Numerical solution of partial differential equations

Finite difference method

Recall the definition of the derivative of a function of one variable. If x is an interior point of the domain of f then **derivative** of the function f in the point x is the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Remark:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h),$$

where $O(h)$ converges to 0 whenever h converges to 0.

The first derivative of f will be substituted by

forward difference (approximation) $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$,

backward difference (approximation) $f'(x) \approx \dfrac{f(x) - f(x-h)}{h}$,

central difference (approximation) $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$.

The second derivative of f will be substituted by

$$f''(x) \approx \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

The finite difference method will be explained for the simpler type of differential equation:

Heat equation

Let us have the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}. \qquad \text{(T)}$$

Let an interval $\langle 0,1\rangle \times \langle 0,T\rangle$ be the domain of the function $f(x,t)$, that means that $x \in \langle 0,1\rangle$, $t \in \times\langle 0,T\rangle$. Divide the interval $\langle 0,1\rangle$ into M intervals with the length equal δx and the interval $\langle 0,T\rangle$ into N intervals with the length equal δt. We obtain **grid** containing $(M+1)\times(N+1)$ points.

Denote $f_{i,j} = f(i\delta x, j\delta t)$ for $i = 0, 1, ..., M$, $j = 0, 1, ..., N$.

Let us have an **initial condition**

$f(x,0) = u(x)$,

and **boundary conditions**

$f(0,t) = v_1(t)$,

$f(1,t) = v_2(t)$, where u is a function of one variable defined on the interval $\langle 0,1\rangle$, v_1, v_2 are functions defined on $\langle 0,T\rangle$.

EXPLICIT METHOD

The first partial derivative of the function f with respect to t in the equation (T) will be substituted by the forward difference. We transform the equation (T):

$$\frac{f_{i,j+1} - f_{i,j}}{\delta t} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\delta x^2}.$$

If we denote $r = \dfrac{\delta t}{\delta x^2}$, the equation will be of the form:

$$f_{i,j+1} = r f_{i-1,j} + (1 - 2r) f_{i,j} + r f_{i+1,j}.$$

We call the method **convergent** if the error (difference between the obtained and exact solutions) converges to zero whenever the lengths $\delta t, \delta x$ converge to zero and M, N converge to infinity (the grid is finer). The explicit method is convergent whenever $r \le \frac{1}{2}$.

Example 1: : Solve the partial differential equation on the interval $\langle 0, 1\rangle \times \langle 0; 0, 1\rangle$

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

with the boundary conditions $f(0,t) = \mathrm{e}^{-\frac{1}{4}\pi^2 t}$, $f(1,t) = 0$,
and the initial condition $f(x,0) = \cos(\frac{1}{2}\pi x)$,
$\delta x = 0,2$, $\delta t = 0,02$.

Denote $r = \dfrac{\delta t}{\delta x^2} = \dfrac{1}{2}$. After substitution into the equation we get

$$f_{i,j+1} = \frac{1}{2}(f_{i-1,j} + f_{i+1,j}).$$

The first and last columns are obtained from the boundary conditions, the last row is from the initial condition. We evaluate values in all points of the grid from below.

For instance: $f_{2,1} = \frac{1}{2}(f_{1,0} + f_{3,0}) = \frac{1}{2}(0,95106 + 0,58779) = 0,769425 \approx 0,76943$.

We put the results into the tablet:

	$0,78134$	$0,74310$	$0,63212$	$0,45926$	$0,24145$	0
$j=5$	$0,78134$	$0,74157$	$0,63013$	$0,45754$	$0,24049$	0
$j=4$	$0,82087$	$0,77927$	$0,66227$	$0,48099$	$0,25282$	0
$j=3$	$0,86239$	$0,81889$	$0,69615$	$0,50564$	$0,26583$	0
$j=2$	$0,90602$	$0,86064$	$0,73177$	$0,53167$	$0,27951$	0
$j=1$	$0,95185$	$0,90451$	$0,76943$	$0,55902$	$0,29390$	0
$j=0$	1	$0,95106$	$0,80902$	$0,58779$	$0,30902$	0
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$

In the first row there are the values of the exact solution.

Example 2: : Solve the equation on $\langle 0, 1\rangle \times \langle 0; \frac{5}{36}\rangle$

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

with the boundary conditions $f(0,t) = 0,\ f(1,t) = 0,$
and the initial condition $f(x, 0) = 1$ for $x = 0,5,\ f(x, 0) = 0$ otherwise ,
$\delta x = \frac{1}{6},\ \delta t = \frac{1}{36}.$

Denote $r = \frac{\delta t}{\delta x^2} = 1$. After substitution we get

$$f_{i,j+1} = f_{i-1,j} - f_{i,j} + f_{i+1,j}.$$

The results are in the tablet:

$j=5$	0	-25	44	-51	44	-25	0
$j=4$	0	9	-16	19	-16	9	0
$j=3$	0	-3	6	-7	6	-3	0
$j=2$	0	1	-2	3	-2	1	0
$j=1$	0	0	1	-1	1	0	0
$j=0$	0	0	0	1	0	0	0
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$

The numerical solution does not converge to the exact solution. The condition n$r \leq \frac{1}{2}$ does not hold. From the previous example it is clear that the values cannot be negative.

IMPLICIT METHOD

In the equation (T) the first partial derivative of f with respect to t will be substituted by the backward difference. The equation will be transformed:

$$\frac{f_{i,j} - f_{i,j-1}}{\delta t} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\delta x^2}.$$

If we denote $r = \frac{\delta t}{\delta x^2}$ we get:

$$-rf_{i-1,j} + (1 + 2r)f_{i,j} - rf_{i+1,j} = f_{i,j-1}.$$

For each $j = 0, ..., N - 1$ we solve the system of $M - 1$ linear equations:

$$\begin{bmatrix} 1+2r & -r & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ -r & 1+2r & -r & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & -r & 1+2r & -r & 0 & ... & 0 & 0 & 0 & 0 \\ ... \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & -r & 1+2r & -r \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & -r & 1+2r \end{bmatrix} \begin{bmatrix} f_{1,j+1} \\ f_{2,j+1} \\ f_{3,j+1} \\ ... \\ f_{M-2,j+1} \\ f_{M-1,j+1} \end{bmatrix} =$$

$$= \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \\ ... \\ f_{M-2,j} \\ f_{M-1,j} \end{bmatrix} + \begin{bmatrix} rf_{0,j} \\ 0 \\ 0 \\ ... \\ 0 \\ rf_{M,j} \end{bmatrix}.$$

This method is convergent for every real r.

Example 3: : On $\langle 0, \frac{4}{6}\rangle \times \langle 0; \frac{4}{36}\rangle = \langle 0, \frac{2}{3}\rangle \times \langle 0; \frac{1}{9}\rangle$ solve

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

with the boundary conditions $f(0,t) = 0$, $f(1,t) = 0$,
and the initial condition $f(x,0) = 1$ for $x = \frac{2}{6} = \frac{1}{3}$, $f(x,0) = 0$ otherwise ,
$\delta x = \dfrac{1}{6}$, $\delta t = \dfrac{1}{36}$.

Denote $r = \dfrac{\delta t}{\delta x^2} = 1$. After substitution we get

$$-f_{i+1,j} + 3f_{i,j} - f_{i-1,j} = f_{i,j-1}.$$

We have M, $N = 4$. For each $j = 0, 1, 2, 3$ we solve the system with 3 linear equations:

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} f_{1,j+1} \\ f_{2,j+1} \\ f_{3,j+1} \end{bmatrix} = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \end{bmatrix} + \begin{bmatrix} f_{0,j} \\ 0 \\ f_{4,j} \end{bmatrix}.$$

In this example the boundary conditions are zeros. If we denote

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \; X_j = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \end{bmatrix},$$

we get

$$AX_{j+1} = X_j \text{ then } X_{j+1} = A^{-1}X_j.$$

We have $A^{-1} = \frac{1}{21} \begin{bmatrix} 8 & 3 & 1 \\ 8 & 24 & 8 \\ 1 & 3 & 8 \end{bmatrix}$.

From the initial condition we get $X_0 = \begin{bmatrix} f_{1,0} \\ f_{2,0} \\ f_{3,0} \end{bmatrix} = \begin{bmatrix} f(\frac{1}{6},0) \\ f(\frac{2}{6},0) \\ f(\frac{3}{6},0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Evaluate

$$X_1 = A^{-1}X_0 = \frac{1}{21}\begin{bmatrix} 8 & 3 & 1 \\ 8 & 24 & 8 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0,14286 \\ 1,14286 \\ 0,14286 \end{bmatrix},$$

$$X_2 = A^{-1}X_1 = \frac{1}{21}\begin{bmatrix} 8 & 3 & 1 \\ 8 & 24 & 8 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0,14286 \\ 1,14286 \\ 0,14286 \end{bmatrix} \approx \begin{bmatrix} 0,22449 \\ 1,41497 \\ 0,22449 \end{bmatrix},$$

$$X_3 = A^{-1}X_2 = \frac{1}{21}\begin{bmatrix} 8 & 3 & 1 \\ 8 & 24 & 8 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0,22449 \\ 1,41497 \\ 0,22449 \end{bmatrix} \approx \begin{bmatrix} 0,29835 \\ 0,37348 \\ 0,29835 \end{bmatrix},$$

$$X_4 = A^{-1}X_3 = \frac{1}{21}\begin{bmatrix} 8 & 3 & 1 \\ 8 & 24 & 8 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0,29835 \\ 0,37348 \\ 0,29835 \end{bmatrix} \approx \begin{bmatrix} 0,18118 \\ 0,65381 \\ 0,18118 \end{bmatrix}.$$

The results are in the tablet:

$j=4$	0	0,18118	0,65381	0,18118	0
$j=3$	0	0,29835	0,37318	0,29835	0
$j=2$	0	0,22449	1,41497	0,22449	0
$j=1$	0	0,14286	1,14286	0,14286	0
$j=0$	0	0	1	0	0
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$

The implicit method is alvays stable (convergent).

Black-Scholes equation

Solve the partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \qquad \text{(BS)}$$

Let us consider an interval $\langle 0, S_{max}\rangle \times \langle 0, T\rangle$, partition the interval $\langle 0, S_{max}\rangle$ into intervals of the same length δS and $\langle 0, T\rangle$ into intervals of the length δt.

Find approximate values of $f(S,t)$ in points $[i\delta S, j\delta t]$, $i = 0,1,2,...,M$, $k = 0,1,2,...,N$.

Denote $f_{i,j} = f(i\delta S, j\delta t)$.

Approximations of the first partial derivatives:

Forward difference: $\frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j}}{\delta t}$ $\frac{\partial f}{\partial S} = \frac{f_{i+1,j} - f_{i,j}}{\delta S}$.

Backward difference: $\frac{\partial f}{\partial t} = \frac{f_{i,j} - f_{i,j-1}}{\delta t}$ $\frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i-1,j}}{\delta S}$.

Central diffeerence: $\frac{\partial f}{\partial t} = \frac{f_{i,j+1} - f_{i,j-1}}{2\delta t}$ $\frac{\partial f}{\partial S} = \frac{f_{i+1,j} - f_{i-1,j}}{2\delta S}$.

Approximation of the second partial derivative:

$$\frac{\partial^2 f}{\partial S^2} = \frac{\frac{f_{i+1,j}-f_{i,j}}{\delta S} - \frac{f_{i,j}-f_{i-1,j}}{\delta S}}{\delta S} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\delta S^2}\ .$$

Boundary conditions:

for European call option:

$f(S,T) = \max\{S-K, 0\}$,

$f(0,t) = 0$,

$f(S_{max}, t) = S_{max} - K\mathrm{e}^{-r(T-t)}$,

for European put option:

$f(S,T) = \max\{K-S, 0\}$,

$f(0,t) = K\mathrm{e}^{-r(T-t)}$,

$f(S_{max}, t) = 0$.

EXPLICIT METHOD

In the equation (SB) we approximate the first partial derivative of f with respect to S by the central difference, the derivative with respect to t is approximated by the backward difference:

$$\frac{f_{i,j} - f_{i,j-1}}{\delta t} + ri\delta S \frac{f_{i+1,j} - f_{i-1,j}}{2\delta S} + \frac{1}{2}\sigma^2 i^2 \delta S^2 \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\delta S^2} = rf_{i,j} \,.$$

Boundary conditions:

For example

for American put option:

$f_{i,N} = \max\{K - i\delta S, 0\}$, $i = 0, 1, ..., M$,
$f_{0,j} = K$, $j = 0, 1, ..., N$,
$f_{M,j} = 0$, $j = 0, 1, ..., N$.

for European call option:

$f_{i,N} = \max\{i\delta S - K, 0\}$, $i = 0, 1, ..., M$,
$f_{0,j} = 0$, $j = 0, 1, ..., N$,
$f_{M,j} = S_{max} - K\mathrm{e}^{-r(N-j)\delta t}$ $j = 0, 1, ..., N$,

for European put option:

$f_{i,N} = \max\{K - i\delta S, 0\}$, $i = 0, 1, ..., M$,
$f_{0,j} = K\mathrm{e}^{-r(N-j)\delta t}$, $j = 0, 1, ..., N$,
$f_{M,j} = 0$, $j = 0, 1, ..., N$.

Transform the equation (SB):

$$f_{i,j-1} = a_i f_{i-1,j} + b_i f_{i,j} + c_i f_{i+1,j}, \; i = 1, 2, ..., M-1, \; j = N-1, N-2, ..., 1, 0,$$

where $a_i = \frac{1}{2}\delta t(\sigma^2 i^2 - ri)$,
$b_i = 1 - \delta t(\sigma^2 i^2 + r)$,
$c_i = \frac{1}{2}\delta t(\sigma^2 i^2 + ri)$.

The explicit method is stable (convergent) provided $a_i \geq 0$, $b_i \geq 0$, $c_i \geq 0$, $i = 1, 2, ..., M-1$, and $\delta t \leq \frac{1}{\sigma^2 M^2}$. If M increases , which means that δS decreases or S_{max} increases, the method need not be more precise.

IMPLICIT METHOD

In (SB) the first derivative of f along S is approximated by the central difference, the derivative along t is approximated by the forward difference:

$$\frac{f_{i,j+1} - f_{i,j}}{\delta t} + ri\delta S \frac{f_{i+1,j} - f_{i-1,j}}{2\delta S} + \frac{1}{2}\sigma^2 i^2 \delta S^2 \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\delta S^2} = rf_{i,j} \,.$$

We get:

$$f_{i,j+1} = a_i f_{i-1,j} + b_i f_{i,j} + c_i f_{i+1,j}, \; i = 1, 2, ..., M-1, \; j = N-1, N-2, ..., 1, 0,$$

where $a_i = \frac{1}{2}\delta t(ri - \sigma^2 i^2)$,
$b_i = 1 + \delta t(\sigma^2 i^2 + r)$,
$c_i = -\frac{1}{2}\delta t(\sigma^2 i^2 + ri)$.

for each $j = N-1, N-2, ..., 0$ we solve the system of $M-1$ linear equations:

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & ... & 0 & 0 & 0 & 0 \\ ... & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & a_{M-2} & b_{M-2} & c_{M-2} \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 & a_{M-1} & b_{M-1} \end{bmatrix} \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \\ ... \\ f_{M-2,j} \\ f_{M-1,j} \end{bmatrix} =$$

$$= \begin{bmatrix} f_{1,j+1} \\ f_{2,j+1} \\ f_{3,j+1} \\ ... \\ f_{M-2,j+1} \\ f_{M-1,j+1} \end{bmatrix} - \begin{bmatrix} a_1 f_{0,j} \\ 0 \\ 0 \\ ... \\ 0 \\ c_{M-1} f_{M,j} \end{bmatrix}.$$
