The relational clone membership problem is hard

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"Mathematical and algorithmic aspects of constraint satisfaction"
June 27 – Aug 27, 2011

Activities (dates are tentative):

Summer school: June 27–30 (following LICS)

Workshops:

- Graph Homomorphisms: July 11–15
- Algebra and CSPs: Aug 2–6
- Approximability of CSPs: Aug 15–19

Other:

- Ongoing seminars
- Lecture series on applied aspects of CSP
- Distinguished lectures

The relational clone membership problem

Starting point: the Galois connection between operations and relations

Definition

Let f be an n-ary operation and r a k-ary relation on a finite set D.

Say that f **preserves** r iff for all $A = [a_{ij}] \in D^{n \times k}$, if each row of A $[a_{11} \ \dots \ a_{1k}]$, $\dots$, $[a_{n1} \ \dots \ a_{nk}]$ is in r, then

$$\left(f\left(\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \right), \dots, f\left(\begin{bmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{bmatrix} \right) \right) \in r.$$

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Other jargon:

- r is **invariant** under f.
- f is a **polymorphism** of r.

Fix D (a finite set).

- $\mathrm{Op}_D = \{$all finitary operations on $D\}$.
- $\mathrm{Rel}_D = \{$all finitary relations on $D\}$.

For $\mathcal{F} \subseteq \mathrm{Op}_D$ and $\mathcal{R} \subseteq \mathrm{Rel}_D$

- $\mathrm{Inv}(\mathcal{F}) = \{r \in \mathrm{Rel}_D : r$ is invariant under every $f \in \mathcal{F}\}$
- $\mathrm{Pol}(\mathcal{R}) = \{f \in \mathrm{Op}_D : f$ is a polymorphism of every $r \in \mathcal{R}\}$.

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$$\begin{tikzcd}
\mathcal{P}(\mathrm{Op}_D) \arrow[rr, bend left, "\textsc{Inv}"] & & \mathcal{P}(\mathrm{Rel}_D) \arrow[ll, bend left, "\textsc{Pol}"]
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$\textsc{Pol} \circ \textsc{Inv}$ (resp. ${\color{red}\textsc{Inv} \circ \textsc{Pol}}$) is a closure operator on Op_D (resp. ${\color{red}\mathrm{Rel}_D}$).

This lecture focuses on the relational side (${\color{red}\mathrm{Rel}_D}$).

Characterizing $\textsc{Inv} \circ \textsc{Pol}$.

$\textsc{Inv}\ \textsc{Pol}$ Theorem (Geiger ‘68; Bodnarčuk *et al* ‘69)

(D finite.) For any $\mathcal{R} \subseteq \mathrm{Rel}_D$,

$$\begin{aligned}\textsc{Inv}(\textsc{Pol}(\mathcal{R})) &= \textsc{Clo}_{rel}(\mathcal{R}), \text{ the } \textbf{relational clone} \text{ generated by } \mathcal{R} \\ &= \{ \text{ relations on } D \text{ definable by } \textbf{pp } \mathcal{R}\textbf{-formulas} \}\end{aligned}$$

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Definition (Primitive Positive Formula)

A **pp** $\mathcal{R}$-formula is a first-order formula $\varphi(\mathbf{x})$ of the form $\exists \mathbf{y}(\alpha_1 \wedge \cdots \wedge \alpha_m)$ where each α_i is either an application of a relation in $\mathcal{R}$ to some variables in $\mathbf{x} \cup \mathbf{y}$, or an equality between two variables in $\mathbf{x} \cup \mathbf{y}$.

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The **relational clone membership problem** is the decision problem which, given D and $\mathcal{R} \cup \{s\} \subseteq \mathrm{Rel}_D$, asks whether $s \overset{?}{\in} \mathrm{CLO}_{rel}(\mathcal{R})$.

$\mathrm{I{\scriptsize NV}}\ \mathrm{P{\scriptsize OL}}$ theorem $\Rightarrow \exists$ **witnesses** to both sides of $s \overset{?}{\in} \mathrm{C{\scriptsize LO}}_{rel}(\mathcal{R})$:

- If $s \in \mathrm{C{\scriptsize LO}}_{rel}(\mathcal{R})$, then it's **witnessed** by a pp $\mathcal{R}$-formula (defining s).
- If $s \notin \mathrm{C{\scriptsize LO}}_{rel}(\mathcal{R})$, then it's **witnessed** by an operation (preserving $\mathcal{R}$ but not s).

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- If $s \notin \textsc{Clo}_{rel}(\mathcal{R})$, then it's **witnessed** by an operation (preserving $\mathcal{R}$ but not s).

Questions:

- How **large** must witnesses be (in the worst case)?
  - (Cohen, Jeavons $\sim$ 1995.)
- How **hard** is the relational clone membership problem?
  - (Dalmau, 2000.)

Start with first question (on size of witnesses).

Proof of $\textsc{Inv}\,\textsc{Pol}$ theorem $\Rightarrow$ upper bounds

Input: $\mathcal{R} \cup \{s\} \subseteq \mathrm{Rel}_D$.
Let $d = |D|$, $n = |s|$.

- If $s \in \textsc{Clo}_{rel}(\mathcal{R})$, then this is witnessed by a pp $\mathcal{R}$-formula having d^n variables.
- If $s \notin \textsc{Clo}_{rel}(\mathcal{R})$, then this is witnessed by a polymorphism of $\mathcal{R}$ of arity n.

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First Question (refined):

- Do pp formulas witnessing $s \in \textsc{Clo}_{rel}(\mathcal{R})$ ever require $\overset{\log}{\sim} d^n$ variables?
- Do polymorphisms witnessing $s \notin \textsc{Clo}_{rel}(\mathcal{R})$ ever require arity $\sim n$?

Answer: YES

Theorem 1 (W., 2010)

For infinitely many n there exist D and $\mathcal{R} \cup \mathcal{R}' \cup \{s\} \subseteq \mathrm{Rel}_D$ such that:

- $|s| = n$;
- $s \in \mathrm{CLO}_{rel}(\mathcal{R})$, yet every witnessing pp formula has $\geq 2^{n/3}$ variables;
- $s \notin \mathrm{CLO}_{rel}(\mathcal{R}')$, yet every witnessing polymorphism has arity $\geq n/3$.

Answer: YES

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Moreover,

- $|D| = 22$
- $|\mathcal{R}| = |\mathcal{R}'| = n$;
- Each relation in $\mathcal{R} \cup \mathcal{R}' \cup \{s\}$ has arity $O(\log n)$.

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Second Question (refined)

- Is there a better algorithm than naive search for a negative witness?
  - (Dalmau, 2000.)
- Is the relational clone membership problem co-*NEXPTIME*-complete?
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Theorem 2 (W., announced at NSAC 2009)

$\exists d > 0$ such that the relational clone membership problem restricted to d-element domains is co-*NEXPTIME*-complete.

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Roughly speaking, a **tiling problem** involves:

- An unlimited supply of **tiles**, each having a **tile type** $\in \{t_1, \ldots, t_k\}$.
- A positive integer N, which determines an $N \times N$ grid.

(0,4)	(1,4)	(2,4)	(3,4)	(4,4)
(0,3)	(1,3)	(2,3)	(3,3)	(4,3)
(0,2)	(1,2)	(2,2)	(3,2)	(4,2)
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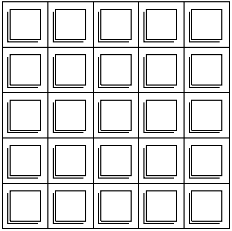

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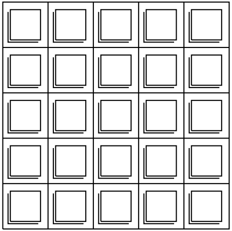

E.g.

types $= \{1, 2, \ldots, 9\}$

constraints:

$$\boxed{i}\,\boxed{j} \quad \text{or} \quad \begin{array}{c}\boxed{j}\\ \boxed{i}\end{array} \quad \Rightarrow \quad i < j$$

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4	5	6	7	8
3	4	5	6	7
2	3	4	5	6
1	2	3	4	5

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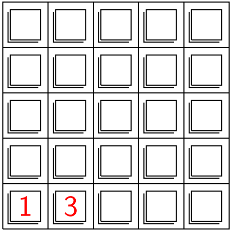

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- Option: can require an **initial condition** (on the first row).

More precisely:

Definition

Fix $N \geq 2$.

1. A **domino system** is a finite relational structure $\mathcal{D} = (\Delta; H, V)$ with H, V binary. ($\Delta =$ "tile types," $H =$ "horizontal," $V =$ "vertical.")

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2. $N = \{0, 1, \ldots, N-1\}$.
3. $\mathcal{B}_N$ denotes the structure $(N \times N; \prec_1, \prec_2)$ where

$$\begin{aligned} \prec_1 &= \{((i,j),(i{+}1,j)) : i,j \in N,\ i \neq N-1\} \\ \prec_2 &= \{((i,j),(i,j{+}1)) : i,j \in N,\ j \neq N-1\}. \end{aligned}$$

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4. A **tiling of** $N \times N$ **by** $\mathcal{D}$ is a homomorphism $\tau : \mathcal{B}_N \to \mathcal{D}$.
5. Given $\mathbf{w} = (w_0, w_1, \ldots, w_{m-1}) \in \Delta^m$ with $m \leq N$, say that a tiling τ of $N \times N$ **satisfies initial condition w** if $\tau(i,0) = w_i \ \ \forall i < m$.

Fix a domino system $\mathcal{D} = (\Delta; H, V)$.

$\underline{\text{EXPTILE}(\mathcal{D})}$, the *Exponential Tiling-by-$\mathcal{D}$ Problem*, is:

Input:

$\mathbf{w} \in \Delta^m$ for some $m \geq 2$.

Question:

Does $\exists$ a tiling of $2^m \times 2^m$ by $\mathcal{D}$ satisfying initial condition $\mathbf{w}$?

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Does $\exists$ a tiling of $2^m \times 2^m$ by $\mathcal{D}$ satisfying initial condition $\mathbf{w}$?

Fact: $\exists$ a "universal" domino system $\mathcal{D}_{\mathrm{u}}$ such that $\textsc{ExpTile}(\mathcal{D}_{\mathrm{u}})$ is *NExpTime*-complete.

Key Construction

For every domino system $\mathcal{D} = (\Delta; H, V)$ there exists a finite set D and a log-space construction

$$\mathbf{w} \in \Delta^m \quad \mapsto \quad \mathcal{R} \cup \{s\} \subseteq \mathrm{Rel}_D$$

such that $s \in \mathrm{CLO}_{rel}(\mathcal{R})$ iff there does **not** exist a tiling of $2^m \times 2^m$ by $\mathcal{D}$ satisfying initial condition $\mathbf{w}$.

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Hence $\mathrm{ExpTile}(\mathcal{D})$ is interpretable into the **complement** of the relational clone membership problem restricted to the domain D.

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Moreover:

- $|s| = 3m$, and s depends only on m (not on $\mathbf{w}$);
- The sizes of witnesses to $s \in [\notin]\ \mathrm{CLO}_{rel}(\mathcal{R})$ are connected to certain properties of tilings/obstructions to tilings.

(With an appropriate $\mathcal{D}$, gives Theorem 1.)

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- Clone membership is known to be hard (*ExpTime*-complete, Bergman *et al*, 1999).
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Variations (on the relational side), e.g.:

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- **Open**: Do there exist fixed D **and** $\mathcal{R}$ such that:
  - Witnesses are large for $s \overset{?}{\in} \mathrm{Clo}_{rel}(\mathcal{R})$?
  - The relational clone membership problem restricted to $D, \mathcal{R}$ is still co-*NEXPTIME*-complete?
- (The latter would nicely complement a result of Kozik 2008 on the operational side.)