1BP453 - Computational Finance

Numerical evaluation of integrals - derivation

We approximate the value of an definite integral $I(f) = \int_a^b f(x)dx$, where f is continuous on the interval $\langle a, b\rangle$.

The interval $\langle a, b\rangle$ is divided into n subintervals such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Let us consider equidistant partitioning where $x_{k+1} - x_k = h = \frac{b-a}{n}$. The integral is partitioned as follows

$$I(f) = \int\limits_a^b f(x)dx = \sum_{k=0}^{n-1} \int\limits_{x_k}^{x_{k+1}} f(x)dx.$$

We substitute the function f by the Lagrange polynomial N-th degree,$L_N(x)$, on the interval $\langle x_k, x_{k+1+N}$ and evaluate its integral.

Rectangle midpoint rule

Error calculation:

In case of midpoint rule cannot be used the first mean value theorem for integrals directly. However, Taylor's formula can be used considering that f has continuous second derivative on the interval $\langle x_k, x_{k+1}\rangle$, i.e., $f \in \mathcal{C}^2_{\langle x_k, x_{k+1}\rangle}$. For each $x \in \langle x_k, x_{k+1}\rangle$ exists $\eta(x) \in \langle x_k, x_{k+1}\rangle$ such that

$$\begin{aligned} f(x) \quad &= \quad f\left(\frac{x_k + x_{k+1}}{2}\right) + f'\left(\frac{x_k + x_{k+1}}{2}\right)\left(x - \frac{x_k + x_{k+1}}{2}\right) \\ &\quad + \frac{f''(\eta(x))}{2}\left(x - \frac{x_k + x_{k+1}}{2}\right)^2. \end{aligned}$$

We can use the first mean value theorem for integrals now and we get that there exists $\xi \in \langle x_k, x_{k+1}\rangle$ such that

1. Simple formula:

$$\begin{aligned}\text{error} &= \frac{f''(\xi)}{2}\int\limits_{x_k}^{x_{k+1}}\left(x-\frac{x_k+x_{k+1}}{2}\right)^2 dx\\ &= \frac{f''(\xi)}{2}\int\limits_{\frac{x_k-x_{k+1}}{2}}^{\frac{x_{k+1}-x_k}{2}} t^2 dt\\ &= f''(\xi)\frac{(x_{k+1}-x_k)^3}{24} = f''(\xi)\frac{h^3}{24}\end{aligned}$$

2. Composed formula:

   For $\xi_k \in \langle a, b\rangle$

$$\begin{aligned}\text{error} &= \sum_{k=0}^{n-1}\int\limits_{x_k}^{x_{k+1}} f''(\xi_k)\left(x-\frac{x_k+x_{k+1}}{2}\right)^2 dx\\ &= \frac{h^3}{24}\sum_{k=0}^{n-1} f''(\xi_k).\end{aligned}$$

   Then exists $\xi \in \langle a, b\rangle$ such that

$$\text{error} = n\frac{h^3}{24}f''(\xi) = \frac{h^2}{24}(b-a)f''(\xi).$$

Trapezoid rule

Integral calculation:

$$\begin{aligned}\int\limits_{x_k}^{x_{k+1}} L_1(x)dx &= \int\limits_{x_k}^{x_{k+1}}\left[f\left(x_k\right)\frac{x-x_{k+1}}{x_k-x_{k+1}} + f\left(x_{k+1}\right)\frac{x-x_k}{x_{k+1}-x_k}\right]dx\\ &= -\int\limits_0^1 f(x_k)(x_k-x_{k+1})tdt + \int\limits_0^1 f(x_{k+1})(x_{k+1}-x_k)tdt\\ &= \frac{f(x_k)(x_{k+1}-x_k)}{2} + \frac{f(x_{k+1})(x_{k+1}-x_k)}{2}\\ &= \frac{x_{k+1}-x_k}{2}\left(f(x_{k+1})+f(x_k)\right).\end{aligned}$$

Error calculation:

1. Simple formula:

$$\begin{aligned}
\text{error} &= \frac{f''(\xi)}{2}\int\limits_{x_k}^{x_{k+1}} (x-x_k)\,(x-x_{k+1})\,dx \\
&= \frac{f''(\xi)}{2}\left[\frac{x_k^3}{6}-\frac{x_k^2x_{k+1}}{2}+\frac{x_kx_{k+1}^2}{2}-\frac{x_{k+1}^3}{6}\right] \\
&= -f''(\xi)\frac{(x_{k+1}-x_k)^3}{12} = -f''(\xi)\frac{h^3}{12}.
\end{aligned}$$

2. Composed formula:

For $\xi_k \in \langle a,b\rangle$

$$\begin{aligned}
\text{error} &= \sum_{k=0}^{n-1}\int\limits_{x_k}^{x_{k+1}} (x-x_k)\,(x-x_{k+1})\,dx \\
&= -\frac{h^3}{12}\sum_{k=0}^{n-1} f''(\xi_k).
\end{aligned}$$

Then exists $\xi \in \langle a,b\rangle$ such that

$$\text{error} = -n\frac{h^3}{12}f''(\xi) = -\frac{h^2}{12}(b-a)f''(\xi).$$

Simpson's rule

Integral calculation:

$$\begin{aligned}
\int\limits_{x_k}^{x_{k+2}} L_2(x)dx &= \int\limits_{x_k}^{x_{k+2}} \Bigg[f\left(x_k\right)\frac{(x-x_{k+1})(x-x_{k+2})}{(x_k-x_{k+1})(x_k-x_{k+2})} \\
&\quad +f\left(x_{k+1}\right)\frac{(x-x_k)(x-x_{k+2})}{(x_{k+1}-x_k)(x_{k+1}-x_{k+2})} \\
&\quad +f\left(x_{k+2}\right)\frac{(x-x_k)(x-x_{k+1})}{(x_{k+2}-x_k)(x_{k+2}-x_{k+1})}\Bigg]dx \\
&= \frac{x_{k+2}-x_k}{6}\left[f(x_k)+4f(x_{k+1})+f(x_{k+2})\right] \\
&= \frac{h}{3}\left[f(x_k)+4f(x_{k+1})+f(x_{k+2})\right].
\end{aligned}$$

Error calculation:

1. Simple formula:

$$\begin{aligned} \text{error} &= \frac{f^{(4)}(\xi)}{24} \int\limits_{x_k}^{x_{k+2}} (x - x_k)\,(x - x_{k+1})^2\,(x - x_{k+2})\,dx \\ &= -\frac{f^{(4)}(\xi)}{180}(x_{k+2} - x_k)^5 = -\frac{f^{(4)}(\xi)}{90}h^5 \end{aligned}$$

2. Composed formula (n is even):

   For $\xi_k \in \langle a, b \rangle$

$$\begin{aligned} \text{error} &= \sum_{k=0}^{n/2-1} \int\limits_{x_{2k}}^{x_{2k+2}} (x - x_{2k})\,(x - x_{2k+1})^2\,(x - x_{2k+2})\,dx \\ &= -\frac{h^5}{90} \sum_{k=0}^{n/2-1} f^{(4)}(\xi_k). \end{aligned}$$

   Then exists $\xi \in \langle a, b \rangle$ such that

$$\text{error} = -n\frac{h^5}{180} f^{(4)}(\xi) = -\frac{h^4}{180}(b - a) f^{(4)}(\xi).$$