outline

1. Introduction
2. Categorically-algebraic topology
3. Lattice-valued categorically-algebraic topology
4. Categorically-algebraic pointless topology
5. Categorically-algebraic soft topology
6. Conclusion
Categorical approach to topology

- There exists a convenient approach to topological spaces:

**Step 1.** The backward powerset operator \( \text{Set} \xrightarrow{(-)^\leftarrow} \text{CBA} \text{Alg}^{\text{op}} \) with \( \text{Set} \) (resp. \( \text{CBA} \text{Alg} \)) the category of sets (resp. complete Boolean algebras) and \((X \xrightarrow{f} Y)^\leftarrow = 2^X \xrightarrow{(f^\leftarrow)^{\text{op}}} 2^Y, f^\leftarrow(\alpha) = \alpha \circ f.\)

**Step 2.** The topological theory, which is just the forgetful functor \( \text{CBA} \text{Alg} \xrightarrow{\| - \|} \text{Frm} \) to the category \( \text{Frm} \) of frames, describing the underlying algebraic structure of topological spaces.

**Step 3.** The category \( \text{Top} \) of topological spaces and continuous maps, whose objects are pairs \((X, \tau)\) for \( \tau \) (topology) a subframe of \( \|2^X\|\), and whose morphisms \((X, \tau) \xrightarrow{f}(Y, \sigma)\) are maps \(X \xrightarrow{f} Y\) with \((\|f^\leftarrow\|)\xrightarrow{\sigma} \subseteq \tau\) (continuity).
1983: S. E. Rodabaugh considers the backward powerset theory

$$\text{Set} \times \text{CBA}l\text{g}^{\text{op}} \xrightarrow{(\cdot)^\leftarrow} \text{CBA}l\text{g}^{\text{op}}$$ defined by the formula

$$((X, L) \xrightarrow{(f, \varphi)} (Y, M))^{\leftarrow} = L^X \xrightarrow{(f, \varphi)^{\leftarrow})^{op}} M^Y,$$

$$(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{\text{op}} \circ \alpha \circ f,$$

and topological spaces $$(X, L, \tau)$$ with $${\tau}$$ a subframe of $$\|L^X\|$$, resulting in variable-basis lattice-valued topology.

1991: S. E. Rodabaugh develops strict categorical foundations for his theory calling it point-set lattice-theoretic (poslat) topology.

1999: Poslat topology becomes a standard in the fuzzy community.
Poslat topology of S. E. Rodabaugh

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1993: Motivated by a promising idea of U. H"ohle, T. Kubiak and A. Šostak consider topological spaces as tuples $(X, L, M, T)$ with $M$ a frame, and $L^X \xrightarrow{T} M$ a map fulfilling several requirements.

2003: C. Guido suggests considering extended topological spaces $(X, L, M, \alpha, T)$ with $\alpha \in L^X$ and $\{\beta \in L^X | \beta \leq \alpha\} \xrightarrow{T} M$.

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Topological systems of S. Vickers

1989: S. Vickers proposes topological systems as a tool for doing pointless topology with. Their category has both the categories $\text{Top}$ and $\text{Frm}^{\text{op}}$ as full subcategories with “nice” properties.

**Definition 1**

A **topological system** is a triple $(X, L, \kappa)$ with $X$ a set, $L$ a frame, and $L \xrightarrow{\kappa} 2^X$ a frame homomorphism.

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Definition 2

Given a set $X$, a soft set over $X$ is a pair $(Y, \lll \lhd)$ with $Y$ a set and $Y \xrightarrow{\lll \lhd} 2^X$ a map.

2000: The process of “softening” of mathematics begins. Such notions as, e.g., soft group, soft ring, soft semiring, soft BCK (resp. BCI)-algebra appear. No link to soft topology available.
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Aim: This talk introduces a new way of approaching topological structures, which is induced by recent developments in lattice-valued topology, and is deemed to incorporate both crisp and many-valued settings.

Machinery: Based in category theory and universal algebra, the framework is called categorically-algebraic (catalg) topology, to underline its motivating theories, and to distinguish it from the poslat topology of S. E. Rodabaugh.

Advantage: The new setting includes all approaches to lattice-valued topology, as well as pointless topology of S. Vickers. It also starts a completely new area of study called soft topology, which is induced by the concept of soft set of D. Molodtsov.
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**Ω-algebras and Ω-homomorphisms**

**Definition 3**

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper) class of cardinal numbers.

- **An Ω-algebra** is a pair $(A, (\omega^A_\lambda)_{\lambda \in \Lambda})$ comprising a set $A$ and a family of maps $A^{n_\lambda} \xrightarrow{\omega^A_\lambda} A$ ($n_\lambda$-ary primitive operations on $A$).

- **An Ω-homomorphism** $(A, (\omega^A_\lambda)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega^B_\lambda)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $f \circ \omega^A_\lambda = \omega^B_\lambda \circ f^{n_\lambda}$ for every $\lambda \in \Lambda$.

- **Alg(Ω)** is the construct of Ω-algebras and Ω-homomorphisms, with the underlying functor denoted by $|-|$. 
Varieties of algebras

Definition 4

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of $\Omega$-homomorphisms with injective (resp. surjective) underlying maps.

- A variety of $\Omega$-algebras is a full subcategory of $\text{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).

- The objects (resp. morphisms) of a variety are called algebras (resp. homomorphisms).

- The categorical dual of a given variety $\mathcal{A}$ is denoted by $\text{LoA}$, whose objects (resp. morphisms) are called localic algebras (resp. homomorphisms).

- Given a subclass $\Omega' \subseteq \Omega$, an $\Omega'$-reduct of $\mathcal{A}$ is a pair $(\|\cdot\|, \mathcal{B})$ with $\mathcal{B}$ a variety of $\Omega'$-algebras and $\mathcal{A} \xrightarrow{\|\cdot\|} \mathcal{B}$ a concrete functor.
Powerset theories

**Definition 5**

A *variety-based backward powerset theory (vbp-theory)* in a given category $\mathbf{X}$ (*ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$.

**Lemma 6**

Given a variety $\mathbf{A}$, every subcategory $\mathbf{C}$ of $\mathbf{LoA}$ induces a functor $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathbf{S}=(\cdot)^\leftarrow} \mathbf{LoA}$, $((X_1,A_1) \xrightarrow{(f,\varphi)} (X_2,A_2)) \leftarrow = A_1^{X_1} \xrightarrow{(f,\varphi)^\leftarrow} A_2^{X_2} \xrightarrow{\varphi^\text{op} \circ \alpha \circ f}$

- $\mathbf{S}_A$ is the subcategory of $\mathbf{LoA}$ with the only morphism $1_A$.
- $\mathbf{Set} \times \mathbf{S}_A \xrightarrow{\mathbf{A}=(\cdot)^\leftarrow} \mathbf{LoA}$ (*fixed-basis approach*, with the full setting being *variable-basis approach*) is denoted by $(\cdot)^\leftarrow_A$. 
Categorically-algebraic topology

Powerset theories

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A variety-based backward powerset theory (vbp-theory) in a given category $X$ (ground category of the theory) is a functor $X \xrightarrow{P} \text{LoA}$. 

**Lemma 6**

Given a variety $A$, every subcategory $C$ of $\text{LoA}$ induces a functor $\text{Set} \times C \xrightarrow{\text{Set} \times (-)^{\leftarrow}} \text{LoA}$, $((X_1,A_1) \xrightarrow{(f,\varphi)} (X_2,A_2))^{\leftarrow} = A_1^{X_1} \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} A_2^{X_2}$

with $(f,\varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

- $S_A$ is the subcategory of $\text{LoA}$ with the only morphism $1_A$.

- $\text{Set} \times S_A \xrightarrow{(-)^{\leftarrow}} \text{LoA}$ (fixed-basis approach, with the full setting being variable-basis approach) is denoted by $(-)^{\leftarrow}_A$. 
Examples of powerset theories

Example 7

1. $\mathbf{Set} \times S_2 \xrightarrow{\mathcal{P}=(-)_2} \mathbf{LoCBAAlg}$, where $2 = \{\bot, \top\}$, provides the above-mentioned backward powerset operator.

2. $\mathbf{Set} \times \mathbf{S}_\mathbb{I} \xrightarrow{\mathcal{Z}=(-)_\mathbb{I}} \mathbf{DmLoc}$ (DeMorgan frames), where $\mathbb{I} = [0,1]$ is the unit interval, provides the fixed-basis fuzzy approach of L. A. Zadeh.

3. $\mathbf{Set} \times S_L \xrightarrow{\mathcal{G}=(-)_L} \mathbf{LoUQuant}$ (unital quantales) provides the fixed-basis $L$-fuzzy approach of J. A. Goguen.

4. $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}=(-)} \mathbf{LoSQuant}$ (semi-quantales) provides the variable-basis poslat approach of S. E. Rodabaugh.
**Definition 8**

Let $\mathbf{X}$ be a category and let $\mathcal{T}_i = ((P_i, (\| - \|_i, B_i)))_{i \in I}$ be a set-indexed family with $\mathbf{X} \xrightarrow{P_i} \text{LoA}_i$ a vbp-theory in $\mathbf{X}$ and $(\| - \|_i, B_i)$ a reduct of $A_i$ for $i \in I$. A **composite variety-based topological theory** (cvt-theory) in $\mathbf{X}$ induced by $\mathcal{T}_i$ is the functor $\mathbf{X} \xrightarrow{\mathcal{T}_i} \prod_{i \in I} \text{LoB}_i$, defined by commutativity of the diagram

$$
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{P_j} & \text{LoA}_j \\
\downarrow\mathcal{T}_i & & \downarrow(\| - \|_j^{op}) \\
\prod_{i \in I} \text{LoB}_i & \xrightarrow{\Gamma_j} & \text{LoB}_j
\end{array}
$$

for $j \in I$, where $\Gamma_j$ is the respective projection functor.

! A cvt-theory induced by a singleton family is denoted by $\mathcal{T}$. 
Definition 9

Let $T_I$ be a cvt-theory in a category $X$. $\mathbf{CTop}(T_I)$ is the concrete category over $X$, whose objects (composite variety-based topological spaces or $T_I$-spaces) are pairs $(X, (\tau_i)_{i \in I})$ with $X$ an $X$-object and $\tau_i$ a subalgebra of $T_i(X)$ for $i \in I$ ($(\tau_i)_{i \in I}$ is called composite variety-based topology or $T_I$-topology on $X$), and whose morphisms $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ are $X$-morphisms $X \xrightarrow{f} Y$, which satisfy $\left( (T_i f)^{\text{op}} \right) \subseteq (\sigma_i)$ for $i \in I$ (composite variety-based continuity or $T_I$-continuity).

The category $\mathbf{CTop}(T)$ is denoted by $\mathbf{Top}(T)$. 
Examples of categorically-algebraic topology

Example 10

1. \( \text{Top}((\mathcal{P}, \text{Frm})) \) is isomorphic to the classical category \( \text{Top} \) of topological spaces and continuous maps.

2. \( \text{Top}((\mathcal{P}, \text{CSL})) \) is isomorphic to the category \( \text{Cls} \) of closure spaces and continuous maps of D. Aerts.

3. \( \text{CTop}(((\mathcal{P}, \text{Frm}))_{i\in\{1,2\}}) \) is isomorphic to the category \( \text{BiTop} \) of bitopological spaces and bicontinuous maps of J. C. Kelly.

4. \( \text{Top}((\mathcal{Z}, \text{Frm})) \) is isomorphic to the category \( \text{I-Top} \) of fixed-basis fuzzy topological spaces of C. L. Chang.

5. \( \text{Top}((\mathcal{G}, \text{UQuant})) \) is isomorphic to the category \( \text{L-Top} \) of fixed-basis \( L \)-fuzzy topological spaces of J. A. Goguen.

6. \( \text{Top}((\mathcal{R}, \text{USQuant})) \) is isomorphic to the category \( \text{C-Top} \) for variable-basis poslat topology of S. E. Rodabaugh.
Lattice-valued algebras

**Definition 11**

Let $A$, $L$ be varieties, let $\text{CSLat}(\bigvee)$ ($\bigvee$-semilattices) be a reduct of $L$ and let $C$ be a subcategory of $L$.

- An $(A, C)$-algebra is a triple $(A, \mu, L)$ with $A$ an $A$-algebra, $L$ a $C$-algebra and $|A| \xrightarrow{\mu} |L|$ a map such that for every $\lambda \in \Lambda$ and every $a_i \in A$ for $i \in n_\lambda$, $\bigwedge_{i \in n_\lambda} \mu(a_i) \leq \mu(\omega^A(\langle a_i \rangle_{n_\lambda}))$.

- An $(A, C)$-homomorphism $(A_1, \mu_1, L_1) \xrightarrow{(\varphi, \psi)} (A_2, \mu_2, L_2)$ is an $A \times C$-morphism $(A_1, L_1) \xrightarrow{(\varphi, \psi)} (A_2, L_2)$ fulfilling the property $\psi \circ \mu_1(a) \leq \mu_2 \circ \varphi(a)$ for every $a \in A_1$.

- $C$-A is the category, concrete over $A \times C$, comprising $(A, C)$-algebras and $(A, C)$-homomorphisms.
### Definition 12

Let $T_I$ be a cvt-theory in a category $\mathbf{X}$, let $(L_i)_{i \in I}$ be a family of extensions of $\texttt{C SLat}(\bigvee)$, and let $C_i$ be a subcategory of $\texttt{LoL}_i$ for $i \in I$. An $\texttt{L}_I$-valued cvt-theory in $\mathbf{X}$ induced by $T_I$ and $(C_i)_{i \in I}$ is the pair $(T_I, \texttt{L}_I)$ with $\texttt{L}_I$ the category $\prod_{i \in I} C_i$.

The category $\texttt{L}_I$ induced by a singleton family is denoted by $\texttt{L}$.

### Remark

The setting of Definition 12 allows not just different underlying lattices for fuzzification (variable-basis framework), but actually different varieties for these lattices to come from.
Lattice-valued topological theories

**Definition 12**

Let $T_I$ be a cvt-theory in a category $X$, let $(L_i)_{i \in I}$ be a family of extensions of $\text{CSLat}(\bigvee)$, and let $C_i$ be a subcategory of $\text{LoL}_i$ for $i \in I$. An $L_I$-valued cvt-theory in $X$ induced by $T_I$ and $(C_i)_{i \in I}$ is the pair $(T_I, L_I)$ with $L_I$ the category $\prod_{i \in I} C_i$.

The category $L_I$ induced by a singleton family is denoted by $L$.

**Remark**

The setting of Definition 12 allows not just different underlying lattices for fuzzification (variable-basis framework), but actually different varieties for these lattices to come from.
Lattice-valued modification

Lattice-valued catalg spaces

**Definition 13**

Let \((T_I, \mathbb{L}_I)\) be an \(\mathbb{L}_I\)-valued cvt-theory in a category \(X\). \(\mathbb{L}_I \text{CTop}(T_I)\) is the concrete category over \(X \times \mathbb{L}_I\), whose

**objects** (\(\mathbb{L}_I\)-valued \(T_I\)-spaces) are triples \((X, (T_i)_{i \in I}, (L_i)_{i \in I})\) with \(X\) in \(X\), \((L_i)_{i \in I}\) in \(\mathbb{L}_I\) and \(T_i(X) \xrightarrow{T_i} L_i\) a \((B_i, \text{LoC}_i)\)-algebra for \(i \in I\) \(((T_i)_{i \in I}\) is called \(\mathbb{L}_I\)-valued \(T_I\)-topology on \(X\)), and whose

**morphisms** \((X, (T_i)_{i \in I}, (L_i)_{i \in I}) \xrightarrow{(f, (\psi_i)_{i \in I})} (Y, (S_i)_{i \in I}, (M_i)_{i \in I})\) are \(X \times \mathbb{L}_I\)-morphisms \((X, (L_i)_{i \in I}) \xrightarrow{(f, (\psi_i)_{i \in I})} (Y, (M_i)_{i \in I})\), for which \((T_i(X), T_i, L_i) \xrightarrow{(T_if, \psi_i)} (T_i(Y), S_i, M_i)\) is a \(\text{Lo}(\text{LoC}_i - B_i)\)-morphism for \(i \in I\) (\(\mathbb{L}_I\)-valued \(T_I\)-continuity).

The underlying functor to the ground category is denoted by \(|-|\).

The category \(\mathbb{L}_I \text{CTop}(T)\) is denoted by \(\mathbb{L}_I \text{Top}(T)\).
Examples of lattice-valued catalg topology

**Example 14**

1. $\mathbb{L} \text{Top}((S_{CLat}^L, Frm, S_{CDCLat}^M))$, with $CLat$ being the variety of complete lattices and $CDCLat$ its subcategory of completely distributive lattices, is the theory of $(L,M)$-fuzzy topological spaces of T. Kubiak and A. Šostak.

2. $\mathbb{L} \text{Top}((\mathcal{P}, Frm, S_{DMLoc}^M))$ provides the approach of U. Höhle.

3. $\mathbb{L} \text{I} \text{CTop}((T_i, \mathbb{L}_i))$ with $C_i = S_{2\text{CSLat}}^{\vee}$ for $i \in I$, is isomorphic to the category $\text{CTop}(T_i)$. 
Main result

**Theorem 15**

The concrete category \((\mathbb{L}_1 \text{CTop}(T_I), |-|)\) is topological over its ground category \(\mathbf{X} \times \mathbb{L}_I\).

- Meta-mathematically restated, one is doing topology when working in the category \(\mathbb{L}_I \text{CTop}(T_I)\).
- Given a topological structure, one can find the variety with the minimum requirements on its algebras, to preserve the “main” properties of the structure (characterizing variety). The corresponding category of lattice-valued catalg spaces (characterizing category) is then topological.
Lattice-valued catalg topological systems . . .

Definition 16

Let \((T_I, \mathbb{L}_I)\) be an \(\mathbb{L}_I\)-valued cvt-theory in \(X\). \(\mathbb{L}_ICTopSys(T_I)\) is the concrete category over \(X \times (\prod_{i \in I} \text{Lo}(\text{LoC}_i-B_i))\), whose objects are triples \((X, (\kappa_i)_{i \in I}, ((A_i, \mu_i, L_i))_{i \in I})\) with \(X\) in \(X\), \(((A_i, \mu_i, L_i))_{i \in I}\) in \(\prod_{i \in I} \text{Lo}(\text{LoC}_i-B_i)\) and \(T_i(X) \xrightarrow{\kappa_i} B_i\) a \(\text{LoB}_i\)-morphism for \(i \in I\) \(((\kappa_i)_{i \in I}\) is called \(\mathbb{L}_I\)-valued composite variety-based satisfaction relation or \(\mathbb{L}_I\)-valued \(T_I\)-satisfaction relation on \((X, ((A_i, \mu_i, L_i))_{i \in I}))\), and whose
... and their morphisms

\[
(X, (\kappa_i)_{i \in I}, ((A_i, \mu_i, L_i))_{i \in I}) \xrightarrow{(f, ((\varphi_i, \psi_i))_{i \in I})} (Y, (\iota_i)_{i \in I}, ((B_i, \nu_i, M_i))_{i \in I})
\]

are \( X \times (\prod_{i \in I} \text{Lo}(\text{LoC}_i - B_i)) \)-morphisms

\[
(X, ((A_i, \mu_i, L_i))_{i \in I}) \xrightarrow{(f, ((\varphi_i, \psi_i))_{i \in I})} (Y, ((B_i, \nu_i, M_i))_{i \in I}),
\]

which make the diagram

\[
\begin{array}{ccc}
T_i(X) & \xrightarrow{T_i f} & T_i(Y) \\
\downarrow \kappa_i & & \downarrow \iota_i \\
A_i & \xrightarrow{\varphi_i} & B_i
\end{array}
\]

commute for \( i \in I \) (\( \mathbb{L}_I \)-valued composite variety-based continuity or \( \mathbb{L}_I \)-valued \( T_I \)-continuity).
Examples of lattice-valued catalg topological systems

! The category \( \mathbb{L}CTopSys(T) \) is denoted by \( \mathbb{L}TopSys(T) \).

Example 17

1. \( \mathbb{L}TopSys((\mathcal{P}, \text{Frm}, S_2^{CSLat(V)})) \) is isomorphic to the category \( \text{TopSys} \) of classical topological systems of S. Vickers.

2. \( \mathbb{L}TopSys((S_{\text{Frm}}^{\text{Frm}}, \text{Frm}, S_2^{CSLat(V)})) \) is isomorphic to the category \( \text{Loc-TopSys} \) of lattice-valued topological systems introduced by J. T. Denniston, A. Melton and S. E. Rodabaugh.

3. \( \mathbb{L}TopSys((S_{\text{Set}}^{S_K}, \text{Set}, S_2^{CSLat(V)})) \) is isomorphic to the category \( \text{Chu(Set, K)} \) of \text{Chu spaces} over a set \( K \) of V. Pratt.

4. \( \text{Chu(Set,2)} \) is the category \( \text{IntSys} \) of \text{interchange systems} of J. T. Denniston, A. Melton and S. E. Rodabaugh. Interchange systems are called \text{contexts} in Formal Concept Analysis.
Theorem 18

There exists a full embedding $\mathbb{L}_I\text{CTop}(T_I) \xhookrightarrow{G_I} \mathbb{L}_I\text{CTopSys}(T_I)$. If the underlying lattices of $\mathbb{L}_I$ are completely distributive, then

1. there exists a functor $\mathbb{L}_I\text{CTopSys}(T_I) \xrightarrow{\text{Spat}_I} \mathbb{L}_I\text{CTop}(T_I)$;
2. $\text{Spat}_I$ is a right-adjoint-left-inverse to $G_I$;
3. $\mathbb{L}_I\text{CTop}(T_I)$ is isomorphic to a full coreflective subcategory of $\mathbb{L}_I\text{CTopSys}(T_I)$.

Theorem 18 provides a lattice-valued catalg analogue for the spatialization procedure of S. Vickers, restoring a significant part of the classical framework.
Soft algebras

**Definition 19**

Let $A$ be a variety, let $A$ be an $A$-algebra and let $X$ be a set. A soft $(A\text{-})$algebra over $A$ is a pair $(\lll, X)$, where $X \xrightarrow{\lll} 2^A$ is a map such that $\lll(x)$ is a subalgebra of $A$ for every $x \in X$.

**Example 20**

The varieties of groups, rings, semirings, as well as quasi-varieties (in the obvious sense) of BCK/BCI-algebras provide the respective soft notions from the literature.
Soft algebras

Definition 19

Let \( A \) be a variety, let \( A \) be an \( A \)-algebra and let \( X \) be a set. A soft \((A-)algebra\) over \( A \) is a pair \( (\lll, X) \), where \( X \xrightarrow{\lll} 2^A \) is a map such that \( \lll(x) \) is a subalgebra of \( A \) for every \( x \in X \).

Example 20

The varieties of groups, rings, semirings, as well as quasi-varieties (in the obvious sense) of BCK/BCI-algebras provide the respective soft notions from the literature.
The next definition is induced by the concept of soft algebra.

**Definition 21**

Let \((T_I, \mathbb{L}_I)\) be an \(\mathbb{L}_I\)-valued cvt-theory in \(X\). \(\mathbb{L}_I\text{CSoftTop}(T_I)\) is the concrete category over \(\mathbb{L}_I\text{CTop}(T_I) \times (\prod_{i \in I} \text{Lo}(\text{LoC}_i - B_i))\), whose objects (soft \(\mathbb{L}_I\)-valued \(T_I\)-spaces) are triples

\[
\left((X, (\mathcal{T}_i)_{i \in I}, (L_i)_{i \in I}), ((\kappa_i, \varphi_i))_{i \in I}, ((B_i, \nu_i, M_i))_{i \in I}\right)
\]

such that

- \((X, (\mathcal{T}_i)_{i \in I}, (L_i)_{i \in I})\) is an \(\mathbb{L}_I\)-valued \(T_I\)-space;
- \(((B_i, \nu_i, M_i))_{i \in I}\) is in \(\prod_{i \in I} \text{Lo}(\text{LoC}_i - B_i)\);
- \((T_i(X), \mathcal{T}_i, L_i) \xrightarrow{\left(\kappa_i, \varphi_i\right)} (B_i, \nu_i, M_i)\) is in \(\text{Lo}(\text{LoC}_i - B_i)\) for \(i \in I\);

\(((\kappa_i, \varphi_i))_{i \in I}\) is called **soft \(\mathbb{L}_I\)-valued \(T_I\)-topology** on \(((X, (\mathcal{T}_i)_{i \in I}, (L_i)_{i \in I}), ((B_i, \nu_i, M_i))_{i \in I})\), and whose
... and soft continuity

morphisms

\[
((X, (T_i)_{i \in I}, (L_i)_{i \in I}), ((\kappa_i, \varphi_i))_{i \in I}, ((B_i, \nu_i, M_i))_{i \in I})) \xrightarrow{((f, (\phi_i)_{i \in I}), ((\xi_i, o_i))_{i \in I})}
((Y, (S_i)_{i \in I}, (N_i)_{i \in I}), ((\psi_i))_{i \in I}, ((C_i, \sigma_i, O_i))_{i \in I})
\]

are \( \mathbb{L}_I \text{CTop}(T_I) \times (\prod_{i \in I} \text{Lo}(\text{LoC}_i-B_i)) \)-morphisms

\[
((X, (T_i)_{i \in I}, (L_i)_{i \in I}), ((B_i, \nu_i, M_i))_{i \in I})) \xrightarrow{((f, (\phi_i)_{i \in I}), ((\xi_i, o_i))_{i \in I})}
((Y, (S_i)_{i \in I}, (N_i)_{i \in I}), ((C_i, \sigma_i, O_i))_{i \in I}),
\]

which make the diagram commute for \( i \in I \) \( \text{(soft } \mathbb{L}_I \text{-valued } T_I \text{-continuity).} \)
Example 22

The (non-full) subcategory $\mathcal{S}$ of the category $\mathbb{L}_I CSoftTop(T_I)$, which comprises all objects

$$(((X, (T_i)_{i \in I}, (L_i)_{i \in I}), ((1_{T_i}(X), 1_{L_i}))_{i \in I}, ((T_i(X), T_i, L_i)))_{i \in I}),$$

together with all morphisms

$$(((X, (T_i)_{i \in I}, (L_i)_{i \in I}), ((1_{T_i}(X), 1_{L_i}))_{i \in I}, ((T_i(X), T_i, L_i)))_{i \in I})$$

$$(f, (\phi_i)_{i \in I}), ((T_i f, \phi_i))_{i \in I})$$

$$(((Y, (S_i)_{i \in I}, (M_i)_{i \in I}), ((1_{T_i}(Y), 1_{M_i}))_{i \in I}, ((T_i(Y), S_i, M_i)))_{i \in I}),$$

is isomorphic to the category $\mathbb{L}_I CTop(T_I)$. 
The talk introduced a new approach to topological structures called **(lattice-valued) categorically-algebraic topology**.  

The framework incorporates crisp and many-valued topology (erasing the border between them in some cases), pointless topology and soft topology.  

It appears that the currently dominating many-valued theory of S. E. Rodabaugh does not deviate significantly from the machinery of the crisp approach, whereas the framework of T. Kubaik and A. Šostak gives a truly lattice-valued setting.
**Definition 23**

Let $A$ be a variety, let $(X_1, ||-1, A_1)$, $(X_2, ||-2, A_2)$ be soft $A$-algebras. A soft $(A)$-algebra homomorphism $(X_1, ||-1, A_1) \xrightarrow{(f, \varphi)} (X_2, ||-2, A_2)$ is a $\text{Set} \times A$-morphism $(X_1, A_1) \xrightarrow{(f, \varphi)} (X_2, A_2)$, which satisfies the following diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
\downarrow ||-1 & & \downarrow ||-2 \\
2^{A_1} & \xrightarrow{\varphi} & 2^{A_2}.
\end{array}
$$

The category $\text{Soft}A$ comprises soft $A$-algebras and soft $A$-algebra homomorphisms, and is concrete over $\text{Set} \times A$.

**Problem:** Develop the theory of soft algebras through investigation of the properties of the category $\text{Soft}A$. 
References I


References II


Thank you for your attention!