FINANČNÍ INŽENÝRSTVÍ
KBP FFÚ & KSTP FIS

3. Iterative methods for solving linear systems

Tento učební text byl podpořen z Operačního programu Praha - Adaptabilita

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System of linear equations

Let us consider a linear system A$\bar{x} = \bar{b}$, (1)*where*A=(a$_{ij}$) is a real *regular, non-singular, invertible* matrix od the order n, $\bar{b} = (b_1, b_2, \ldots, b_n)'$ is a column vector of right sides, the solution is denoted by $\bar{x}^{(*)} = (x_1^{(*)}, x_2^{(*)}, \ldots, x_n^{(*)})'$.

The system (1) can be put into the equation

$$\bar{x} = M\bar{x} + \bar{v}, \tag{2}$$

where M is an *iteration* matrix, $\bar{v}$ is a column vector. The iterative method means that we choose an initial iteration $\bar{x}^{(0)}$ and according to the formula

$$\bar{x}^{(k+1)} = M\bar{x}^{(k)} + \bar{v}, \qquad k = 0, 1, \ldots \tag{3}$$

we get a sequence of vectors $\bar{x}^{(k)}$ that converges to the solution of the system $\bar{x}^{(*)}$.

The process finishes if

$$\|\bar{x}^{(k+1)} - \bar{x}^{(k)}\| < \varepsilon$$

or after the given number of iterations.

Conditions for convergence:

(1) "a)" *necessary and sufficient condition*
The method (3) converges for any initial iteration $\bar{x}^{(0)}$ iff the spectral radius of the iteration matrix $\rho(M) < 1$.

(2) "b)" *sufficient condition*
If for at least one of the norms of M we get $\|M\| < 1$, then the method (3) converges for any initial iteration $\bar{x}^{(0)}$.

If a sufficietn condition holds then we can estimate the error in the k-th step:

$$\|\bar{x}^* - \bar{x}^{(k)}\| \leq \frac{\|M\|}{1 - \|M\|}\|\bar{x}^{(k)} - \bar{x}^{(k-1)}\|$$

or

$$\|\bar{x}^* - \bar{x}^{(k)}\| \leq \frac{\|M\|^k}{1 - \|M\|}\|\bar{x}^{(1)} - \bar{x}^{(0)}\|.$$

We will study two iterative methods. We need to put the matrix of the system A into

$$A = D + L + U, \tag{4}$$

where D a diagonal matrix, L is strictly (with zeros on the diagonal) lower triangular and U is strictly upper triangular, i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \ldots & 0 \\ a_{21} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & \ldots & a_{1n} \\ 0 & 0 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

Jacobi iterative method

Put the matrix of a system A into $D+L+U$. And then ste system (1) can be expressed:

$$\begin{aligned} A\bar{x} &= \bar{b} \\ (D+L+U)\,\bar{x} &= \bar{b} \\ D\bar{x} &= -(L+U)\,\bar{x} + \bar{b} \\ \bar{x} &= -D^{-1}(L+U)\,\bar{x} + D^{-1}\bar{b}, \end{aligned}$$

this means that in (2) we put

$$M_J = -D^{-1}(L+U), \qquad \bar{v}_J = D^{-1}\bar{b}.$$

Choose $\bar{x}^{(0)}$ and calculate $k = 0, 1, 2, \dots$

$$\bar{x}^{(k+1)} = -D^{-1}(L+U)\,\bar{x}^{(k)} + D^{-1}\bar{b} \tag{J}$$

for particular coordinates:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}} \sum_{j=1,\ j\neq i}^{n} a_{ij} x_j^{(k)} + \frac{b_i}{a_{ii}}, \qquad i = 1, \dots, n, \quad k = 0, 1, \dots \tag{4}$$

Conditions for convergence:

(1) "a)" $\rho(M_J) < 1 \quad \Leftrightarrow \quad$ the Jacobi method converges .
The eigenvalues of M_J are the solutions of $\det(L + \lambda D + U) = 0$.
(2) "b)" There exists a norm with $\|M_J\| < 1 \quad \Rightarrow \quad$ the Jacobi method converges.
(3) "c)" A is strictly diagonally dominant $\quad \Rightarrow \quad$ the Jacobi method converges.

Pklad 1: Let us have the system

$$\begin{aligned} 4x_1 + \ x_2 - 2x_3 &= 4 \\ 2x_1 + 5x_2 + 2x_3 &= 5 \\ x_1 + \ x_2 - 5x_3 &= 4. \end{aligned} \tag{5}$$

We will prove that the Jacobi method converges, choose an intial iteration and compute the first two iterations. The matrix of the system

$$A = \begin{pmatrix} 4 & 1 & -2 \\ 2 & 5 & 2 \\ 1 & 1 & -5 \end{pmatrix}$$

is strictly diagonally dominant, because (for the rows)

$$\begin{array}{ccccc} |\,4\,| & > & |\,1\,| & + & |-2| \\ |\,5\,| & > & |\,2\,| & + & |\,2\,| \\ |-5| & > & |\,1\,| & + & |\,1\,|. \end{array}$$

The sufficient condition c) is satisfied, then the Jacobi method converges.

We transform (5) to get x_1 from the first equation, x_2 from the second one and x_3 from the third one.

$$\begin{aligned} x_1 &= -\tfrac{1}{4}(\quad x_2 - 2x_3 - 4) \\ x_2 &= -\tfrac{1}{5}(2x_1 \quad + 2x_3 - 5) \\ x_3 &= \ \ \tfrac{1}{5}(\ x_1 + \ x_2 \quad - 4). \end{aligned}$$

Put $\bar{x}^{(0)} = (0,0,0)$ and from (4) get the components of $\bar{x}^{(1)}$:

$$\begin{aligned} x_1^{(1)} &= -\tfrac{1}{4}(x_2^{(0)} - 2x_3^{(0)} - 4) = -\tfrac{1}{4}(0 - 0 - 4) = 1 \\ x_2^{(1)} &= -\tfrac{1}{5}(2x_1^{(0)} + 2x_3^{(0)} - 5) = -\tfrac{1}{5}(0 + 0 - 5) = 1 \\ x_3^{(1)} &= \ \ \tfrac{1}{5}(\ x_1^{(0)} + \ x_2^{(0)} - 4) = \ \tfrac{1}{5}(0 + 0 - 4) = -\tfrac{4}{5}. \end{aligned}$$

The first iteration is $\bar{x}^{(1)} = (1, 1, -\frac{4}{5})' = (1; 1; -0,8)'$. Compute the second iteration:

$$\begin{aligned} x_1^{(2)} &= -\tfrac{1}{4}(x_2^{(1)} - 2x_3^{(1)} - 4) = -\tfrac{1}{4}(1 + 2 \cdot \tfrac{4}{5} - 4) = \tfrac{7}{20} \\ x_2^{(2)} &= -\tfrac{1}{5}(2x_1^{(1)} + 2x_3^{(1)} - 5) = -\tfrac{1}{5}(2 - 2 \cdot \tfrac{4}{5} - 5) = \tfrac{23}{25} \\ x_3^{(2)} &= \ \ \tfrac{1}{5}(\ x_1^{(1)} + \ x_2^{(1)} - 4) = \ \tfrac{1}{5}(1 + 1 - 4) = -\tfrac{2}{5}. \end{aligned}$$

The second iteration is $\bar{x}^{(2)} = (\frac{7}{20}, \frac{23}{25}, -\frac{2}{5})' = (0,35; 0,92; -0,4)'$.

The exact solution of the system is $\bar{x}^* = (\frac{1}{2}, 1, -\frac{1}{2})' = (0,5; 1; -0,5)'$.
We estimate the error for the norm $\|\cdot\|_\infty$

$$\|\bar{x}^* - \bar{x}^{(2)}\|_\infty \leq \frac{\|M_J\|_\infty}{1 - \|M_J\|_\infty} \|\bar{x}^{(k)} - \bar{x}^{(k-1)}\|_\infty = \frac{0,8}{1 - 0,8} \cdot 0,65 = 2,6.$$

The estimate is not very accurate because $\|\bar{x}^* - \bar{x}^{(2)}\|_\infty = 0,15$.

MATLAB – Jacobi iterative method

A – matrix of system, b – vector of right sides, $x^{(0)}$ – initial iteration

computing M_J

```
>> U=triu(A,1)
>> L=tril(A,-1)
>> D=A-U-L
>> MJ=(-1)*inv(D)*(U+L)
```

computing v_J

```
>> vJ=inv(D)*b
```

computing iterations $x^{(1)}$, $x^{(2)}$, atd.

```
>> x1=MJ*x0+vJ
>> x2=MJ*x1+vJ
```

and so on.

Gauss–Seidel iterative method

If for computiong the i-th component of the vector of $(k+1)$-th iteration, i.e. x_i^{k+1}, we use the components $x_1^{k+1}, \ldots, x_{i-1}^{k+1}$ computed before, we get the Gauss-Seidel method. We get it from (2) by substitution

$$M_G = -(D+L)^{-1}U, \qquad \bar{v}_G = (D+L)^{-1}\bar{b}$$

and can be expressed:

$$\bar{x}^{(k+1)} = -(D+L)^{-1}U\,\bar{x}^{(k)} + (D+L)^{-1}\bar{b}.$$

For particular components:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}}\left(\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right) + \frac{b_i}{a_{ii}}, \qquad i=1,\ldots,n, \quad k=0,1,\ldots \tag{6}$$

Conditions for convergence:

(1) "a)" $\rho(M_G) < 1 \quad \Leftrightarrow \quad$ the G-S method converges.
The eigenvalues of M_G are the solutions of $\det(\lambda L + \lambda D + U) = 0$.
(2) "b)" There exists a norm with $\|M_G\| < 1 \quad \Rightarrow \quad$ the G-S method converges.
(3) "c)" A is strictly diagonally dominant $\quad \Rightarrow \quad$ the G-S method converges.
(4) "d)" A is symmetric and positively definite $\quad \Rightarrow \quad$ the G-S method converges.

Example 2: For the system (5) compute the first two iterations by using G-S method.

The matrix A is strictly diagonally dominant, the condition c) for convergence is satisfied and the G-S method converges. We transform the system:

$$\begin{aligned} x_1 &= -\tfrac{1}{4}(\phantom{2x_1 +{}} x_2 - 2x_3 - 4) \\ x_2 &= -\tfrac{1}{5}(2x_1 \phantom{{}+x_2} + 2x_3 - 5) \\ x_3 &= \tfrac{1}{5}(\ x_1 + \ x_2 \phantom{{}+2x_3} - 4), \end{aligned}$$

but for computing we use (6), this means that for the second and third components of $\bar{x}^{(1)}$ we use the computed components.

$$\begin{aligned} x_1^{(1)} &= -\tfrac{1}{4}(x_2^{(0)} - 2x_3^{(0)} - 4) = -\tfrac{1}{4}(0 - 0 - 4) = 1 \\ x_2^{(1)} &= -\tfrac{1}{5}(2\boldsymbol{x_1^{(1)}} + 2x_3^{(0)} - 5) = -\tfrac{1}{5}(2 + 0 - 5) = \tfrac{3}{5} \\ x_3^{(1)} &= \tfrac{1}{5}(\ \boldsymbol{x_1^{(1)}} + \ \boldsymbol{x_2^{(1)}} - 4) = \ \tfrac{1}{5}(1 + \tfrac{3}{5} - 4) = -\tfrac{12}{25}. \end{aligned}$$

The first iteration is $\bar{x}^{(1)} = (1, \frac{3}{5}, -\frac{12}{25})' = (1; 0,6; -0,48)'$. We compute the second iteration:

$$\begin{aligned} x_1^{(2)} &= -\tfrac{1}{4}(x_2^{(1)} - 2x_3^{(1)} - 4) = -\tfrac{1}{4}(\tfrac{3}{5} + 2\cdot\tfrac{12}{25} - 4) = \tfrac{61}{100} \\ x_2^{(2)} &= -\tfrac{1}{5}(2\boldsymbol{x_1^{(2)}} + 2x_3^{(1)} - 5) = -\tfrac{1}{5}(2\cdot\tfrac{61}{100} - 2\cdot\tfrac{12}{25} - 5) = \tfrac{474}{500} \\ x_3^{(2)} &= \tfrac{1}{5}(\ \boldsymbol{x_1^{(2)}} + \ \boldsymbol{x_2^{(2)}} - 4) = \ \tfrac{1}{5}(\tfrac{61}{100} + \tfrac{474}{500} - 4) = -\tfrac{1221}{2500}. \end{aligned}$$

The second iteration is $\bar{x}^{(2)} = (\frac{61}{100}, \frac{474}{500}, -\frac{1221}{2500})' = (0,61; 0,948; -0,4884)'$.

We estimate the error:

$$\|\bar{x}^* - \bar{x}^{(2)}\|_\infty \le \frac{\|M_G\|_\infty}{1 - \|M_G\|_\infty}\|\bar{x}^{(k)} - \bar{x}^{(k-1)}\|_\infty = \frac{0,75}{1 - 0,75}\cdot 0,39 = 1,17.$$

In fact the error is $\|\bar{x}^* - \bar{x}^{(2)}\|_\infty = 0,11$.

If both methods – Jacobi and Gauss–Seidel – converge, Gauss–Seidel method converges faster.

MATLAB – Gauss–Seidel method

A – matrix of system, b – vector of right sides, $x^{(0)}$ – initial iteration

computing M_G

```
>> U=triu(A,1)
>> L=tril(A,-1)
>> D=A-U-L
>> MG=(-1)*inv(D+L)*U
```

computing v_G

```
>> vG=inv(D+L)*b
```

computing iterations $x^{(1)}$, $x^{(2)}$, and so on.

```
>> x1=MG*x0+vG
>> x2=MG*x1+vG
```

and so on.