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4.5 Iterative methods for solving nonlinear systems

Tento učební text byl podpořen z Operačního programu Praha - Adaptabilita

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Systems of nonlinear equations – Newton's method

Let us consider a nonlinear system

$$F(\bar{x}) = \bar{o}, \tag{7}$$

that is

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ f_2(x_1, \ldots, x_n) &= 0 \\ &\cdots \\ f_n(x_1, \ldots, x_n) &= 0 \end{aligned}$$

and assume that $f_1, \ldots, f_n$ have continuous second derivatives.

Let us describe the method for two functions of two variables

$$\begin{aligned} f_1(x_1, x_2) &= 0 \\ f_2(x_1, x_2) &= 0. \end{aligned}$$

In a neighborhood of $\bar{c} = (c_1, c_2)'$ we substitute functions f_1, f_2 by the Taylor polynomial of the first degree (tangent plane)

$$\begin{aligned} f_1(x_1, x_2) &\approx f_1(c_1, c_2) + \frac{\partial f_1}{\partial x_1}(c_1, c_2)(x_1 - c_1) + \frac{\partial f_1}{\partial x_2}(c_1, c_2)(x_2 - c_2) = 0 \\ f_2(x_1, x_2) &\approx f_2(c_1, c_2) + \frac{\partial f_2}{\partial x_1}(c_1, c_2)(x_1 - c_1) + \frac{\partial f_2}{\partial x_2}(c_1, c_2)(x_2 - c_2) = 0. \end{aligned}$$

This system can be expressed:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(c_1, c_2) & \frac{\partial f_1}{\partial x_2}(c_1, c_2) \\ \frac{\partial f_2}{\partial x_1}(c_1, c_2) & \frac{\partial f_2}{\partial x_2}(c_1, c_2) \end{pmatrix} \begin{pmatrix} x_1 - c_1 \\ x_2 - c_2 \end{pmatrix} = - \begin{pmatrix} f_1(c_1, c_2) \\ f_2(c_1, c_2) \end{pmatrix} \tag{8}$$

Denote F' **Jacobian (matrix)**

$$F' = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \qquad \text{a} \qquad \bar{d} = \bar{x} - \bar{c},$$

the system (8) is then of the form

$$F'(\bar{c})\,\bar{d} = -F(\bar{c}) \qquad \text{a} \qquad \bar{x} = \bar{c} + \bar{d}.$$

Choose an initial iteration $\bar{x}^{(0)}$ and for $k = 0, 1, 2, \ldots$ we evaluate in two steps:

i) find $\bar{d}^{(k)}$ as the solution of the system

$$F'(\bar{x}^{(k)})\,\bar{d}^{(k)} = -F(\bar{x}^{(k)}), \tag{9}$$

ii) put

$$\bar{x}^{(k+1)} = \bar{x}^{(k)} + \bar{d}^{(k)}.$$

For every vector $\bar{x}^{(k)}$ the matrix $F'(\bar{x}^{(k)})$ must be regular (non-singular, invertible) so that the system (9) has the unique solution. If this condition is not satisfied we take another vector $\bar{x}^{(0)}$ and calculate again.

Newton's method converges to the exact solution if $\bar{x}^{(0)}$ is near the exact solution.

Example 3: For the system

$$\begin{aligned} x_1^2 + x_2^2 &= 4 \\ x_2 &= x_1^3 + 1 \end{aligned}$$

evaluate two iterations using Newton's method with the initial iteration $\bar{x}^{(0)} = (1, 2)'$.

The system can be expressed:

$$F'(\bar{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 - 4 \\ -x_1^3 + x_2 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \bar{o}, \qquad \text{where} \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

form the Jacobian

$$F'(\bar{x}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 \\ -3x_1^2 & 1 \end{pmatrix}.$$

For evaluation the *first* iteration $\bar{x}^{(1)} = (x_1^{(1)}, x_2^{(1)})'$:

i) first find $\bar{d}^{(0)} = (d_1^{(0)}, d_2^{(0)})'$ solving the system

$$F'(\bar{x}^{(0)})\,\bar{d}^{(0)} = -F(\bar{x}^{(0)}),$$

after substitution $\bar{x}^{(0)} = (1, 2)'$

$$\begin{pmatrix} 2 \cdot 1 & 2 \cdot 2 \\ -3 \cdot 1^2 & 1 \end{pmatrix} \begin{pmatrix} d_1^{(0)} \\ d_2^{(0)} \end{pmatrix} = - \begin{pmatrix} 1^2 + 2^2 - 4 \\ -1^3 + 2 - 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} d_1^{(0)} \\ d_2^{(0)} \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we get $\bar{d}^{(0)} = (-\frac{1}{14}, -\frac{3}{14})'$.

ii) in the second step e find the first iteration

$$\bar{x}^{(1)} = \bar{x}^{(0)} + \bar{d}^{(0)},$$

i.e.

$$\bar{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{14} \\ -\frac{3}{14} \end{pmatrix} = \begin{pmatrix} \frac{13}{14} \\ \frac{25}{14} \end{pmatrix} \approx \begin{pmatrix} 0,9286 \\ 1,7857 \end{pmatrix}$$

Similarly evaluate the *second* iteration $\bar{x}^{(2)} = (x_1^{(2)}, x_2^{(2)})'$:

i) first find $\bar{d}^{(1)} = (d_1^{(1)}, d_2^{(1)})'$ solving the system

$$F'(\bar{x}^{(1)})\,\bar{d}^{(1)} = -F(\bar{x}^{(1)}),$$

after substitution $\bar{x}^{(1)} = (\frac{13}{14}, \frac{25}{14})'$

$$\begin{pmatrix} 2 \cdot \frac{13}{14} & 2 \cdot \frac{25}{14} \\ -3 \cdot (\frac{13}{14})^2 & 1 \end{pmatrix} \begin{pmatrix} d_1^{(1)} \\ d_2^{(1)} \end{pmatrix} = - \begin{pmatrix} (\frac{13}{14})^2 + (\frac{25}{14})^2 - 4 \\ -(\frac{13}{14})^3 + \frac{25}{14} - 1 \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \begin{pmatrix} \frac{13}{7} & \frac{25}{7} \\ -\frac{507}{196} & 1 \end{pmatrix} \begin{pmatrix} d_1^{(1)} \\ d_2^{(1)} \end{pmatrix} = - \begin{pmatrix} \frac{5}{98} \\ -\frac{41}{2744} \end{pmatrix}$$

we get $\bar{d}^{(1)} = (-\frac{105}{11161}, -\frac{11}{1171})'$.

ii) in the second step we get

$$\bar{x}^{(2)} = \bar{x}^{(1)} + \bar{d}^{(1)},$$

i.e.

$$\bar{x}^{(2)} = \begin{pmatrix} \frac{13}{14} \\ \frac{25}{14} \end{pmatrix} + \begin{pmatrix} -\frac{105}{11161} \\ -\frac{11}{1171} \end{pmatrix} = \begin{pmatrix} \frac{1319}{1435} \\ \frac{4876}{2745} \end{pmatrix} \approx \begin{pmatrix} 0,9192 \\ 1,7763 \end{pmatrix}.$$

MATLAB – Newton's method

F – given functions (column vector), $x^{(0)}$ – initial iteration

symbolic variables

```
>> syms x,y;
```

initial iteration $x^{(0)}$ and function F

```
>> x0=[1; 2]
>> F=[x^2+y^2-4; y-x^3-1]
```

Jacobian

```
>> J=jacobian(F,[x,y])
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substitution $x = x_1^{(0)}$, $y = x_2^{(0)}$ into Jacobian and F

```
>> J0=subs(J,[x,y],x0)
>> F0=subs(F,[x,y],x0)
```

evaluation of the first iteration $x^{(1)}$ ($d^{(0)} = -\text{inv}(J0) * F0$)

```
>> x1=x0-inv(J0)*F0
```

výpočet druhé aproximace $x^{(2)}$ ($d^{(1)} = -\text{inv}(J1) * F1$)

```
>> J1=subs(J,[x,y],x1)
>> F1=subs(F,[x,y],x1)
>> x2=x1-inv(J1)*F1
```

and so on.
