. Numerical solution of ordinary differential equations

Let us have a differential equation of the 1st order

$$y' = f(x,y) \quad \text{with an initial condition } y(x_0) = y_0, \tag{1}$$

where f is a continuous function of two variables.

Theorem: : Let f be a continuous function on a closed 2-dimensional interval $\Omega = \langle x_0 - a, x_0 + a\rangle \times \langle y_0 - b, y_0 + b\rangle$, a, $b > 0$. Let there exist a real constant L such that

$$|f(x,u) - f(x,v)| \leq L|u - v| \quad \text{for each}[x,u],\ [x,v] \in \Omega,$$

(f is called **Lipschitz continuous** with respect to y in Ω.) Then there exists a unique function $y = \phi(x)$ that is a solution of (1), and, moreover, it is continuous in some interval $\langle x_0 - k, x_0 + k\rangle$, $k > 0$.

Remark: : The function f is continuous in Ω. Then we can show that f is bounded in Ω, that means that there is such a constant M that $|f(x,y)| \leq M$ for each $[x,y] \in \Omega$.

Remark: : The number k must be chosen to be smaller than a, $\frac{b}{M}$, $\frac{1}{L}$.

A function ϕ is solution of (1) iff it is solution of **integral equation**

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t,\phi(t))\mathrm{d}t\,. \tag{2}$$

The set $\mathcal{C} = \mathcal{C}_{\langle x_0-k,\ x_0+k\rangle}$ of all continuous functions on $\langle x_0 - k,\ x_0 + k\rangle$ with the norm

$$||\phi|| = \max_{x\in\langle x_0-k,\ x_0+k\rangle} |\phi(x)|$$

form a complete normed space.

The subspace $\mathcal{C}^b$ of the space $\mathcal{C}$ consisting of all ϕ z $\mathcal{C}$ with
$\phi(x) \in \langle y_0 - b, y_0 + b\rangle$ for each $x \in \langle x_0 - k,\ x_0 + k\rangle$ form a closed, and thus complete normed subspace of $\mathcal{C}$.

The mapping (transformation) T that assigns g_ϕ to ϕ, i.e. $T(\phi) = g_\phi$, where

$$g_\phi(x) = y_0 + \int_{x_0}^{x} f(t,\phi(t))\mathrm{d}t,$$

maps $\mathcal{C}^b$ into $\mathcal{C}^b$ continuously for some k.

For an arbitrary pair ϕ, ψ from $\mathcal{C}^b$ we have

$$||g_\phi - g_\psi|| \leq Lk\,||\phi - \psi||.$$

If $k < \dfrac{1}{L}$, i.e. $0 < \alpha = kL < 1$, the assumptions of Banach's fixed point theorem are satisfied. Then there exists a unique function ϕ in $\mathcal{C}^b$ that is a fixed point of T, which means that $T(\phi) = g_\phi = \phi$. This means that

$$\phi(x) = g_\phi(x) = y_0 + \int_{x_0}^{x} f(t,\phi(t))\mathrm{d}t \quad \text{for all} x \in \langle x_0 - k,\ x_0 + k\rangle.$$

Iterative method

(1) (1) $\phi_0(x) = y_0$,

(2) (n) $\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))\mathrm{d}t$.

Then the sequence (ϕ_n) converges uniformly (in $\mathcal{C}$) to ϕ in $\langle x_0 - k,\ x_0 + k\rangle$.

Error estimation:

$$||\phi_n - \phi|| \le \frac{kL}{1-kL}|||\phi_n - \phi_{n-1}||, \quad ||\phi_n - \phi|| \le \frac{(kL)^n}{1-kL}\,||\phi_0 - \phi_1||.$$

Remark: : Let us have a linear equation

$$y' + y\,P(x) = Q(x) \text{ i.e. } y' = Q(x) - y\,P(x) \text{ with an initial condition } y(x_0) = y_0, \qquad \text{(LDR)}$$

where P, Q are functions that are continuous on $\langle x_0 - a, x_0 + a\rangle$, $a > 0$.

The function $f(x,y) = Q(x) - y\,P(x)$ is continuous on every $\Omega = \langle x_0 - a, x_0 + a\rangle \times \langle y_0 - b, y_0 + b\rangle$ for every $b > 0$.

The function P is continuous, and thus bounded on every closed interval, i.e. there is a real number L such that $|P(x)| \le L$ for all $x \in \langle x_0 - a, x_0 + a\rangle$.

Thus for all $[x,u]$, $[x,v]$ $\in \Omega$ we get

$$|f(x,u) - f(x,v)| = |v\,P(x) - u\,P(x)| = |P(x)||v - u| \le L|v - u|.$$

The function f is Lipschitz continuous and (LDR) has a unique solution x_0.

Example :: Solve $y' - y = 2x$ with $y(0) = 1$ on $\langle 0; 0,3\rangle$, $n = 4$.

Transform: $y' = y + 2x$. The function $f(x,y) = y + 2x$ is continuous on $\mathbb{R}^2$. Then it is Lipschitz continuous, $P(x) = 1$ for all x, and we can take $L = |P(x)| = 1$.

Take $\Omega = \langle 0 - a, 0 + a\rangle \times \langle 1 - b, 1 + b\rangle$, $a = 1$ so that $\langle 0 - a, 0 + a\rangle$ includes $\langle 0; 0,3\rangle$.

We have $|f(x,y)| = |y + 2x| \le b + 3$ for $x \in \langle -1, 1\rangle$. Take $M = b + 3$. Take k and b so that k is smaller than $a = 1$, $\frac{b}{M} = \frac{b}{b+3}$, $\frac{1}{L} = 1$. If $k = 0,4$ and $b = 3$ the condition holds because $0,4 < \frac{3}{3+3} = 0,5$.

The iteration converge to the solution that is continuous on $\langle -0,4; 0,4\rangle$.

Iterations:

$\phi_0(x) = 1$,

$\phi_1(x) = 1 + \int_0^x f(t,1)\mathrm{d}t = 1 + \int_0^x (1 + 2t)\mathrm{d}t = 1 + x + x^2$,

$\phi_2(x) = 1 + \int_0^x f(t, 1 + t + t^2)\mathrm{d}t = 1 + \int_0^x (1 + 3t + t^2)\mathrm{d}t = 1 + x + \frac{3x^2}{2} + \frac{x^3}{3}$,

$\phi_3(x) = 1 + \int_0^x f\left(t, 1 + t + \frac{3t^2}{2} + \frac{t^3}{3}\right)\mathrm{d}t = 1 + \int_0^x \left(1 + 3t + \frac{3t^2}{2} + \frac{t^3}{3}\right)\mathrm{d}t = 1 + x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{12}$,

$\phi_4(x) = 1 + \int_0^x f\left(t, 1 + t + \frac{3t^2}{2} + \frac{t^3}{2} + \frac{t^4}{12}\right)\mathrm{d}t = 1 + \int_0^x \left(1 + 3t + \frac{3t^2}{2} + \frac{t^3}{2} + \frac{t^4}{12}\right)\mathrm{d}t = 1 + x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \frac{x^5}{60}$.

Using $\phi_4(x)$ evaluate values in 0; 0,1; 0,2; 0,3:

$\phi_4(0) = 1$,

$\phi_4(0,1) = 1,115513,$
$\phi_4(0,2) = 1,264205,$
$\phi_4(0,3) = 1,449525.$

Error estimations:
$d(\phi_4, \phi) \leq \frac{Lk}{1-Lk} d(\phi_4, \phi_5) = \frac{0,4}{1-0,4} \left(\frac{0,3^5}{60} + \frac{0,3^4}{24} \right) = 0,000252.$
$d(\phi_4, \phi) \leq \frac{(Lk)^4}{1-Lk} d(\phi_0, \phi_1) = \frac{0,4^4}{1-0,4} (0,3^2 + 0,3) = 0,01664.$

The exact solution is $\phi(x) = -2x - 2 + 3\mathrm{e}^x$. Thus
$\phi(0) = 1,$
$\phi(0,1) \approx 1,115513,$
$\phi(0,2) \approx 1,264208,$
$\phi(0,3) \approx 1,449576.$

Runge-Kutta methods

Let us have an equation

$$y' = f(x,y) \quad \text{with} y(x_0) = y_0, \tag{1}$$

where f is a real function of 2 variables.

Take a positive number h. Denote
$x_1 = x_0 + h,$
$x_2 = x_1 + h = x_0 + 2h,$
.... ,
$x_n = x_0 + nh$ for $n \in \mathbb{N},$
....
Approximate values of ϕ that is the solution of (1) in x_n are
$\phi(x_0) = y_0,$
$\phi(x_1) \approx y_1 = y_0 + hz_0,$
$\phi(x_2) \approx y_2 = y_1 + hz_1,$
.......,
$\phi(x_{n+1}) \approx y_{n+1} = y_n + hz_n,$ (RK)
.......,
where z_n is a function of the variables h, x_n, y_n and depends on f.

A method is **convergent of order** $p = 1, 2, ...$ if for the error estimation (difference between the exact and approximate solution) ϵ_h we have:

$$\epsilon_h = O(h^{p+1}),$$

i.e.

$$\epsilon_h \leq Kh^{p+1} \text{ for some } K.$$

(Runge-Kutta method of the 1st order) Euler method

Let us have a differential equation of the 1st order

$$y' = f(x,y) \quad \text{with an initial condition} y(x_0) = y_0, \tag{1}$$

where f is a real function of two variables.

Take a positive number h. Denote

$x_1 = x_0 + h$,
$x_2 = x_1 + h = x_0 + 2h$,
.......,
$x_n = x_0 + nh$ for $n \in \mathbb{N}$,
....
Approximate values of the function ϕ that is the solution of (1) in the points x_n are
$\phi(x_0) = y_0$,
$\phi(x_1) \approx y_1 = y_0 + hf(x_0, y_0)$,
$\phi(x_2) \approx y_2 = y_1 + hf(x_1, y_1)$,
.......,
$\phi(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n)$,
....
The integral curve is substituted by the polyline. With increasing n the error is increasing, as well.

Example :: Solve $y' = y + 2x$ with $y(0) = 1$ on $\langle 0; 0,3 \rangle$, $h = 0,1$.

Put the solutions into the tablet:

x_n	y_n	$2x_n$	$f(x_n, y_n) = y_n + 2x_n$	$hf(x_n, y_n)$	$y_n + hf(x_n, y_n)$
0	1	0	1	$0,1$	$1,1$
$0,1$	$1,1$	$0,2$	$1,3$	$0,13$	$1,23$
$0,2$	$1,23$	$0,4$	$1,63$	$0,163$	$1,393$
$0,3$	$1,393$				

Euler method is convergent of order 1.

(Runge-Kutta method of the 2nd order) Modified Euler method

Let us have an equation

$$y' = f(x, y) \quad \text{with } y(x_0) = y_0, \tag{1}$$

where f is a rela function of two variables.

Take a positive number h. Denote
$x_1 = x_0 + h$,
$x_2 = x_1 + h = x_0 + 2h$,
.......,
$x_n = x_0 + nh$, pro $n \in \mathbb{N}$,
....
Approximate values of the solution ϕ:
$\phi(x_0) = y_0$,
$\phi(x_1) \approx y_1 = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$,
$\phi(x_2) \approx y_2 = y_1 + hf(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1))$,
.......,
$\phi(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_1, y_0))$,
....
This method is convergent of order 2.

Four point RK method (of the 4th order)

The most used RK method is **four point method**, where z_n are given:

$$z_n = w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4,$$

where
$k_1 = f(x_n, y_n)$,
$k_2 = f(x_n + \alpha_2 h, y_n + \beta_{21}hk_1)$,
$k_3 = f(x_n + \alpha_3 h, y_n + \beta_{31}hk_1 + \beta_{32}hk_2)$,
$k_4 = f(x_n + \alpha_4 h, y_n + \beta_{41}hk_1 + \beta_{42}hk_2 + \beta_{43}hk_3)$.

We can get many formulas, the most used formula is:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where
$k_1 = f(x_n, y_n)$,
$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$,
$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$,
$k_4 = f(x_n + h, y_n + hk_3)$.

Example :: Solve $y' = y + 2x$ with $y(0) = 1$ on $\langle 0; 0,2\rangle$, $h = 0,1$.

Evaluate y_1:
$k_1 = f(0,1) = 1$,
$k_2 = f(0 + 0,05; 1 + 0,05 \cdot 1) = f(0,05; 1,05) = 1,15$,
$k_3 = f(0 + 0,05; 1 + 0,05 \cdot 1,15) = f(0,05; 1,0575) = 1,1575$,
$k_4 = f(0 + 0,1; 1 + 0,1 \cdot 1,1575) = f(0,1; 1,11575) = 1,31575$.

$$y_1 = 1 + \frac{0,1}{6}(1 + 2 \cdot 1,15 + 2 \cdot 1,1575 + 1,31575) = 1,1155125.$$

This result coincides with the exact solution up to six decimal places.
Evaluate y_2:
$k_1 = f(0,1; 1,1155125) = 1,3155125$,
$k_2 = f(0,1 + 0,05; 1,1155125 + 0,05 \cdot 1,3155125) = f(0,15; 1,181288125) =$
$= 1,481288125$,
$k_3 = f(0,1 + 0,05; 1,1155125 + 0,05 \cdot 1,481288125) = f(0,15; 1,18957690625) =$
$= 1,48957690625$,
$k_4 = f(0,1 + 0,1; 1,1155125 + 0,1 \cdot 1,48957690625) = f(0,2; 1,264470190625) =$
$= 1,664470190625$.

$$y_2 = 1,1155125 + \frac{0,1}{6}(1,3155125 + 2 \cdot 1,481288125 + 2 \cdot 1,48957690625 + 1,664470190625) =$$

$$= 1,2642077125521.$$

MATLAB - command beggining *ode*.
For instance *ode*23 or *ode*45.
Syntax: $[t, Y] = ode23(odefun, [t0, tf], y0)$, where $odefun$ is f from above, $[t0, tf]$ is the given interval, $y0$ is the initial condition.
Command $ode23(nazevfce, [t0, tf], y0)$ makes the plot of the solution.
