1BP453 - Computational Finance

Numerical evaluation of derivatives - derivation

Let us recall that $L_n(x)$ is Lagrange interpolating polynomial of degree n and f is the original function. We differentiate $L_n^{'}(x)$ in poles $x_0, \dots, x_n$, $n \in \mathbb{N}$, to approximate $f'(x)$.

Let f be smooth function on the interval $\langle x_0, x_n \rangle$ such that f has $n+1$ continuous derivatives, i.e.,$f \in \mathcal{C}^{n+1}_{\langle x_0, x_n \rangle}$. Then the estimation error can be expressed as

$$f'(x_i) - L_n'(x_i) = \frac{f^{(n+1)}(\theta_i)}{(n+1)!}\omega_{n+1}'(x_i), \quad i = 0, \dots, n$$

where $\theta \in \langle x_0, x_n \rangle$ and $\omega_{n+1}(x) = (x - x_0)(x - x_1)\dots(x - x_n)$.

Consider equidistant division of the interval $\langle x_0, x_n \rangle$, i.e., $x_i = x_0 + ih$, and therefore $h = x_{i+1} - x_i$.

First forward difference

1. For $n = 1$, poles $x, x + h$:

$$\begin{aligned}
L_1(x) &= \sum_{i=0}^{1} f(x_i) l_i(x) \\
&= f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0}, \\
L_1'(x_0) &= f(x_0)\frac{1}{x_0 - x_1} + f(x_1)\frac{1}{x_1 - x_0} \\
&= -\frac{f(x_0)}{h} + \frac{f(x_0 + h)}{h} = \frac{f(x_0 + h) - f(x_0)}{h}, \\
\omega_2'(x_0) &= \left[(x - x_0)(x - x_1)\right]_{x = x_0}^{'} = -h, \\
f'(x) &= \underbrace{L_1'(x)}_{D_P(x,h)} + \underbrace{\frac{f''(\theta)}{2}(-h)}_{\text{error}}, \quad \theta \in \langle x, x + h \rangle
\end{aligned}$$

2. For $n = 2$, poles $x, x+h, x+2h$:

$$\begin{aligned}
L_2(x) &= \sum_{i=0}^{2} f(x_i) l_i(x) \\
&= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\
&\quad + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)},
\end{aligned}$$

$$\begin{aligned}
L_2'(x_0) &= f(x_0)\frac{2x_0-x_1-x_2}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{x_0-x_2}{(x_1-x_0)(x_1-x_2)} \\
&\quad + f(x_2)\frac{x_0-x_1}{(x_2-x_0)(x_2-x_1)} \\
&= -\frac{f(x_0)3h}{2h^2} + \frac{f(x_0+h)2h}{h^2} - \frac{f(x_0+2h)h}{2h^2} \\
&= \frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h},
\end{aligned}$$

$$\omega_2'(x_0) = \left[(x-x_0)(x-x_1)(x-x_2)\right]'_{x=x_0} = 2h^2,$$

$$f'(x) = L_2'(x) + \frac{f'''(\theta)}{3}h^2, \quad \theta \in \langle x, x+2h \rangle$$

First backward difference

1. For $n = 1$, poles $x - h, x$:

$$
\begin{aligned}
L_1(x) &= \sum_{i=0}^{1} f(x_i) l_i(x) \\
&= f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0}, \\
L_1'(x_1) &= f(x_0)\frac{1}{x_0 - x_1} + f(x_1)\frac{1}{x_1 - x_0} \\
&= -\frac{f(x_1 - h)}{h} + \frac{f(x_1)}{h} = \frac{f(x_1) - f(x_1 - h)}{h}, \\
\omega_2'(x_1) &= \left[(x - x_0)(x - x_1)\right]'_{x=x_1} = h, \\
f'(x) &= \underbrace{L_1'(x)}_{D_L(x,h)} + \frac{f''(\theta)}{2}h, \quad \theta \in \langle x - h, x\rangle
\end{aligned}
$$

2. For $n=2$, poles $x-2h, x-h, x$:

$$\begin{aligned}
L_2(x) &= \sum_{i=0}^{2} f(x_i) l_i(x) \\
&= f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\
&\quad + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}, \\
L_2'(x_2) &= f(x_0)\frac{x_2-x_1}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{x_2-x_0}{(x_1-x_0)(x_1-x_2)} \\
&\quad + f(x_2)\frac{2x_2-x_0-x_1}{(x_2-x_0)(x_2-x_1)} \\
&= \frac{f(x_2-2h)h}{2h^2} - \frac{f(x_2-h)2h}{h^2} + \frac{f(x_2)3h}{2h^2} \\
&= \frac{3f(x_2)-4f(x_2-h)+f(x_2-2h)}{2h}, \\
\omega_2'(x_2) &= \left[(x-x_0)(x-x_1)(x-x_2)\right]'_{x=x_2} = 2h^2, \\
f'(x) &= L_2'(x) + \frac{f'''(\theta)}{3}h^2, \quad \theta \in \langle x-2h, x\rangle
\end{aligned}$$

First central difference

For $n=2$, poles $x-h, x, x+h$:

$$\begin{aligned}
L_2'(x_1) &= f(x_0)\frac{x_1-x_2}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{2x_1-x_2-x_0}{(x_1-x_0)(x_1-x_2)} \\
&\quad + f(x_2)\frac{x_1-x_0}{(x_2-x_0)(x_2-x_1)} \\
&= -\frac{f(x_1-h)h}{2h^2} + \frac{f(x_1+h)h}{2h^2} \\
&= \frac{f(x_1+h)-f(x_1-h)}{2h}, \\
\omega_2'(x_1) &= \left[(x-x_0)(x-x_1)(x-x_2)\right]'_{x=x_1} = -h^2, \\
f'(x) &= \underbrace{L_2'(x)}_{D_C(x,h)} + \frac{f'''(\theta)}{6}(-h^2), \quad \theta \in \langle x-h, x+h\rangle
\end{aligned}$$

Second central difference

For $n = 2$, poles $x - h, x, x + h$:

$$\begin{aligned}
L_2''(x_1) &= f(x_0)\frac{2}{(x_0 - x_1)(x_0 - x_2)} + f(x_1)\frac{2}{(x_1 - x_0)(x_1 - x_2)} \\
&\quad + f(x_2)\frac{2}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{2f(x_1 - h)}{2h^2} - \frac{2f(x_1)}{h^2} + \frac{2f(x_1)}{2h^2} \\
&= \frac{f(x_1 + h) - 2f(x_1) + f(x_1 - h)}{h^2},
\end{aligned}$$

$$f'(x) = \underbrace{L_2'(x)}_{D_{2C}(x,h)} + \frac{f^{(4)}(\theta)}{12}(-h^2), \quad \theta \in \langle x - h, x + h \rangle$$