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# 1. Numerical mathematics - introduction

Tento učební text byl podpořen z Operačního programu Praha - Adaptabilita

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## Absolute and relative error

Denote $x^*$ the *exact* quantity and $x$ its approximation, then **absolute error** $e_a$ is equal to the absolute value of their difference

$$e_a = |x^* - x|.$$

Usually we do not know $x^*$ , and thus we look for **estimation** $\varepsilon(x)$ of the absolute error $x$

$$|x^* - x| \le \varepsilon(x).$$

Accuracy of a method is better characterized by **relative error** $e_r$, ratio of the absolute error and absolute value of the exact quantity

$$e_r = \frac{|x^* - x|}{|x^*|}.$$

**Estimation of the relative error** $\delta(x)$ is such a real number that

$$\frac{|x^* - x|}{|x^*|} \le \delta(x).$$

## Norm of vector

Arithmetic vector $x$ is an ordered $n$-tuple of real numbers, i.e. $x = (x_1, \ldots, x_n)$. **Norm of the arithmetic vector** $x \in \mathbb{R}^n$ is a non-negative function (denoted $\|\cdot\|$) that satisfies:

(1) (i) $\|x\| > 0$ for every non-zero vector and $\|x\| = 0$ only for $x = o$,

(2) (ii) $\|c \cdot x\| = |c| \cdot \|x\|$ for every real number $c$,

(3) (iii) $\|x + y\| \le \|x\| + \|y\|$ for every two vectors from $\mathbb{R}^n$.

Examples of norms:

(1) a) $\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, **maximum** norm,

(2) b) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, **octahedron, taxicab, Manhattan** norm,

(3) c) $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$, **euclidean** norm.

**Example 2**: Evaluate all the previous norms of $x = (1, -2, 0, 3)$.

$$\begin{aligned}
\|x\|_\infty &= \max\{|\ 1\ |, |-2|, |\ 0\ |, |\ 3\ |\} = \max\{1, 2, 0, 3\} = 3\\
\|x\|_1 &= |\ 1\ | + |-2| + |\ 0\ | + |\ 3\ | = 1 + 2 + 0 + 3 = 6\\
\|x\|_2 &= \sqrt{1^2 + (-2)^2 + 0^2 + 3^2} = \sqrt{1 + 4 + 0 + 9} = \sqrt{14}
\end{aligned}$$

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MATLAB – norm of vector $v$:

maximum:

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>> norm(v,inf)
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octahedron:

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>> norm(v,1)
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euclidean:

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>> norm(v,2)
>> norm(v)
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Norms in other spaces:

(1) i) In the space of all real numbers $\mathbb{R}$ norm means the absolute value

$$\|x\| = |x|, \qquad x \in \mathbb{R},$$

(2) ii) in the space of all continuous functions on a closed interval $\mathcal{C}_{\langle a,b\rangle}$ we can define a norm in the following way

$$\|f\| = \max_{a \le x \le b} |f(x)|, \qquad f \in \mathcal{C}_{\langle a,b\rangle}.$$

Spaces with norms are called **normed spaces**.

*Remark*: For $u$, $v$ from a normed space $\mathbb{M}$ the number $\|u - v\|$ is called the **distance** of $u$ and $v$.

## Convergence

Let $\mathbb{M}$ be a normed space and let us have a sequence $v_n \in \mathbb{M}$, $n = 1, 2, \dots$.
We say that the sequence $(v_n)$ **converges** to $v \in \mathbb{M}$, i.e. $v_n \longrightarrow v$ for $n \longrightarrow \infty$ (resp. $\lim\limits_{n\to\infty} v_n = v$), if

$$\|v_n - v\| \longrightarrow 0 \qquad \text{pro} \quad n \longrightarrow \infty.$$

We say that the sequence $(v_n)$ is **Cauchy** if

$$\|v_n - v_m\| \longrightarrow 0 \qquad \text{pro} \quad n, m \longrightarrow \infty.$$

*Remark*: If a sequence is convergent in $\mathbb{M}$ the it is Cauchy in $\mathbb{M}$.

A normed space $\mathbb{M}$ is called **complete** if every Cauchy sequence converges in $\mathbb{M}$.
The following normed spaces are complete:

(1) "1)" $\mathbb{M} = \mathbb{R}$ with the norm $\|x\| = |x|$,

(2) "2)" $\mathbb{M} = \mathbb{R}^n$ with the norms $\|x\|_\infty$, $\|x\|_1$, $\|x\|_2$,

(3) "3)" $\mathbb{M} = \mathcal{C}_{\langle a,b\rangle}$ with the norm $\|f\| = \max\limits_{a \le x \le b} |f(x)|$ (the space of continuous functions on a closed interval $\langle a, b\rangle$ ).

**Example 3**: Let us have the space $\mathbb{M} = (0, 1)$ with the norm $\|x\| = |x|$. Then $\mathbb{M}$ is not complete.

# Banach fixed point theorem

Many of the methods we will study are based on the following theorem

**Banach fixed point theorem.**
*Let* $\mathbb{M}$ *be a complete normed space, let us have* $\alpha \in (0,1)$ *and let* $f$ *be a mapping* $\mathbb{M}$ *into* $\mathbb{M}$ *where*

$$\|f(x) - f(y)\| \le \alpha \|x - y\| \qquad \text{for each} \quad x, y \in \mathbb{M}.$$

*Then*

(1) *"a)" There exists a unique point* $x_p$ *with*

$$f(x_p) = x_p.$$

(2) *"b)" For every* $x_0 \in \mathbb{M}$ *the sequence of points*

$$x_n = f(x_{n-1}) \qquad \text{for} \quad n = 1, 2, \ldots$$

*converges to* $x_p$*, i.e.*

$$x_p = \lim_{n\to\infty} x_n.$$

(3) *"c)" And we have the estimations:*

$$\|x_k - x_p\| \le \frac{\alpha^k}{1-\alpha}\|x_0 - x_1\| \qquad \text{for} \quad k > 1,$$

$$\|x_k - x_p\| \le \frac{\alpha}{1-\alpha}\|x_{k-1} - x_k\| \qquad \text{for} \quad k > 1.$$

*Remark*: The point $x_p$ is called the **fixed point** of $f$.