Lattice Universal Semigroup Varieties

Inna Mikhailova Mark Sapir Mikhail Volkov

Ural State University, Ekaterinburg, Russia

Jardafest, June 24th, 2010

Π_∞ is the partition lattice on a countably infinite set.

Lattice Universality

Π_∞ is the partition lattice on a countably infinite set.

Π_∞ is universal in the following sense: it contains the congruence lattice of every countable algebra as a sublattice.

Lattice Universality

Π_∞ is the partition lattice on a countably infinite set.

Π_∞ is universal in the following sense: it contains the congruence lattice of every countable algebra as a sublattice.

In particular, Π_∞ contains the dual of the lattice of all varieties of algebras of any fixed finite or countably infinite type τ

Π_∞ is the partition lattice on a countably infinite set.

Π_∞ is universal in the following sense: it contains the congruence lattice of every countable algebra as a sublattice.

In particular, Π_∞ contains the dual of the lattice of all varieties of algebras of any fixed finite or countably infinite type τ since the latter lattice is the dual of the lattice of fully invariant congruences on the free countably generated algebra of type τ.

Lattice Universality

Π_∞ is the partition lattice on a countably infinite set.

Π_∞ is universal in the following sense: it contains the congruence lattice of every countable algebra as a sublattice.

In particular, Π_∞ contains the dual of the lattice of all varieties of algebras of any fixed finite or countably infinite type τ since the latter lattice is the dual of the lattice of fully invariant congruences on the free countably generated algebra of type τ.

$$L(\mathcal{V})^d \cong \mathrm{Con}_{fi}(\mathbf{F}_\infty(\mathcal{V})) \hookrightarrow \mathrm{Con}(\mathbf{F}_\infty(\mathcal{V})) \hookrightarrow \Pi_\infty$$

Lattice Universal Varieties

A variety of semigroups is called lattice universal if its subvariety lattice contains an interval dual to Π_∞.

A variety of semigroups is called lattice universal if its subvariety lattice contains an interval dual to Π_∞.

If $\mathcal{U}$ is a lattice universal semigroup variety, then $L(\mathcal{U})$ contains an isomorphic copy of $L(\mathcal{V})$ for every variety $\mathcal{V}$ of algebras of any finite or countably infinite type τ.

Lattice Universal Varieties

A variety of semigroups is called lattice universal if its subvariety lattice contains an interval dual to Π_∞.

If $\mathcal{U}$ is a lattice universal semigroup variety, then $L(\mathcal{U})$ contains an isomorphic copy of $L(\mathcal{V})$ for every variety $\mathcal{V}$ of algebras of any finite or countably infinite type τ.

Ježek's Example

Do lattice universal semigroup varieties exist?

Ježek's Example

Do lattice universal semigroup varieties exist?

Yes!

Ježek's Example

Do lattice universal semigroup varieties exist?

Yes! The first example is due to Burris and Nelson (1971): the variety defined by the identity $x^2 \doteq x^3$ is lattice universal.

Ježek's Example

Do lattice universal semigroup varieties exist?

Yes! The first example is due to Burris and Nelson (1971): the variety defined by the identity $x^2 \doteq x^3$ is lattice universal.

Ježek (1976) strengthened this result by showing that the variety defined by the identities $x^2y \doteq yx^2 \doteq x^2$ is lattice universal.

Ježek's Example

Do lattice universal semigroup varieties exist?

Yes! The first example is due to Burris and Nelson (1971): the variety defined by the identity $x^2 \doteq x^3$ is lattice universal.

Ježek (1976) strengthened this result by showing that the variety defined by the identities $x^2y \doteq yx^2 \doteq x^2$ is lattice universal. As the identities mean that x^2 is zero, one abbreviates them as $x^2 \doteq 0$.

Ježek's Example

Do lattice universal semigroup varieties exist?

Yes! The first example is due to Burris and Nelson (1971): the variety defined by the identity $x^2 \doteq x^3$ is lattice universal.

Ježek (1976) strengthened this result by showing that the variety defined by the identities $x^2y \doteq yx^2 \doteq x^2$ is lattice universal. As the identities mean that x^2 is zero, one abbreviates them as $x^2 \doteq 0$.

Ježek's result was quite important from the viewpoint of proof techniques used.

Burris and Nelson used a classic combinatorial result by Axel Thue (1906, 1912):

Combinatorics on Words Comes into the Play

Burris and Nelson used a classic combinatorial result by Axel Thue (1906, 1912): there exists an infinite square-free sequence on three symbols.

Combinatorics on Words Comes into the Play

Burris and Nelson used a classic combinatorial result by Axel Thue (1906, 1912): there exists an infinite square-free sequence on three symbols.

For Ježek's result, the mere existence of such a sequence wasn't sufficient; he needed an infinite sequence of square-free words with an additional combinatorial property.

Avoidability

Let Σ and Δ be two alphabets (which may be equal), and let Σ^+ and Δ^+ stand for the sets of non-empty words over Σ and Δ respectively.

Avoidability

Let Σ and Δ be two alphabets (which may be equal), and let Σ^+ and Δ^+ stand for the sets of non-empty words over Σ and Δ respectively.

Definition

A word $v \in \Sigma^+$ encounters a word $p \in \Delta^+$ if there is a map $h : \Delta \to \Sigma^+$ such that $h(p)$ is a factor of v.

Avoidability

Let Σ and Δ be two alphabets (which may be equal), and let Σ^+ and Δ^+ stand for the sets of non-empty words over Σ and Δ respectively.

Definition

A word $v \in \Sigma^+$ encounters a word $p \in \Delta^+$ if there is a map $h : \Delta \to \Sigma^+$ such that $h(p)$ is a factor of v.

- The word *banana* encounters the word x^2.

Let Σ and Δ be two alphabets (which may be equal), and let Σ^+ and Δ^+ stand for the sets of non-empty words over Σ and Δ respectively.

Definition

A word $v \in \Sigma^+$ encounters a word $p \in \Delta^+$ if there is a map $h : \Delta \to \Sigma^+$ such that $h(p)$ is a factor of v.

- The word *ba na na* encounters the word x^2.

Avoidability

Let Σ and Δ be two alphabets (which may be equal), and let Σ^+ and Δ^+ stand for the sets of non-empty words over Σ and Δ respectively.

Definition

A word $v \in \Sigma^+$ encounters a word $p \in \Delta^+$ if there is a map $h : \Delta \to \Sigma^+$ such that $h(p)$ is a factor of v.

- The word *ba na na* encounters the word x^2.

Otherwise the word v is said to avoid the word p.

Avoidability

Let Σ and Δ be two alphabets (which may be equal), and let Σ^+ and Δ^+ stand for the sets of non-empty words over Σ and Δ respectively.

Definition

A word $v \in \Sigma^+$ encounters a word $p \in \Delta^+$ if there is a map $h : \Delta \to \Sigma^+$ such that $h(p)$ is a factor of v.

- The word *ba na na* encounters the word x^2.

Otherwise the word v is said to avoid the word p.

In particular, a word is square-free iff it avoids x^2.

Antichains

Definition

A set $A \subseteq \Sigma^+$ is an antichain if every two distinct words in A avoid each other.

Antichains

Definition

A set $A \subseteq \Sigma^+$ is an antichain if every two distinct words in A avoid each other.

The combinatorial result that underlines Ježek's theorem is: there is an infinite antichain consisting of square-free words on three symbols.

Antichains

Definition

A set $A \subseteq \Sigma^+$ is an antichain if every two distinct words in A avoid each other.

The combinatorial result that underlines Ježek's theorem is: there is an infinite antichain consisting of square-free words on three symbols.

Ježek constructed such an antichain. He wrote that Thue's paper was not accessible to him so he did not know whether or not Thue already had this result.

Antichains

Definition

A set $A \subseteq \Sigma^+$ is an antichain if every two distinct words in A avoid each other.

The combinatorial result that underlines Ježek's theorem is: there is an infinite antichain consisting of square-free words on three symbols.

Ježek constructed such an antichain. He wrote that Thue's paper was not accessible to him so he did not know whether or not Thue already had this result. In fact, Thue had several constructions but none of them yield Ježek's result.

Rest of the Proof

The rest is easy.

Rest of the Proof

The rest is easy. Let A be an infinite antichain consisting of square-free words on three symbols. Given a partition π of A, we define a semigroup variety $\mathcal{V}_\pi$ by the following identities:

Rest of the Proof

The rest is easy. Let A be an infinite antichain consisting of square-free words on three symbols. Given a partition π of A, we define a semigroup variety $\mathcal{V}_\pi$ by the following identities:

- $x^2 \simeq 0$,

Rest of the Proof

The rest is easy. Let A be an infinite antichain consisting of square-free words on three symbols. Given a partition π of A, we define a semigroup variety $\mathcal{V}_\pi$ by the following identities:

- $x^2 \simeq 0$,
- all identities $w \simeq 0$ where w encounters some word in A,

Rest of the Proof

The rest is easy. Let A be an infinite antichain consisting of square-free words on three symbols. Given a partition π of A, we define a semigroup variety $\mathcal{V}_\pi$ by the following identities:

- $x^2 \doteq 0$,
- all identities $w \doteq 0$ where w encounters some word in A,
- all identities $u \doteq v$ such that $u, v \in A$ lie in the same π-class.

Rest of the Proof

The rest is easy. Let A be an infinite antichain consisting of square-free words on three symbols. Given a partition π of A, we define a semigroup variety $\mathcal{V}_\pi$ by the following identities:

- $x^2 \simeq 0$,
- all identities $w \simeq 0$ where w encounters some word in A,
- all identities $u \simeq v$ such that $u, v \in A$ lie in the same π-class.

Then $\pi \mapsto V_\pi$ is a dual isomorphism between Π_∞ and an interval in the subvariety lattice of the variety $\mathcal{J}^2$ defined by $x^2 \simeq 0$.

Further Questions

Thus, $\mathcal{J}^2$ is lattice universal and so is every variety containing $\mathcal{J}^2$.

Further Questions

Thus, $\mathcal{J}^2$ is lattice universal and so is every variety containing $\mathcal{J}^2$. Are there any further examples?

Further Questions

Thus, $\mathcal{J}^2$ is lattice universal and so is every variety containing $\mathcal{J}^2$. Are there any further examples? Is it possible to somehow classify lattice universal varieties?

Further Questions

Thus, $\mathcal{J}^2$ is lattice universal and so is every variety containing $\mathcal{J}^2$. Are there any further examples? Is it possible to somehow classify lattice universal varieties? Given a finite set of identities, can one decide if these identities define a lattice universal variety?

Further Questions

Thus, $\mathcal{J}^2$ is lattice universal and so is every variety containing $\mathcal{J}^2$. Are there any further examples? Is it possible to somehow classify lattice universal varieties? Given a finite set of identities, can one decide if these identities define a lattice universal variety?

Addressing these question has become possible only after a crucial progress in combinatorics on words was achieved by Bean, Ehrenfeucht, McNulty (1979) and Zimin (1982).

Avoidability Revisited

A word w is called avoidable if there exists an infinite sequence of words $\{u_i\}$ over a finite alphabet such that each word u_i avoids w.

Avoidability Revisited

A word w is called avoidable if there exists an infinite sequence of words $\{u_i\}$ over a finite alphabet such that each word u_i avoids w.

Thue's result can be stated as saying that x^2 is avoidable (over an alphabet with three symbols).

Avoidability Revisited

A word w is called avoidable if there exists an infinite sequence of words $\{u_i\}$ over a finite alphabet such that each word u_i avoids w.

Thue's result can be stated as saying that x^2 is avoidable (over an alphabet with three symbols).

Bean, Ehrenfeucht, McNulty and Zimin found efficient descriptions of (un)avoidable words.

Avoidability Revisited

A word w is called avoidable if there exists an infinite sequence of words $\{u_i\}$ over a finite alphabet such that each word u_i avoids w.

Thue's result can be stated as saying that x^2 is avoidable (over an alphabet with three symbols).

Bean, Ehrenfeucht, McNulty and Zimin found efficient descriptions of (un)avoidable words. The BEM description is easier to use while Zimin's description is easier to formulate.

Zimin's words $Z_n \in \{x_1, x_2, \ldots\}^+$ are defined as follows:

$$\begin{aligned} Z_1 &= x_1, \\ Z_2 &= x_1 x_2 x_1, \\ &\ldots \\ Z_{n+1} &= Z_n x_{n+1} Z_n, \\ &\ldots \end{aligned}$$

Zimin's Words

Zimin's words $Z_n \in \{x_1, x_2, \ldots\}^+$ are defined as follows:

$$Z_1 = x_1,$$
$$Z_2 = x_1 x_2 x_1,$$
$$\ldots$$
$$Z_{n+1} = Z_n x_{n+1} Z_n,$$
$$\ldots$$

Theorem (Zimin, 1982)

A word w involving n different letters is unavoidable iff the word Z_n encounters w.

Generalization

It is relatively easy to extend Ježek's theorem to any variety defined by an identity of the form $w \doteq 0$ where w is an avoidable word or even by a system of identities $w_i \doteq 0$ where all w_i are avoidable and involve only finitely many letters.

Generalization

It is relatively easy to extend Ježek's theorem to any variety defined by an identity of the form $w \doteq 0$ where w is an avoidable word or even by a system of identities $w_i \doteq 0$ where all w_i are avoidable and involve only finitely many letters.

For this, one should only show that for every avoidable word w, there exists an infinite antichain A_w such that all words in A_w involve only finitely many letters and avoid w; then Ježek's argument applies.

Generalization

It is relatively easy to extend Ježek's theorem to any variety defined by an identity of the form $w \doteq 0$ where w is an avoidable word or even by a system of identities $w_i \doteq 0$ where all w_i are avoidable and involve only finitely many letters.

For this, one should only show that for every avoidable word w, there exists an infinite antichain A_w such that all words in A_w involve only finitely many letters and avoid w; then Ježek's argument applies.

A construction for such an antichain A_w was found by Mikhailova in 2009.

Counter-Example

This generalization is not yet sufficient:

Counter-Example

This generalization is not yet sufficient:

Example

The variety $\mathcal{M}_3$ defined by the identity

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq bxa \cdot y \cdot axb \cdot z \cdot axb \cdot y \cdot bxa$$

is lattice universal while its subvariety defined by

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq 0$$

is locally finite, and therefore, is not lattice universal.

Counter-Example

This generalization is not yet sufficient:

Example

The variety $\mathcal{M}_3$ defined by the identity

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq bxa \cdot y \cdot axb \cdot z \cdot axb \cdot y \cdot bxa$$

is lattice universal while its subvariety defined by

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq 0$$

is locally finite, and therefore, is not lattice universal.

The local finiteness of the subvariety follows from the fact that the word $axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb$ is unavoidable.

Counter-Example

This generalization is not yet sufficient:

Example

The variety $\mathcal{M}_3$ defined by the identity

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq bxa \cdot y \cdot axb \cdot z \cdot axb \cdot y \cdot bxa$$

is lattice universal while its subvariety defined by

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq 0$$

is locally finite, and therefore, is not lattice universal.

The local finiteness of the subvariety follows from the fact that the word $axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb$ is unavoidable. The subvariety lattice of a locally finite variety is algebraic and hence it cannot contain Π_∞ as an interval.

Further Generalization

Theorem 1

Suppose that a semigroup variety $\mathcal{V}$ is defined by identities depending on at most n letters and satisfies no non-trivial identity of the form $Z_{n+1} \simeq w$. Then $\mathcal{V}$ is lattice universal.

Further Generalization

Theorem 1

Suppose that a semigroup variety $\mathcal{V}$ is defined by identities depending on at most n letters and satisfies no non-trivial identity of the form $Z_{n+1} \simeq w$. Then $\mathcal{V}$ is lattice universal.

In particular, the lattice universality of $\mathcal{M}_3$ follows as soon as one verifies that $\mathcal{M}_3$ satisfies no non-trivial identity of the form $Z_6 \simeq w$.

Further Generalization

Theorem 1

Suppose that a semigroup variety $\mathcal{V}$ is defined by identities depending on at most n letters and satisfies no non-trivial identity of the form $Z_{n+1} \doteq w$. Then $\mathcal{V}$ is lattice universal.

In particular, the lattice universality of $\mathcal{M}_3$ follows as soon as one verifies that $\mathcal{M}_3$ satisfies no non-trivial identity of the form $Z_6 \doteq w$.

Our proof of Theorem 1 relies on constructing an infinite antichain $A_{\mathcal{V}}$ such that $\mathcal{V}$ satisfies no non-trivial identity of the form $u \doteq v$ with $u \in A_{\mathcal{V}}$.

Further Generalization

Theorem 1

Suppose that a semigroup variety $\mathcal{V}$ is defined by identities depending on at most n letters and satisfies no non-trivial identity of the form $Z_{n+1} \simeq w$. Then $\mathcal{V}$ is lattice universal.

In particular, the lattice universality of $\mathcal{M}_3$ follows as soon as one verifies that $\mathcal{M}_3$ satisfies no non-trivial identity of the form $Z_6 \simeq w$.

Our proof of Theorem 1 relies on constructing an infinite antichain $A_{\mathcal{V}}$ such that $\mathcal{V}$ satisfies no non-trivial identity of the form $u \simeq v$ with $u \in A_{\mathcal{V}}$. Then Ježek's argument applies mutatis mutandis.

Construction

Let $r' = \lceil \log_2(6n-1) \rceil$ and $r = 2^{r'}$.

Construction

Let $r' = \lceil \log_2(6n-1) \rceil$ and $r = 2^{r'}$. Consider $r^2 \times r$ matrix

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r & \cdots & 1 & r \\ 2 & 1 & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & r & \cdots & 2 & r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & 1 & \cdots & r & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & r & \cdots & r & r \end{pmatrix}$$

Construction

Let $r' = \lceil \log_2(6n-1) \rceil$ and $r = 2^{r'}$. Consider $r^2 \times r$ matrix

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r & \cdots & 1 & r \\ 2 & 1 & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & r & \cdots & 2 & r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & 1 & \cdots & r & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & r & \cdots & r & r \end{pmatrix} \qquad P_\Sigma = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r-1} & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{11} & a_{r2} & \cdots & a_{1r-1} & a_{rr} \\ a_{21} & a_{12} & \cdots & a_{2r-1} & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{21} & a_{r2} & \cdots & a_{2r-1} & a_{rr} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{12} & \cdots & a_{rr-1} & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr-1} & a_{rr} \end{pmatrix}$$

Whenever i occurs in the column j, we substitute i by a_{ij}.

Construction

Let $r' = \lceil \log_2(6n-1) \rceil$ and $r = 2^{r'}$. Consider $r^2 \times r$ matrix

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & r & \cdots & 1 & r \\ 2 & 1 & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & r & \cdots & 2 & r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & 1 & \cdots & r & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r & r & \cdots & r & r \end{pmatrix} \qquad P_\Sigma = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r-1} & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{11} & a_{r2} & \cdots & a_{1r-1} & a_{rr} \\ a_{21} & a_{12} & \cdots & a_{2r-1} & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{21} & a_{r2} & \cdots & a_{2r-1} & a_{rr} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{12} & \cdots & a_{rr-1} & a_{1r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr-1} & a_{rr} \end{pmatrix}$$

Whenever i occurs in the column j, we substitute i by a_{ij}.

$$\Sigma = \{a_{ij} \mid 1 \leq i, j \leq r\}.$$

Construction Continued

Let v_i be the word in $\imath$-th row of the matrix P_Σ.

Construction Continued

Let v_i be the word in $\imath$-th row of the matrix P_Σ. We define the map $\gamma : \Sigma \to \Sigma^+$ by $\gamma(a_{ij}) = v_{(i-1)r+j}$.

Let v_i be the word in $\imath$-th row of the matrix P_Σ. We define the map $\gamma : \Sigma \to \Sigma^+$ by $\gamma(a_{ij}) = v_{(i-1)r+j}$.

Theorem (Sapir, 1987)

Each word of the sequence $\gamma^m(a_{11})$ avoids all avoidable words over n letters.

Construction Continued

Let v_i be the word in $\imath$-th row of the matrix P_Σ. We define the map $\gamma : \Sigma \to \Sigma^+$ by $\gamma(a_{ij}) = v_{(i-1)r+j}$.

Theorem (Sapir, 1987)

Each word of the sequence $\gamma^m(a_{11})$ avoids all avoidable words over n letters.

If $w = x_1 x_2 \ldots x_\ell$ is a word, we denote by $\overleftarrow{w} = x_\ell \ldots x_2 x_1$ the mirror image of w. A word w is a palindrome if $\overleftarrow{w} = w$.

Construction Completed

Theorem (Mikhaylova, Volkov, 2007)

Take $d_1 \notin \Sigma$. Then each word in the sequence of palindromes $\sigma_m = \gamma^m(a_{11})d_1\overleftarrow{\gamma^m(a_{11})}$ avoids all avoidable words over n letters.

Construction Completed

Theorem (Mikhaylova, Volkov, 2007)

Take $d_1 \notin \Sigma$. Then each word in the sequence of palindromes $\sigma_m = \gamma^m(a_{11})d_1\overleftarrow{\gamma^m(a_{11})}$ avoids all avoidable words over n letters.

Construction Completed

Theorem (Mikhaylova, Volkov, 2007)

Take $d_1 \notin \Sigma$. Then each word in the sequence of palindromes $\sigma_m = \gamma^m(a_{11})d_1\overleftarrow{\gamma^m(a_{11})}$ avoids all avoidable words over n letters.

Take $d_2 \notin \{d_1\} \cup \Sigma$. Then the sequence $\eta_m = d_2\sigma_m d_2 d_1$ where $m \geq r^2$ is our antichain $A_{\mathcal{V}}$.

Converse Statement

Theorem 2

Suppose that a lattice universal semigroup variety $\mathcal{V}$ is defined by identities depending on at most n letters and all periodic groups in $\mathcal{V}$ are locally finite. Then $\mathcal{V}$ satisfies no non-trivial identity of the form $Z_{n+1} \doteq w$.

Theorem 2

Suppose that a lattice universal semigroup variety $\mathcal{V}$ is defined by identities depending on at most n letters and all periodic groups in $\mathcal{V}$ are locally finite. Then $\mathcal{V}$ satisfies no non-trivial identity of the form $Z_{n+1} \doteq w$.

In other words, we have obtained a complete characterization of lattice universal semigroup varieties in the class of varieties defined by identities in finitely many letters and containing no infinite periodic groups with finitely many generators.

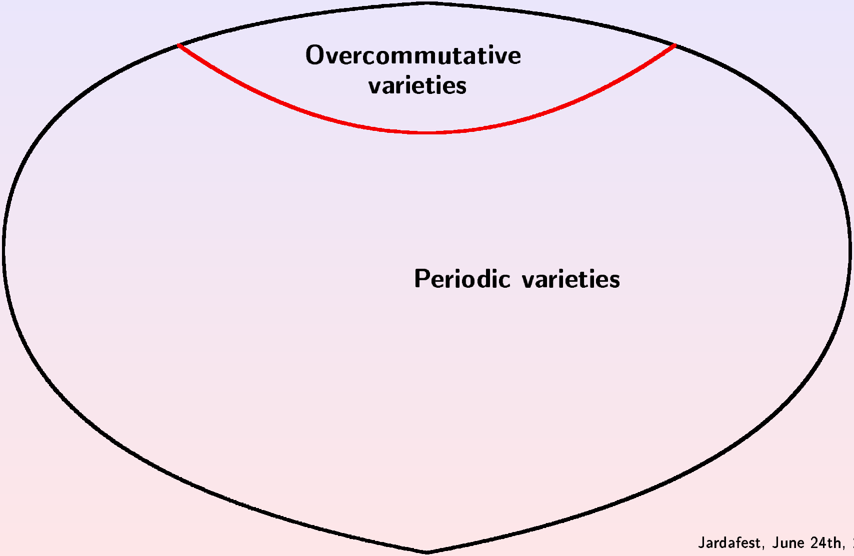
Converse Statement

Theorem 2

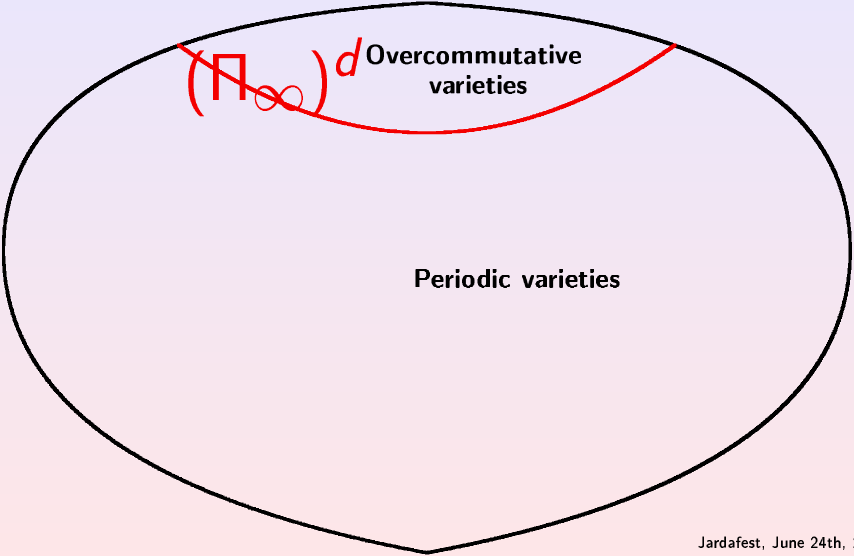
Suppose that a lattice universal semigroup variety $\mathcal{V}$ is defined by identities depending on at most n letters and all periodic groups in $\mathcal{V}$ are locally finite. Then $\mathcal{V}$ satisfies no non-trivial identity of the form $Z_{n+1} \simeq w$.

In other words, we have obtained a complete characterization of lattice universal semigroup varieties in the class of varieties defined by identities in finitely many letters and containing no infinite periodic groups with finitely many generators. Given a finite set of identities that force periodic groups to be locally finite, we can effectively check whether or not these identities define a lattice universal variety.

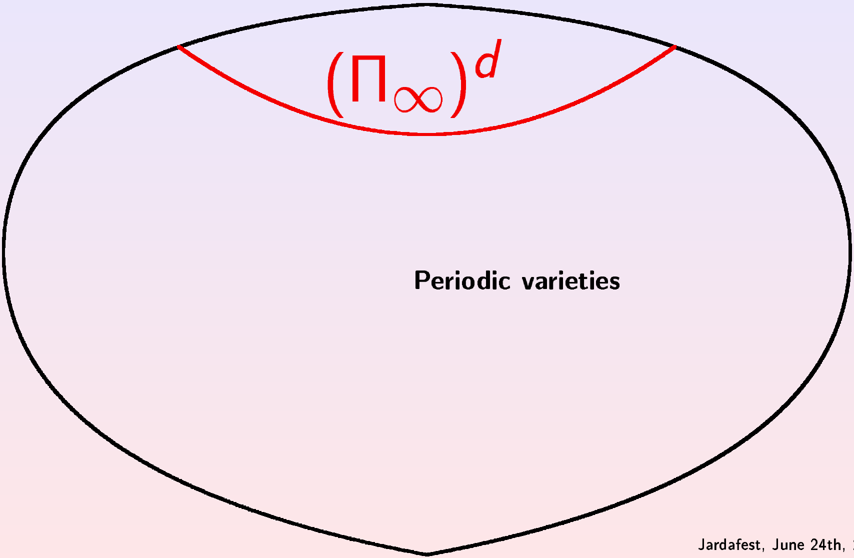
Embedding $(\Pi_\infty)^d$ into the Lattice of Semigroup Varieties



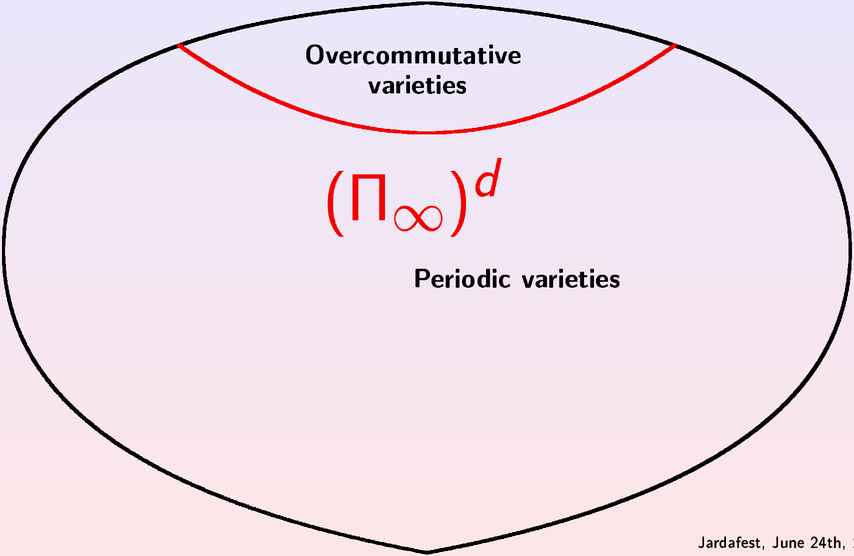
Embedding $(\Pi_\infty)^d$ into the Lattice of Semigroup Varieties



Embedding $(\Pi_\infty)^d$ into the Lattice of Semigroup Varieties



Embedding $(\Pi_\infty)^d$ into the Lattice of Semigroup Varieties



Embedding $(\Pi_\infty)^d$ into the Lattice of Periodic Varieties

Thus, if $(\Pi_\infty)^d$ embeds into $L(\mathcal{V})$ for some semigroup variety $\mathcal{V}$, it must embed into $L(\mathcal{P})$ where $\mathcal{P}$ is a periodic subvariety of $\mathcal{V}$.

Embedding $(\Pi_\infty)^d$ into the Lattice of Periodic Varieties

Thus, if $(\Pi_\infty)^d$ embeds into $L(\mathcal{V})$ for some semigroup variety $\mathcal{V}$, it must embed into $L(\mathcal{P})$ where $\mathcal{P}$ is a periodic subvariety of $\mathcal{V}$.

Theorem (Sapir, 1987)

Suppose that a periodic semigroup variety $\mathcal{P}$ is defined by identities depending on at most n letters and all groups in $\mathcal{P}$ are locally finite. Then $\mathcal{P}$ is locally finite iff $\mathcal{P}$ satisfies a non-trivial identity of the form $Z_{n+1} \simeq w$.

Embedding $(\Pi_\infty)^d$ into the Lattice of Periodic Varieties

Thus, if $(\Pi_\infty)^d$ embeds into $L(\mathcal{V})$ for some semigroup variety $\mathcal{V}$, it must embed into $L(\mathcal{P})$ where $\mathcal{P}$ is a periodic subvariety of $\mathcal{V}$.

Theorem (Sapir, 1987)

Suppose that a periodic semigroup variety $\mathcal{P}$ is defined by identities depending on at most n letters and all groups in $\mathcal{P}$ are locally finite. Then $\mathcal{P}$ is locally finite iff $\mathcal{P}$ satisfies a non-trivial identity of the form $Z_{n+1} \doteq w$.

We have already seen that a lattice universal variety cannot be locally finite.

Thus, if $(\Pi_\infty)^d$ embeds into $L(\mathcal{V})$ for some semigroup variety $\mathcal{V}$, it must embed into $L(\mathcal{P})$ where $\mathcal{P}$ is a periodic subvariety of $\mathcal{V}$.

Theorem (Sapir, 1987)

Suppose that a periodic semigroup variety $\mathcal{P}$ is defined by identities depending on at most n letters and all groups in $\mathcal{P}$ are locally finite. Then $\mathcal{P}$ is locally finite iff $\mathcal{P}$ satisfies a non-trivial identity of the form $Z_{n+1} \mathrel{\hat{=}} w$.

We have already seen that a lattice universal variety cannot be locally finite. Thus, $\mathcal{P}$ (and hence $\mathcal{V}$) satisfies no non-trivial identity of the form $Z_{n+1} \mathrel{\hat{=}} w$.

Future Work

A semigroup variety $\mathcal{V}$ is said to be finitely universal if for each n, the lattice $L(\mathcal{V})$ contains an interval dual to Π_n, the partition lattice on an n-element set.

Future Work

A semigroup variety $\mathcal{V}$ is said to be finitely universal if for each n, the lattice $L(\mathcal{V})$ contains an interval dual to Π_n, the partition lattice on an n-element set. Examples: the variety of all commutative semigroups, the variety defined by $x^2 \doteq yxy$.

Future Work

A semigroup variety $\mathcal{V}$ is said to be finitely universal if for each n, the lattice $L(\mathcal{V})$ contains an interval dual to Π_n, the partition lattice on an n-element set. Examples: the variety of all commutative semigroups, the variety defined by $x^2 \simeq yxy$.

Problem

Classify all finitely universal semigroup varieties.

Future Work

A semigroup variety $\mathcal{V}$ is said to be finitely universal if for each n, the lattice $L(\mathcal{V})$ contains an interval dual to Π_n, the partition lattice on an n-element set. Examples: the variety of all commutative semigroups, the variety defined by $x^2 \doteq yxy$.

Problem

Classify all finitely universal semigroup varieties.

Problem

Classify all semigroup varieties whose subvariety lattice satisfies a non-trivial lattice identity.

Conclusion

"We can see only a short distance ahead, but we can see plenty there that needs to be done." (Alan M. Turing)

Conclusion

"We can see only a short distance ahead, but we can see plenty there that needs to be done." (Alan M. Turing)

Thank you very much for your attention!

Thank you very much for your attention!

Thank you very much for your attention!