Elementary problems in number theory

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n+1 is enough:

There are two consecutive numbers among them.
Proof: Pigeon-holes: $\{1, 2\}$, $\{3, 4\}$, $\ldots$, $\{2n-1, 2n\}$,

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- $2n$ is enough (contains 1 and 2)
- n is not enough: $n+1, n+2, \ldots, 2n$

n+1 is enough:

Proof: Pigeon-holes: $\{1 \cdot 2^t\}, \{3 \cdot 2^t\}, \ldots, \{(2n-1) \cdot 2^t\}$, labelled by odd numbers.

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Proof: Pigeon-holes: residue classes

Pigeons: a_1, $a_1 + a_2$, ..., $a_1 + a_2 + \cdots + a_n$

Two in the same pigeon-hole:

$$\sum_1^l a_i - \sum_1^k a_i = \sum_k^l a_i$$

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$(n-1)^2 + 1$ is enough:

Is there a better bound?

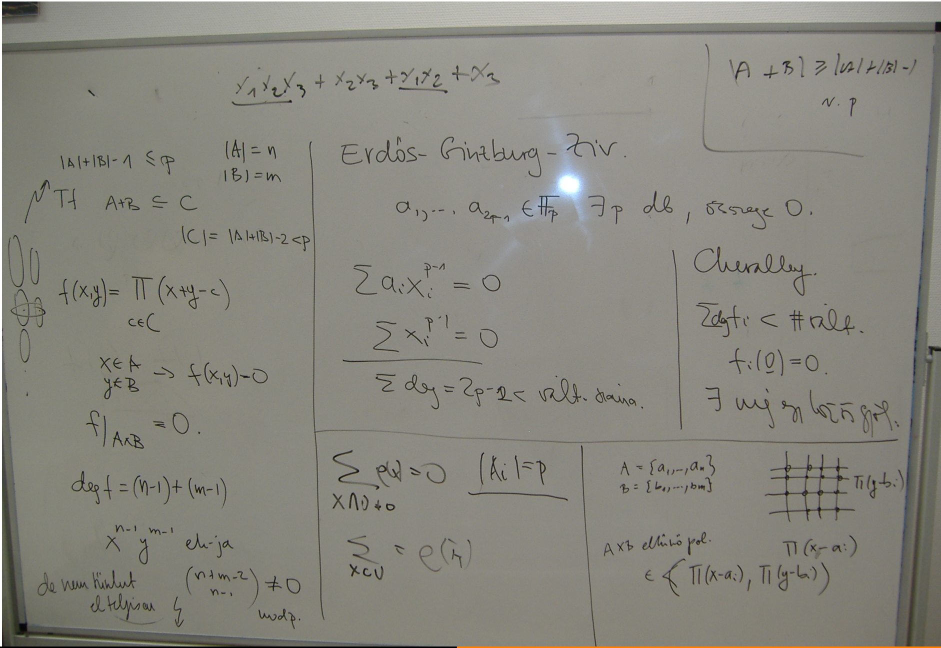
Chevalley's Theorem

Lemma

Let $A_1, \ldots, A_n$ *be subsets of* F_p*, the* p*-element field, and* $f \in F_p[x_1, \ldots, x_n]$ *such that*

$$\sum_{i=1}^{n}(|A_i| - 1) > (p-1)\deg f.$$

If the set $\{a \in A_1 \times \cdots \times A_n | f(a) = 0\}$ *is not empty, then it has at least two different elements.*


Erdős-Ginzburg-Ziv.
|A| = n
|B| = m
Tf A+B ⊆ C
Chevalley.
f(x,y) = Π (x+y-c)
deg f = (n-1)+(m-1)
A×B eltűnő pol.

Why am I talking about these problems?

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Always decidable: check every substitution

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Rings

- ID-CHECK **R**: Is $t = t_1 - t_2$ identically 0?
- POL-SAT **R**: Does $t = t_1 - t_2$ have a root?

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- $x_1^{k_1} \dots x_n^{k_n} \equiv 1 \Longleftrightarrow \forall m\colon\ \exp \mathbf{A} \mid k_m$

Nilpotent rings

Idziak- Szabó

Let A be a nilpotent algebra of size r and of nilpotency class k, and $f(\bar{x}) \in R[x_1, x_2, \ldots, x_n]$ be a polynomial over A.
Then for every $\bar{a} \in R^n$ there is a $\bar{b} \in R^n$ such that

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G. Horváth

same bound, simpler proof for groups and rings

Nilpotent rings

Let $F(\bar{a}) = F(a_1, \ldots, a_n) = b$.

For $H \subseteq \{1, 2, \ldots, n\}$ let $a_H = \begin{cases} a_i \text{ if} & i \in H \\ 0 \text{ if} & i \notin H \end{cases}$

$$\varphi(H) = \text{see board}$$

$$\overline{\varphi}(H) = \sum_{X \subseteq H} \varphi(X)$$

$$f(x) = \sum_{H} \varphi(H) \prod_{i \in H} x_i$$

Clearly , $\overline{\varphi}(H) = F(\overline{a}_H)$

recall

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$g(\bar{x}) = f(\bar{x}) - f(\overline{1})$
$g(\overline{1}) = 0$
$g(\chi(H)) = 0 \iff F(\bar{a}_H) = b$

Chevalley's Theorem, again

Recall

Let $A_1, \ldots, A_n$ be subsets of F_p, the p-element field, and $f \in F_p[x_1, \ldots, x_n]$ such that

$$\sum_{i=1}^{n} (|A_i| - 1) > (p-1) \deg f.$$

If the set $\{a \in A_1 \times \cdots \times A_n | f(a) = 0\}$ is not empty, then it has at least two different elements.

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Apply Lemma for $g(\bar{x})$ and $A_i = \{0, 1\}$. $g(\bar{1}) = 0$.
If $n > k(p-1)$, there is an other root.